Ivan Chajda

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*Archivum Mathematicum*, Vol. 13 (1977), No. 3, 135--140

Persistent URL: http://dml.cz/dmlcz/106969

ARCH. MATH. 3, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XIII: 135—140, 1977

# WEAK HOMOMORPHISMS OF SYSTEMS OF EQUATIONS

IVAN CHAJDA, Přerov

(Received October 6, 1976)

1. The concept of a weak isomorphism for Post algebras and, in particular, Boolean algebras was introduced by T. Traczyk in [3]. In [1], there is generalized the concept of a weak isomorphism for arbitrary classes of abstract algebras and the concept of a weak homomorphism is defined there.

Systems of equations over algebras were introduced and investigated in [2] by J. Słominski. We can now introduce solutions of systems of equations and investigate relations between the solutions of the original system and the solutions of its weakly homomorphic maps. This is a purpose of the paper.

2. By the symbol $\mathfrak{A} = (A, \Omega)$ an algebra with a set of elements $A$ and a set of *fundamental* operations $\Omega$ is denoted. Denote by $\bar{\Omega}$ the set of all *algebraic* operations of this algebra. Let $\mathfrak{A} = (A, \Omega)$ and $\mathfrak{B} = (B\ \Omega^*)$ be two algebras and $\varphi$ be a mapping of $A$ into $B$. For $n$-ary $\omega \in \bar{\Omega}$, the result of $\omega$ for elements $a_1, \dots, a_n \in A$ will be denoted by $\omega(a_1, \dots, a_n)$. Making use of the mapping $\varphi$ we define a relation $R_\varphi$ between $\bar{\Omega}$ and $\bar{\Omega}^*$ setting for $\omega \in \bar{\Omega}$ and $\omega^* \in \bar{\Omega}^*$:

$$\omega R_\varphi \omega^* \qquad \text{if and only if} \qquad \omega^* . \varphi = \varphi . \omega, \tag{R}$$

i.e. $\omega^*(\varphi(a_1), \dots, \varphi(a_n)) = \varphi(\omega(a_1, \dots, a_n))$ for each $n$-tuple $a_1, \dots, a_n \in A$. The three following properties of the relation $R_\varphi$ are proved in [1]:

(a) If $\omega \in \bar{\Omega}$ is $n$-ary and $\omega R_\varphi \omega^*$ for $\omega^* \in \bar{\Omega}^*$, then $\omega^*$ is also $n$-ary with the same $n$.

(b) If $\omega \in \bar{\Omega}$ is $n$-ary, $\omega_1, \dots, \omega_n \in \bar{\Omega}$ and $\omega R_\varphi \omega^*$, $\omega_i R_\varphi \omega_i^*$ for each $i = 1, \dots, n$, then $\omega(\omega_1, \dots, \omega_n)\ R_\varphi \omega^*(\omega_1^*, \dots, \omega_n^*)$, where $\omega(\omega_1, \dots, \omega_n)$ denotes the superposition of operations (see [1]).

(c) If $\mathfrak{A} = (A, \Omega)$, $\mathfrak{B} = (B, \Omega^*)$, $\mathfrak{C} = (C, \Omega^{**})$ are three algebras and $\varphi_1$: $A \to B$, $\varphi_2$: $B \to C$ mappings, if $\omega \in \bar{\Omega}$, $\omega^* \in \bar{\Omega}^*$, $\omega^{**} \in \bar{\Omega}^{**}$ and $\omega R_{\varphi_1} \omega^*$, $\omega^* R_{\varphi_2} \omega^{**}$, then $\omega R_{\varphi_2 \varphi_1} \omega^{**}$ (where $\varphi_2 \varphi_1$ denotes the composition of the mappings $\varphi_1, \varphi_2$).

