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LATTICES OF COMPATIBLE RELATIONS

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As it is shown in [3], [4], [5] and [6], various results on congruences on algebras can be generalized also for other types of relations. The aim of this paper is to show some of common properties of lattices of relations and give mutual interrelations among these lattices.

By $\mathfrak{A} = \langle A, F \rangle$ denote an algebra with a base set A and the set of fundamental operations F. A binary relation R on A (i.e. $R \subseteq A \times A$) is called to be *compatible on A*, if for each n-ary $f \in F$ and arbitrary a_i, $b_i \in A$ $(i = 1, \ldots, n)$ the following implication is true:

$$\langle a_i, b_i \rangle \in R \quad \text{for} \quad i = 1, \ldots, n \Rightarrow \langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \rangle \in R.$$

By $\mathsf{Comp}(\mathfrak{A})$ or $\mathscr{R}(\mathfrak{A})$ or $\mathscr{S}(\mathfrak{A})$ or $\mathscr{T}(\mathfrak{A})$ or $\mathscr{L}\mathscr{T}(\mathfrak{A})$ or $\mathscr{Q}(\mathfrak{A})$ or $\mathsf{In}(\mathfrak{A})$ or $\mathscr{C}(\mathfrak{A})$ denote the set of all *compatible* or *reflexive and compatible* or *symmetric and compatible* or *transitive and compatible* or *compatible reflexive and symmetric* (so called *tolerance*) or *compatible reflexive and transitive* (i.e. *quasiorder*) or *compatible symmetric and transitive* (so called *quasiequivalences* or *congruences "in"*) or *congruence relations* on $\mathfrak{A}$, respectively. Further, denote by $\Lambda = \{\mathsf{Comp}, \mathscr{R}, \mathscr{S}, \mathscr{T}, \mathscr{L}\mathscr{T}, \mathscr{Q}, \mathsf{In}, \mathscr{C}\}$ and agree with a convention: $\mathscr{P}(\mathfrak{A})$ for $\mathscr{P} \in \Lambda$ means that $\mathscr{P}(\mathfrak{A}) = \mathsf{Comp}(\mathfrak{A})$ or $\mathscr{P}(\mathfrak{A}) = \mathscr{R}(\mathfrak{A})$ etc.

By ε the so called *empty relation on A* is denoted, i.e. $\langle a, b \rangle \in \varepsilon$ for *no elements* $a, b \in A$; by Δ the *diagonal* is denoted, i.e. $\langle a, b \rangle \in \Delta$ if $a = b \in A$, by ∇ the *Cartesian square* is denoted, i.e. $\langle a, b \rangle \in \Delta$ *for each* $a, b \in A$. Clearly, ε, Δ, ∇ are compatible relations on every algebra $\mathfrak{A}$.

In [1] it is proved that $\mathscr{C}(\mathfrak{A})$ is an algebraic lattice for every algebra $\mathfrak{A}$. This result is extended also to $\mathscr{L}\mathscr{T}(\mathfrak{A})$ in [3]. Here it will be generalized also for other lattices of relations.

Theorem 1. *Let $\mathfrak{A} = \langle A, F \rangle$ be an algebra. Then for each $\mathscr{P} \in \Lambda$ the set $\mathscr{P}(\mathfrak{A})$ is a complete lattice with respect to the set inclusion. The greatest element in $\mathscr{P}(\mathfrak{A})$ is equal to ∇. The least element of $\mathscr{P}(\mathfrak{A})$ is equal to Δ for $\mathscr{P} \in \{\mathscr{R}, \mathscr{L}\mathscr{T}, \mathscr{Q}, \mathscr{C}\}$. The meet in the lattice $\mathscr{P}(\mathfrak{A})$ is equal to the set intersection for all $\mathscr{P} \in \Lambda$.*