Now, we can accept a concept of weak homomorphism taken from [1]:

**Definition 1.** Let $\mathfrak{A} = (A, \Omega)$, $\mathfrak{B} = (\mathfrak{B}, \Omega^*)$ be two algebras. The mapping $\varphi$ of $A$ into $B$ is called a *weak homomorphism* of $\mathfrak{A}$ into $\mathfrak{B}$ if to each fundamental operation $\omega \in \Omega$ there exists an algebraic operation $\omega^* \in \bar{\Omega}^*$ such that $\omega R_\varphi \omega^*$ and to each fundamental operation $\omega_0 \in \Omega^*$ there exists an algebraic operation $\omega_0 \in \bar{\Omega}$ such that $\omega_0 R_\varphi \omega_0^*$, where $R_\varphi$ is the relation corresponding to $\varphi$ by the rule (R).

Therefore, every homomorphism $\varphi$ of algebras in the usual sense is, at the same time, a weak homomorphism. In this case, there is a fixed one-to-one correspondence between fundamental operations $\omega \in \Omega$ and $\omega^* \in \Omega^*$ fulfilling $\omega R_\varphi \omega^*$.

**Definition 2.** Let $\mathfrak{A} = (A, \Omega)$, $\mathfrak{B} = (B, \Omega^*)$ be two algebras and let a mapping $\varphi$ of $A$ into $B$ be a weak homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. If $\varphi$ is a one-to-one mapping of $A$ onto $B$, then $\varphi$ is called a *weak isomorphism* of $\mathfrak{A}$ onto $\mathfrak{B}$.

From the definitions, it is evident that the inverse mapping of a weak isomorphism is also a weak isomorphism (Theorem 4 in [1]). Further, for a weak isomorphism $\varphi$, the relation $R_\varphi$ defined by (R) is a one-to-one correspondence $\omega \leftrightarrow \omega^*$ between $\bar{\Omega}$ and $\bar{\Omega}^*$ (see [1], p. 164). For a weak homomorphism $\varphi$ of an algebra $\mathfrak{A}$, the restriction $\varphi \mid \mathfrak{A}_0$ onto a subalgebra $\mathfrak{A}_0 \subseteq \mathfrak{A}$ is also a weak homomorphism and a composition of two weak homomorphisms is a weak homomorphism, again.

3. Let $\mathfrak{A} = (A, \Omega)$ be an algebra and $X$ be a set of elements $x_\mu$, i.e. $X = \{x_\mu, \mu < s\}$, where $s$ is an ordinal number. Let $X \cap A = \emptyset$, $X \cap \Omega = \emptyset$. Denote by $\mathfrak{A}(X)$ the set of all expressions consisting of elements of the algebra $\mathfrak{A} = (A, \Omega)$, of $\Omega$ and of the set $X$, which would give elements of $A$ if the elements of $X$ were replaced by elements of $A$; that is, expressions with the right number of elements (from $A$ or $X$) after each operation-symbol.

Let us introduce formal operations on $\mathfrak{A}(X)$ by the natural way; namely, if $\omega \in \Omega$ is $n$-ary and $\tau_1, \ldots, \tau_n \in \mathfrak{A}(X)$, then, by the definition of $\mathfrak{A}(X)$, also $\omega(\tau_1, \ldots, \tau_n) \in \mathfrak{A}(X)$. Clearly, $(\mathfrak{A}(X), \Omega)$ is also an algebra.

If $\tau$ is an element of $\mathfrak{A}(X)$ and $a = \{a_\mu, \mu < s\}$ is a sequence of elements $a_\mu \in A$, then $\bar{\tau}_a$ denotes an element of $A$ obtained from $\tau$ by replacing all elements of $X$ generating $\tau$ by elements of $A$ such that each $x_\mu \in X$ is replaced by the same $a_\mu$ in all places in $\tau$. Let us introduce the equivalence relation $\Theta$ on $\mathfrak{A}(X)$ by:

$\tau, \vartheta \in \mathfrak{A}(X)$, $\tau \Theta \vartheta$ if and only if $\bar{\tau}_a = \bar{\vartheta}_a$ for each $a = \{a_\mu, \mu < s\}$.

It is clear that $\Theta$ is a congruence relation on $(\mathfrak{A}(X), \Omega)$.