Proof. Clearly $\nabla \in \mathscr{P}(\mathfrak{A})$ for each $\mathscr{P} \in \Lambda$, it is compatible, i.e. it is the greatest element in $\mathscr{P}(\mathfrak{A})$ with respect to the set inclusion. If $R_\gamma \in \mathscr{P}(\mathfrak{A})$ for $\gamma \in \Gamma$, then clearly also $R = \cap\{R_\gamma; \gamma \in \Gamma\} \in \mathscr{P}(\mathfrak{A})$, thus, by Theorem 17 in [2], $\mathscr{P}(\mathfrak{A})$ is a complete lattice. Evidently, R is the infimum of the family $\{R_\gamma; \gamma \in \Gamma\}$. Let $\mathscr{P} \in \{\mathscr{R}, \mathscr{L}\mathscr{T}, \mathscr{Q}, \mathscr{C}\}$, then $\Delta \subseteq R$ for each $R \in \mathscr{P}(\mathfrak{A})$, Δ is compatible, i.e. it is the least element of $\mathscr{P}(\mathfrak{A})$ for other $\mathscr{P} \in \Lambda$.

Notation. Let $\mathscr{P} \in \Lambda$ and $R_\gamma \in \mathscr{P}(\mathfrak{A})$ for $\gamma \in \Gamma$. Denote by *pol* $\mathfrak{A}$ the set of all *polynomials of the algebra* $\mathfrak{A}$ (see [1]). Introduce the following two operators $^{\mathrm{C}}$, $^{\mathrm{T}}$ on the family $\{R_\gamma; \gamma \in \Gamma\}$:

$\langle a, b\rangle \in (\cup\{R_\gamma; \gamma \in \Gamma\})^{\mathrm{C}}$ if and only if there exist an n-ary $p \in pol\, \mathfrak{A}$ and elements a_i, b_i $(i = 1, \dots, n)$ from A with $a = p(a_1, \dots, a_n)$, $b = p(b_1, \dots, b_n)$ and $\langle a_i, b_i\rangle \in R_{\gamma_i}$ for $\gamma_i \in \Gamma$, $i = 1, \dots, n$.

Further,

$\langle a, b\rangle \in (\cup\{R_\gamma; \gamma \in \Gamma\})^{\mathrm{T}}$ if and only if there exist $a_0, \dots, a_n \in A$, $\gamma_1, \dots, \gamma_n \in \Gamma$ with $a_0 = a$, $a_n = b$ and $\langle a_{i-1}, a_i\rangle \in R_{\gamma_i}$ for $i = 1, \dots, n$.

If Γ is a one-element set and $R_\gamma = R$, abbreviate it by R^{C}, R^{T}. If the index set Γ is given, abbreviate $(\cup\{R_\gamma; \gamma \in \Gamma\})^{\mathrm{C}}$ or $(\cup\{R_\gamma; \gamma \in \Gamma\})^{\mathrm{T}}$ only by $(\cup R_\gamma)^{\mathrm{T}}$ or $(\cup R_\gamma)^{\mathrm{T}}$ respectively.

Remark. It is clear that $(\cup R_\gamma)^{\mathrm{C}}$ is the least compatible relation containing $\cup R_\gamma$ and $(\cup R_\gamma)^{\mathrm{T}}$ is the least transitive relation containing $\cup R_\gamma$, i.e. R^{C} or R^{T} is the *compatible* or the *transitive hull* of the relation R, respectively.

Denote by $\vee_{\mathscr{P}}$ the lattice join in $\mathscr{P}(\mathfrak{A})$, $\mathscr{P} \in \Lambda$.

Lemma 1. *Let $\mathfrak{A}$ be an algebra, $\mathscr{P} \in \Lambda$ and $R_\gamma \in \mathscr{P}(\mathfrak{A})$ for $\gamma \in \Gamma$. Then $(\cup R_\gamma)^{\mathrm{C}} \subseteq \vee_{\mathscr{P}}\{R_\gamma; \gamma \in \Gamma\}$.*

The proof is clear.

Theorem 2. *Let $\mathfrak{A}$ be an algebra and $\mathscr{P} \in \{$Comp, $\mathscr{R}, \mathscr{S}, \mathscr{L}\mathscr{T}\}$. Then*

$$\vee_{\mathscr{P}}\{R_\gamma; \gamma \in \Gamma\} = (\cup\{R_\gamma; \gamma \in \Gamma\})^{\mathrm{C}} \qquad \textit{for } R_\gamma \in \mathscr{P}(\mathfrak{A}).$$

Proof. Let $\mathscr{P} =$ Comp. For $p(x) = {}^{\bullet}x$ we have $R_\gamma \in (\cup R_\gamma)^{\mathrm{C}}$, thus $(\cup R_\gamma)^{\mathrm{C}}$ is the compatible relation containing every R_γ for $\gamma \in \Gamma$, i.e. $(\cup\{R_\gamma; \gamma \in \Gamma\})^{\mathrm{C}} \supseteq \vee_{\mathscr{P}}\{R_\gamma; \gamma \in \Gamma\}$. The converse inclusion is given by Lemma 1. If $\mathscr{P} = \mathscr{R}$, then $\Delta \subseteq R$ implies $\Delta \subseteq (\cup R_\gamma)^{\mathrm{C}}$, thus $(\cup R_\gamma)^{\mathrm{C}} \in \mathscr{R}(\mathfrak{A})$, i.e. also $(\cup R_\gamma)^{\mathrm{C}} = \vee_{\mathscr{P}}\{R_\gamma; \gamma \in \Gamma\}$. Let $\mathscr{P} = \mathscr{S}$. Thus $R_\gamma = R_\gamma^{-1}$ and

$$\langle a, b\rangle \in (\cup R_\gamma)^{\mathrm{C}} \quad \text{iff} \quad \langle b, a\rangle \in (\cup R_\gamma^{-1})^{\mathrm{C}}$$

implies the symetry of $(\cup R_\gamma)^{\mathrm{C}}$, i. e. also the assertion is proved. By the combination of the two previous results, we can prove the assertion also for $\mathscr{L}\mathscr{T}(\mathfrak{A})$.

Theorem 3. *Let $\mathfrak{A}$ be an algebra, $\mathscr{P} \in \{\mathscr{Q}, \mathscr{C}\,In\}$ and $R_\gamma \in \mathscr{P}(\mathfrak{A})$ for $\gamma \in \Gamma$. Then $\vee_{\mathscr{P}}\{R_\gamma; \gamma \in \Gamma\} = (\cup\{R_\gamma; \gamma \in \Gamma)^{\mathrm{T}}$.*

Proof. Clearly, $R_\gamma \subseteqq (\cup R_\gamma)^T$ for each $\gamma \in \Gamma$. By the definition, $(\cup R_\gamma)^T$ is transitive and the reflexivity of R_γ $(\gamma \in \Gamma)$ implies the reflexivity of $(\cup R_\gamma)^T$. By the proof of Theorem 84 in [2], the reflexivity, transitivity and compatibility of R_γ for $\gamma \in \Gamma$ imply the compatibility of $(\cup R_\gamma)^T$. If R_γ are symmetric, it is also true for $(\cup R_\gamma)^T$. Thus also $(\cup R_\gamma)^T \in \mathscr{P}(\mathfrak{A})$. Now, the assertion is a direct consequence of it.

Remark. For $\mathscr{P}(\mathfrak{A})$, $\mathscr{P}\{\mathscr{T}\}$ the assertion analogous to Theorem 2 or Theorem 3 cannot be stated, because $\vee_{\mathscr{P}}$ need not be constructed by the using of operators ^{C}, ^{T} in a finite number of steps. It follows from the fact that the $((\cup R_\gamma)^C)^T$ need not be compatible (see Example 2 in [5]) and $((\cup R_\gamma)^T)^C$ need not be transitive.