**Definition 3.** The factor-algebra $(\mathfrak{A}(X), \Omega)/\Theta$ is called a *formal* $\mathfrak{A}$*-polynomial algebra* and it is denoted by $For(\mathfrak{A}, X)$. Each element of $For(\mathfrak{A}, X)$ is called $\mathfrak{A}$*-term* (or briefly *term*).

**Convention.** Any term $\tau \in For(\mathfrak{A}, X)$ generated only by the set $\{x_\mu, \mu < k\} \cup A$ and by operations $\Omega'$, where $k \leq s$ and $\Omega' \subseteq \Omega$, is denoted by $\tau(x_\mu, k, \Omega')$.

**Remark.** $\mathfrak{A}$ is a subalgebra of $For(\mathfrak{A}, X)$ because the implication $Y \subseteq X \Rightarrow$ $\Rightarrow For(\mathfrak{A}, Y)$ is a subalgebra of $For(\mathfrak{A}, X)$ is true and $\mathfrak{A} = For(\mathfrak{A}, \emptyset)$.

4. Let $\mathfrak{A} = (A, \Omega)$ be an algebra, $X = \{x_\mu, \mu < s\}$ be a set such that $A \cap X =$ $= \emptyset = \Omega \cap X$ and $S$ be the set of all elements from $For(\mathfrak{A}, X)$, i.e. so called *support* of $For(\mathfrak{A}, X)$.

**Definition 4.** The subset $E$ of the Cartesian product $S \times S$ is said to be a *system of equations over* $\mathfrak{A}$, each pair $\langle \tau, \vartheta \rangle \in E$ of $\mathfrak{A}$-terms $\tau$, $\vartheta$ is called an *equation*. Elements of $X$ generating $\tau$, $\vartheta$ for $\langle \tau, \vartheta \rangle \in E$ are called *unknowns* of the equation $\langle \tau, \vartheta \rangle \in E$.

This conception of system of equations over $\mathfrak{A}$ is taken from [2].

**Definition 5.** Let $\mathfrak{A}$ be an algebra, $E$ be a system of equations over $\mathfrak{A}$ and $X = \{x_\mu, \mu < s\}$ be a set of unknowns of $E$. Let $\sim$ be a congruence relation on $\mathfrak{A}$. Any sequence $v = \{V_\mu, \mu < s\}$ of elements $V_\mu \in A$ such that $\bar{\tau}_v \sim \bar{\vartheta}_v$ for each $\langle \tau, \vartheta \rangle \in E$ is called a *solution of E with the regularizer* $\sim$. If $\sim$ is the *identity relation* on $\mathfrak{A}$, the solution $v$ with the regularizer $\sim$ is called *proper*.

**Remark.** The proper solution is a solution in the sense of the classical definition. The above mentioned definition is, however, more general than the classical one.

5. Each weak homomorphism transforms congruence relations as it is shown in the following:

**Theorem 1.** *Let* $\mathfrak{A} = (A, \Omega)$, $\mathfrak{B} = (B, \Omega^*)$ *and* $\varphi$ *be a weak homomorphism of* $\mathfrak{A}$ *into* $\mathfrak{B}$. *If* $\sim$ *is a congruence relation on* $\mathfrak{B}$, *the relation* $\sim_\varphi$ *on* $\mathfrak{A}$ *defined by*

$$a_1 \sim_\varphi a_2 \quad \text{for } a_1, a_2 \in A \quad \text{if and only if} \quad \varphi(a_1) \sim \varphi(a_2) \qquad \text{(P)}$$

*is a congruence relation on* $\mathfrak{A}$.