Notation. Denote $\Lambda_0 = \Lambda - \{\mathscr{T}, \mathsf{In}\}$.

Definition. Let $\mathfrak{A} = \langle A, F\rangle$ be an algebra, $\emptyset \neq H \subseteqq A \times A$ and $\mathscr{P} \in \Lambda$. Denote by $R_{\mathscr{P}}(H) = \cap\{R \in \mathscr{P}(\mathfrak{A});\ H \subseteqq R\}$. For $H = \{\langle a, b\rangle\}$ abbreviate $R_{\mathscr{P}}(\{\langle a, b\rangle\})$ by $R_{\mathscr{P}}(a, b)$ and call it the *principal $\mathscr{P}$-relation generated by a, b*

Remark. Evidently, $R_{\mathscr{P}}(H) \in \mathscr{P}(\mathfrak{A})$ for every $\emptyset \neq H \subseteqq A \times A$ and the principal $\mathscr{P}$-relation is a generalization of a principal congruence (see [1]) and a minimal tolerance (see [4]).

Theorem 4. *Let $\mathfrak{A} = \langle A, F\rangle$ be an algebra, $a, b \in A$ and $\mathscr{P} \in \Lambda_0$. Then $\langle x, y\rangle \in R_{\mathscr{P}}(a, b)$ if and only if*

(1) *there exist n-ary $p \in pol\mathfrak{A}$ and unary $t_i \in pol\mathfrak{A}$ $(i = 1, \dots, n)$ such that $x = p(a_1, \dots, a_n)$, $y = p(b_1, \dots, b_n)$, where for $i = 1, \dots, n$*
 (a) *$a_i = t_i(a)$, $b_i = t_i(b)$ for $\mathscr{P} = \mathsf{Comp}$,*
 (b) *$a_i = b_i$ or $a_i = t_i(a)$, $b_i = t_i(b)$ for $\mathscr{P} = \mathscr{R}$,*
 (c) *$\{a_i, b_i\} = \{t_i(a), t_i(b)\}$ for $\mathscr{P} = \mathscr{S}$,*
 (d) *$a_i = b_i$ or $\{a_i, b_i\} = \{t_i(a), t_i(b)\}$ for $\mathscr{P} = \mathscr{L}\mathscr{T}$.*

(2) *there exist $a_0, \dots, a_n \in A$ and unary algebraic functions* (see [1]) *$\varphi_1, \dots, \varphi_n$ such that $a_0 = x$, $a_n = y$ and for $i = 1, \dots, n$,*
 (e) *$a_{i-1} = \varphi_i(a)$, $a_i = \varphi_i(b)$ for $\mathscr{P} = \mathscr{Q}$,*
 (f) *$\{a_{i-1}, a_i\} = \{\varphi_i(a), \varphi_i(b)\}$ for $\mathscr{P} = \mathscr{C}$.*

Proof. Let $\mathscr{P} \in \{\mathsf{Comp}, \mathscr{R}, \mathscr{S}, \mathscr{L}\mathscr{T}\}$ and R be the relation defined by (1). For $n = 1$, $t_1(x) = x$ we obtain $\langle a, b\rangle \in R$. Let $\langle x_j, y_j\rangle \in R$ for $j = 1, \dots, m$ and $r \in F$ be m-ary. By (1), there exist t_j^i, p_j $(j = 1, \dots, m, i = 1, \dots, n)$ such that $x_j = p_j(t_j^1(a), \dots, t_{n_j}^j(a))$, $y_j = p_j(t_1^j(b), \dots, t_{n_j}^j(b))$, thus

$$x = r(x_1, \dots, x_n) = r(p_1(t_1^1(a), \dots, t_{n_1}^1(a)), \dots, p_m(t_1^m(a), \dots, t_{n_m}^m(a))),$$
$$y = r(p_1(t_1^1(b), \dots, t_{n_1}^1(b)), \dots, p_m(t_1^m(b), \dots, t_{n_m}^m(b)))$$

and by (1) it implies $\langle x, y\rangle \in R$, thus R is compatible on $\mathfrak{A}$. If S is a compatible relation with $\langle a, b\rangle \in S$, thus $\langle t_i(a), t_i(b)\rangle \in S$ for each unary $t_i \in pol\mathfrak{A}$, i.e. $R \subseteqq S$.

Thus $R = R_{\mathscr{P}}(a, b)$ for $\mathscr{P} = \mathsf{Comp}$. From (1b) we can clearly prove the reflexivity of R, from (1c) the symmetry of R and from (1d) both of these properties, i.e. $R = R_{\mathscr{P}}(a, b)$ for all four these $\mathscr{P}$.

For (2f) see Theorem 10.3 in [1], for (2e) this proof can be also used.

Theorem 5. *Let $\mathfrak{A} = \langle A, F \rangle$ be an algebra, $\mathscr{P} \in \Lambda$ and $\emptyset \neq H \subseteq A \times A$. Then $R_{\mathscr{P}}(H) = \vee_{\mathscr{P}}\{R_{\mathscr{P}}(a, b); \langle a, b \rangle \in H\}$.*

Proof. As $\langle a, b \rangle \in H$ implies $R_{\mathscr{P}}(a, b) \subseteq R_{\mathscr{P}}(H)$, we have

$$\vee_{\mathscr{P}}\{R_{\mathscr{P}}(a, b); \langle a, b \rangle \in H\} \subseteq R_{\mathscr{P}}(H).$$

Further, if $X \subseteq Y \subseteq R$, then evidently $R_{\mathscr{P}}(X) \subseteq R_{\mathscr{P}}(Y)$ and $Z \in \mathscr{P}(\mathfrak{A})$ implies $R_{\mathscr{P}}(Z) = Z$. Then

$$H \subseteq \vee_{\mathscr{P}}\{R_{\mathscr{P}}(a, b); \langle a, b \rangle \in H\} \in \mathscr{P}(\mathfrak{A})$$

implies

$$R_{\mathscr{P}}(H) \subseteq R_{\mathscr{P}}(\vee_{\mathscr{P}}\{R_{\mathscr{P}}(a, b); \langle a, b \rangle \in H\}) = \vee_{\mathscr{P}}\{R_{\mathscr{P}}(a, b); \langle a, b \rangle \in H\},$$

which is the converse inclusion.

Corollary. *Let $\mathfrak{A}$ be an algebra and $\mathscr{P} \in \Lambda$. Then*

$$R = \vee_{\mathscr{P}}\{R_{\mathscr{P}}(a, b); \langle a, b \rangle \in R\}$$

for each $R \in \mathscr{P}(\mathfrak{A})$.

Theorem 6. *Let $\mathfrak{A} = \langle A, F \rangle$ be an algebra, $\emptyset \neq H \subseteq A \times A$ and $\mathscr{P} \in \Lambda_0$. Then $\langle x, y \rangle \in R_{\mathscr{P}}(H)$ if and only if*

(1) *there exist n-ary $p \in pol\mathfrak{A}$, unary $t_i \in pol\mathfrak{A}$ and $\langle a_i, b_i \rangle \in H$ with $x = p(x_1, \ldots, x_n)$, $y = p(y_1, \ldots, y_n)$ and for $i = 1, \ldots, n$*
   (a) *$x_i = t_i(a_i)$, $y_i = t_i(b_i)$ for $\mathscr{P} = \mathsf{Comp}$,*
   (b) *$x_i = y_i$ or $x_i = t_i(a_i)$, $y_i = t_i(b_i)$ for $\mathscr{P} = \mathscr{R}$,*
   (c) *$\{x_i, y_i\} = \{t_i(a_i), t_i(b_i)\}$ for $\mathscr{P} = \mathscr{S}$,*
   (d) *$x_i = y_i$ or $\{x_i, y_i\} = \{t_i(a_i), t_i(b_i)\}$ for $\mathscr{P} = \mathscr{LT}$.*