Proof. It is evident that $\sim_\varphi$ defined by (P) is an equivalence relation on $\mathfrak{A}$ because the reflexivity, transitivity and symmetry of $\sim_\varphi$ follow directly from (P). Let $\sim_\varphi$ not be a congruence relation on $\mathfrak{A}$. Then there exists at least one sequence $\{(a_i, b_i), i = 1, \ldots, n\}$, where $a_i, b_i \in A$, and at least one $n$-ary operation $\omega \in \Omega$ such that $a_i \sim_\varphi b_i$ for each $i$ but $\omega(a_1, \ldots, a_n) \sim_\varphi \omega(b_1, \ldots, b_n)$ is not true. Let $\omega^* \in \bar{\Omega}^*$ be an operation fulfilling (R) with $\omega$. From this it follows by (P) that $\varphi(a_i) \sim \varphi(b_i)$ and $\omega^*(\varphi(a_i), \ldots, \varphi(a_n)) = \varphi(\omega(a_1, \ldots, a_n))$, $\omega^*(\varphi(b_1), \ldots, \varphi(b_n)) =$ $= \varphi(\omega(b_1, \ldots, b_n))$, thus it is not true

$$\varphi(\omega(a_1, \ldots, a_n)) \sim \varphi(\omega(b_1, \ldots, b_n))$$

which is a contradiction, because $\sim$ is a congruence.

**Definition 6.** Let $\mathfrak{A} = (A, \Omega)$, $\mathfrak{B} = (B, \Omega^*)$ be algebras. A mapping $\psi$ of $For(\mathfrak{A}, X)$ into $For(\mathfrak{B}, X)$ is called a *transformation*, if there exists $\omega^* \in \bar{\Omega}^*$ for each $\omega \in \Omega$ such that $\omega R_\psi \omega^*$ (where $\omega$, $\omega^*$ are performed on $For(\mathfrak{A}, X)$, $For(\mathfrak{B}, X)$, respectively).

**Theorem 2.** *Let $\mathfrak{A}$, $\mathfrak{B}$ be algebras and $\varphi$ a weak homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Let $h$ be a mapping of $X$ into itself. Then there exists a transformation $\varphi_h$ of $For(\mathfrak{A}, X)$ into $For(\mathfrak{B}, X)$ such that*

(1) $$\varphi_h(x) = h(x) \qquad \textit{for each } x \in X$$

(2) $$\varphi_h \mid \mathfrak{A} = \varphi.$$

Proof. If $\omega_\alpha \in \Omega$, then we put $\Omega_\alpha = \{\omega \in \bar{\Omega}^*;\ \omega_\alpha R_\varphi \omega\}$.

As $\varphi$ is a weak homomorphism, $\Omega_\alpha \neq \emptyset$ exists for each $\omega_\alpha \in \Omega$. Let $\xi$ be a choice function on $\{\Omega_\alpha,\ \omega_\alpha \in \Omega\}$. Then $\omega_\alpha R_\varphi \omega_\xi$, where $\omega_\xi = \xi(\Omega_\alpha)$.

Now, we can construct the mapping $\varphi_h$ of $For(\mathfrak{A}, X)$ into $For(\mathfrak{B}, X)$ by the induction of the rank of polynomial (see e.g. [4], p. 40, Remark before Lemma 5):

(i) $\varphi_h(a) = \varphi(a)$ for each $a \in A$

(ii) $\varphi_h(x) = h(x)$ for each $x \in X$

(iii) if $\omega \in \Omega$ is $n$-ary, $\omega R_\varphi \omega_\xi$ and, for $\tau_1, \ldots, \tau_n \in For(\mathfrak{A}, X)$, we have $\varphi_h(\tau_i) = \sigma_i \in For(\mathfrak{B}, X)$ $(i = 1, \ldots, n)$, then $\varphi_h(\omega(\tau_1, \ldots, \tau_n)) = \omega_\xi(\sigma_1, \ldots, \sigma_n)$.

Clearly, (ii) is equal to (1). If $a_1, \ldots, a_n \in A$, $\omega_\alpha \in \Omega$ is $n$-ary, $\omega_\alpha R_\varphi \omega^*$, then $\omega^* \in \Omega_\alpha$, clearly $\omega^*(\varphi_h(a_1), \ldots, \varphi_h(a_n)) = \omega_\xi(\varphi_h(a_1), \ldots, \varphi_h(a_n))$, thus clearly $\varphi_h \mid \mathfrak{A} = \varphi$, i.e. (2) is also true. By the induction of the rank of polynomial, from (iii), it follows that $\varphi_h$ is a transformation.