(2) *there exist $a_0, \ldots, a_n \in A$, unary algebraic functions φ_i and $\langle x_i, y_i \rangle \in H$ $(i = 1, \ldots, n)$ such that $x = a_0$, $y = a_n$ and for $i = 1, \ldots, n$*
   (e) *$a_{i-1} = \varphi_i(x_i)$, $a_i = \varphi_i(y_i)$ for $\mathscr{P} = \mathscr{G}$*
   (f) *$\{a_{i-1}, a_i\} = \{\varphi_i(x_i), \varphi_i(y_i)\}$ for $\mathscr{P} = \mathscr{C}$.*

Proof. The assertion follows directly from Theorems 2, 3, 4, 5.

An element c of the lattice L is said to be *compact*, if $x \leq \vee\{x_i; i \in I\}$ implies the existence of finite $I_0 \subseteq I$ such that $c \leq \vee\{x_i; i \in I_0\}$. The lattice L is called *algebraic*, if it is complete and each its element is a join of compact elements.

Theorem 7. *Let $\mathfrak{A}$ be an algebra, $\mathscr{P} \in \Lambda_0$ or $\mathscr{P} = \mathsf{In}$ and $R \in \mathscr{P}(\mathfrak{A})$. Then R is a compact element of $\mathscr{P}(\mathfrak{A})$ if and only if $R = \vee_{\mathscr{P}}\{R_{\mathscr{P}}(a_i, b_i); i = 1, \ldots, n\}$.*

Proof. Let $R_{\mathscr{P}}(a, b) \subseteq \vee_{\mathscr{P}}\{R_\gamma; \gamma \in \Gamma\}$. Then $\langle a, b\rangle \in \vee_{\mathscr{P}}\{R_\gamma; \gamma \in \Gamma\}$ and, by Theorem 2 or Theorem 3, there exists $\Gamma_0 = \{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ with $\langle a, b\rangle \in \in \vee_{\mathscr{P}}\{R_\gamma; \gamma \in \Gamma_0\}$, thus also $R_{\mathscr{P}}(a, b) \subseteq \vee_{\mathscr{P}}\{R_\gamma; \gamma \in \Gamma_0\}$, i.e. $R_{\mathscr{P}}(a, b)$ is a compact element in $\mathscr{P}(\mathfrak{A})$. Hence, every join of finitely many principal $\mathscr{P}$-relations is a compact element in $\mathscr{P}(\mathfrak{A})$.

Let R be a compact element in $\mathscr{P}(\mathfrak{A})$. By Corollary of Theorem 5, we have $R = \vee_{\mathscr{P}}\{R_{\mathscr{P}}(a, b); \langle a, b\rangle \in R\}$. As R is compact, there exists a finite subset $\{\langle a_i, b_i\rangle; i = 1, \ldots, n\} \subseteq R$ such that $R = \vee_{\mathscr{P}}\{R_{\mathscr{P}}(a_i, b_i); i = 1, \ldots, n\}$, thus the converse statement is proved.

Theorem 8. *Let $\mathfrak{A}$ be an algebra and $\mathscr{P} \in \Lambda_0$* or *$\mathscr{P} = In$. Then $\mathscr{P}(\mathfrak{A})$ is an algebraic lattice.*

Proof. By Theorem 1, $\mathscr{P}(\mathfrak{A})$ is complete and, by Theorem 7 and by Corollary of Theorem 5, every element of $\mathscr{P}(\mathfrak{A})$ is the join of compact elements.