**Notation.** Let $\varphi$ be a weak homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ and $h$ a mapping of $X$ into itself. Denote by $\varphi_h$ an arbitrary (fix choosen) transformation of $For(\mathfrak{A}, X)$ into $For(\mathfrak{B}, X)$ such that (1) and (2) of Theorem 2 is true.

Now, we can precisely formulate our problem.

**Assumptions:**

(1) $\mathfrak{A} = (A, \Omega)$, $\mathfrak{B} = (\mathfrak{B}, \Omega^*)$ are two algebras.

(2) $E$ is a system of equations over $\mathfrak{A}$ with the set of unknowns $X = \{x_\mu,\ \mu < s\}$.

(3) $\varphi$ is a weak homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$.

(4) the mapping $h$ of $X$ into itself is the identity mapping on $X$.

(5) $\sim$ is a congruence relation on $\mathfrak{B}$.

**Problem:** Let the Assumption (1)–(5) be valid and $\varphi_h$ be a transformation of $For(\mathfrak{A}, X)$ into $For(\mathfrak{B}, X)$. We put

$$\varphi_h(E) = \{\langle \varphi_h(\tau), \varphi_h(\vartheta) \rangle;\ \text{for } \langle \tau, \vartheta \rangle \in E\}.$$

Which is a relationship between the solutions of the system of equations $(E)$ over $\mathfrak{B}$ with the regularizer $\sim$ and solutions of the original system $E$ over $\mathfrak{A}$?

**Theorem 3.** *Let Assumptions* (1)−(5) *be true and* $v = \{V_\mu, \mu < s\}$ *be a solution of the system* $\varphi_h(E)$ *with the regularizer* $\sim$. *Let* $W_\mu$ be an arbitrary element of $A$ fulfilling $\varphi(W_\mu) = V_\mu$ for each $\mu < s$. Then $w = \{W_\mu; \mu < s\}$ is a solution of $E$ with the regularizer $\sim_\varphi$ given by the rule (P).

Proof. From $\varphi_h \mid \mathfrak{A} = \varphi$, it follows $\sim_{\varphi_h} = \sim_\varphi$ on $A$ and, by Theorem 1, $\sim_\varphi$ is a congruence on $\mathfrak{A}$. Let $v = \{V_\mu, \mu < s\}$ be a solution of $\varphi_h(E)$ with the regularizer $\sim$, i.e.

$$\overline{\varphi_h(\tau)_v} \sim \overline{\varphi_h(\vartheta)_v} \tag{K}$$

for each $\langle\varphi_h(\tau), \varphi_h(\vartheta)\rangle \in \varphi_h(E)$. Let $w = \{W_\mu, \mu < s\}$, where $W_\mu \in A$ and $\varphi(W_\mu) = V_\mu$. Then evidently

$$\varphi_h(W_\mu) = V_\mu. \tag{L}$$

Denote by $\sigma_a$ the mapping: $\tau \to \bar{\tau}_a$. As $h$ is the identity mapping on $X$ (by the Assumption (4)), from (L), we obtain the commutativity of the following diagram

$$\begin{array}{ccc} \tau(x_\mu, k, \Omega) & \xrightarrow{\sigma_w} & \tau(W_\mu, k, \Omega) = \bar{\tau}_w \\ \downarrow \varphi_h & & \downarrow \varphi_h \\ \varphi_h(\tau)\,(x_\mu, k, \bar{\Omega}') & \xrightarrow{\sigma_v} & \varphi_h(\tau)\,(V_\mu, k, \bar{\Omega}') \end{array}$$

for an arbitrary $\tau \in For(\mathfrak{A}, X)$, if we use the notation by the above mentioned convention. Accordingly, we have