Theorem 9. *Let $\mathfrak{A} = \langle A, F\rangle$ be an algebra, $\mathscr{P} \in \Lambda_0$ and $a, b \in A$, $a \neq b$. Then there exists the maximal element $R_{ab} \in \mathscr{P}(\mathfrak{A})$ with $\langle a, b.\rangle \notin R_{ab}$.*

Proof. Let $\mathscr{W} = \{R \in \mathscr{P}(\mathfrak{A}); \langle a, b\rangle \notin R\}$. Clearly, $\mathscr{W} \neq \emptyset$, because $\nabla \in \mathscr{W}$. Let $\mathscr{D}$ be a chain in $\langle \mathscr{W}, \subseteq\rangle$. Then $S = \vee_{\mathscr{P}}\{R'; R' \in \mathscr{D}\} \in \mathscr{P}(\mathfrak{A})$. Evidently, $S = \cup\{R'; R' \in \mathscr{D}\}$, hence $\langle x, y\rangle \in S$ is and only if $\langle x, y\rangle \in R'$ for some $R' \in \mathscr{D}$. This implies $\langle a, b\rangle \notin S$ and, by Kuratowski–Zorn lemma the assertion is obtained.

The aim of the rest of this paper is to show for which $\mathscr{P}, \mathscr{P}' \in \Lambda$ the lattice $\mathscr{P}(\mathfrak{A})$ is a sublattice of $\mathscr{P}'(\mathfrak{A})$.

Lemma 3. *For every algebra $\mathfrak{A}$, the lattice $\mathscr{LT}(\mathfrak{A})$ is a sublattice of the lattices $\mathscr{R}(\mathfrak{A})$, $\mathscr{S}(\mathfrak{A})$ and these lattices are sublattices of* $Comp(\mathfrak{A})$.

Proof. The set inclusions $\mathscr{LT}(\mathfrak{A}) \subseteq \mathscr{R}(\mathfrak{A}) \subseteq Comp(\mathfrak{A})$ and $\mathscr{LT}(\mathfrak{A}) \subseteq \mathscr{S}(\mathfrak{A}) \subseteq \subseteq Comp(\mathfrak{A})$ are evident and, by Theorem 1 and 2, the join and the meet is the same in all of these lattices.

Lemma 4. *For every algebra $\mathfrak{A}$, the lattice $\mathscr{C}(\mathfrak{A})$ is a sublattice of $\mathscr{Q}(\mathfrak{A})$ $In(\mathfrak{A})$ and these lattices are sublattices of $\mathscr{T}(\mathfrak{A})$.*

Proof. The set inclusions $\mathscr{C}(\mathfrak{A}) \subseteq \mathscr{Q}(\mathfrak{A}) \subseteq \mathscr{T}(\mathfrak{A})$, $\mathscr{C}(\mathfrak{A}) \subseteq In(\mathfrak{A}) \subseteq \mathscr{T}(\mathfrak{A})$ are evident and, by Theorem 1, the meet is the same in all of these lattices. By Theorem 3, $\vee_{\mathscr{C}} = \vee_{\mathscr{Q}}$, thus $\mathscr{C}(\mathfrak{A})$ is a sublattice of $\mathscr{Q}(\mathfrak{A})$. Let $R_\gamma \in \mathscr{C}(\mathfrak{A})$ for $\gamma \in \Gamma$ and $R = 4 \vee_{In} = \vee_{In}\{R_\gamma; \gamma \in \Gamma\}$. Then $\Delta \subseteq R_\gamma \subseteq R$ implies the reflexivity of R, i.e. R is a congruence on $\mathfrak{A}$ containing every R_γ, thus

$$\vee_{\mathscr{C}}\{R_\gamma; \gamma \in \Gamma\} \subseteq \vee_{In}\{R_\gamma; \gamma \in \Gamma\}.$$