$$\overline{\varphi_h(\tau)_v} = \varphi_h(\bar{\tau}_w). \tag{M}$$

From $\bar{\tau}_w \in A$, it follows $\varphi_h(\bar{\tau}_w) = \varphi(\bar{\tau}_w)$ and by (M), we obtain

$$\overline{\varphi_h(\tau)_v} = \varphi(\bar{\tau}_w). \tag{N}$$

Now, from (K) and (N), we have

$$\varphi(\bar{\tau}_w) \sim \varphi(\bar{\vartheta}_w) \qquad \text{for each } \langle\tau, \vartheta\rangle \in E$$

and, by the theorem 1, we obtain

$$\bar{\tau}_w \sim_\varphi \bar{\vartheta}_w \qquad \text{for each } \langle\tau, \vartheta\rangle \in E$$

which complete the proof.

6. It is possible that the system $E$ has other solutions which cannot be obtained from solutions of $\varphi_h(E)$ by Theorem 3. The solution of $E$ obtained from the solution $v = \{V_\mu, \mu < s\}$ of $\varphi_h(E)$ by Theorem 3 is called *induced by the solution v*. The solution of $E$ induced by a proper solution of $\varphi_h(E)$ need not be proper. However, from Theorem 3 we obtain the following:

**Corollary 1.** *Let the Assumptions* (1)−(5) *be true and* $v = \{V_\mu, \mu < s\}$ *be a solution of the system* $\varphi_h(E)$ *with the regularizer* $\sim$. *Then each solution of the original* *system E induced by v is proper if and only if the two following conditions are true:*

(a) *$v$ is a proper solution of $\varphi_k(E)$,*
(b) *$\varphi$ is a weak isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$.*

Proof. The sufficiency is evident. Necessity: let $v = \{V_\mu, \mu < s\}$ not be a proper solution of $\varphi_h(E)$, i.e. $\sim$ is not the identity on $B$. Further, let the induced solution $w$ be proper. Then from $\bar{\tau}_w = \bar{\vartheta}_w$ we obtain directly by Theorem 3

$$\varphi_h(\tau)_v = \varphi_h(\vartheta)_v \qquad \text{for each } \langle \varphi_h(\tau), \varphi_h(\vartheta) \rangle \in \varphi_h(E),$$

which is a contradiction. Let $\varphi$ not be a weak isomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, then for at least one element $b \in \varphi(\mathfrak{A})$ there exist $a_1, a_2 \in A$, $a_1 \neq a_2$ such that $\varphi(a_1) = \varphi(a_2) = b$. Then $\sim_\varphi$ is not the identity on $A$, which is a contradiction again. q.e.d.

It is possible that the system $\varphi_k(E)$ has not any solution in $\mathfrak{B}$ but it has a solution in $\mathfrak{B}'$ such that $\mathfrak{B}$ is a subalgebra of $\mathfrak{B}'$. By Theorem 3, we can easily prove the following:

**Corollary 2.** *Let the Assumptions* (1)–(5) *be true and $\mathfrak{B}'$ be an algebra such that $\mathfrak{B}$ is a subalgebra of $\mathfrak{B}'$ and $\varphi_h(E)$ has a solution $v = \{V_\mu, \mu < s\}$ with the regularizer $\sim'$ in $\mathfrak{B}'$, where $\sim'$ is a congruence on $\mathfrak{B}'$ such that $\sim' \mid \mathfrak{B} = \sim$. If there exists an algebra $\mathfrak{A}'$ such that $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}' = (A', \Omega)$ and a weak homomorphism $\psi$ of $\mathfrak{A}'$ into $\mathfrak{B}'$ such that $\psi \mid \mathfrak{A} = \varphi$, then each sequence $w = \{W_\mu, \mu < s\}$ of elements $W_\mu \in A'$ fulfilling $\psi(W_\mu) = V_\mu$ is the solution of the original system E over $\mathfrak{A}'$ with the regularizer $\sim'_\psi$ given by* (P).

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*I. Chajda*
*750 00 Přerov, třída Lidových milicí 290*
*Czechoslovakia*