However, for every $R_\gamma \in \mathscr{C}(\mathfrak{A})$ the converse inclusion is clear, hence $\mathscr{C}(\mathfrak{A})$ is a sublattice of $In(\mathfrak{A})$. Analogously, the reflexivity of $R_\gamma \in \mathscr{Q}(\mathfrak{A})$ implies the reflexivity of $\vee_{\mathscr{T}}\{R_\gamma; \gamma \in \Gamma\}$, hence $\mathscr{Q}(\mathfrak{A})$ is a sublattice of $\mathscr{T}(\mathfrak{A})$. If $R_\gamma \in In(\mathfrak{A})$, $R = \vee_{\mathscr{T}}\{R_\gamma; \gamma \in \Gamma\}$, then $\langle a, b\rangle \in R$ iff $\langle b, a\rangle \in \vee_{\mathscr{T}}\{R_\gamma^{-1}; \gamma \in \Gamma\}$. As $R_\gamma = R_\gamma^{-1}$, also R is symmetric and again $In(\mathfrak{A})$ is a sublattice of $\mathscr{T}(\mathfrak{A})$.

Lemma 5. *There exists an algebra* $\mathfrak{A}$ *such that* $\mathscr{C}(\mathfrak{A})$ *is not a sublattice of* $\mathscr{L}\mathscr{T}(\mathfrak{A})$, $\mathscr{Q}(\mathfrak{A})$ *is not a sublattice of* $\mathscr{R}(\mathfrak{A})$, $\mathsf{In}(\mathfrak{A})$ *is not a sublattice of* $\mathscr{S}(\mathfrak{A})$ *and* $\mathscr{T}(\mathfrak{A})$ *is not a sublattice of* $\mathsf{Comp}(\mathfrak{A})$.

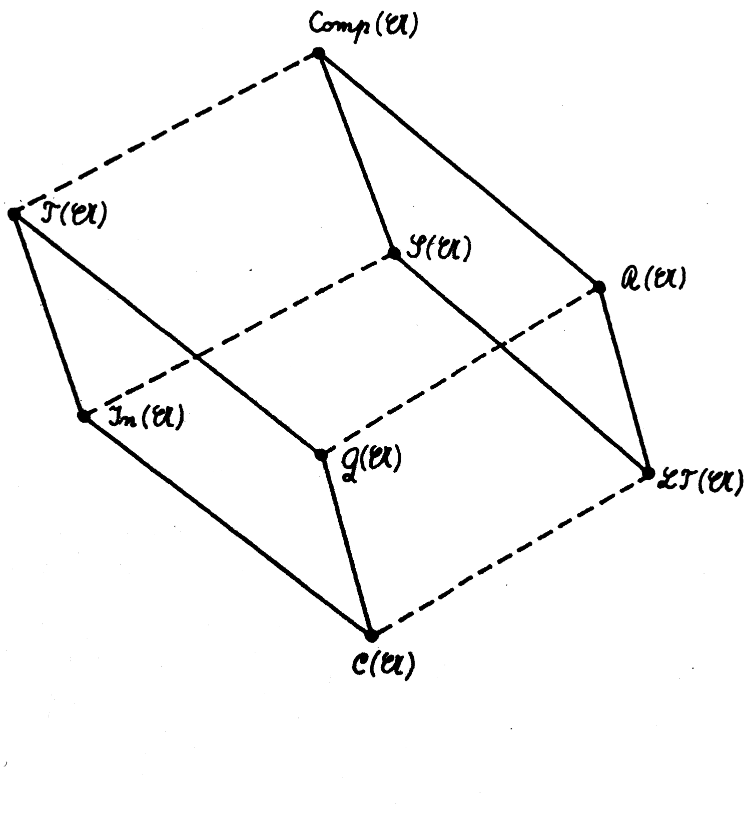
Proof. Let $\mathfrak{A}$ be a distributive lattice which is not relatively complemented. By Corollary 2 in [7], there exists a compatible tolerance T on $\mathfrak{A}$ which is not a congruence. By Corollary in [4], for every distributive lattice, $\mathscr{C}(\mathfrak{A})$ is a sublattice of $\mathscr{L}\mathscr{T}(\mathfrak{A})$ if and only if $\mathscr{C}(\mathfrak{A}) = \mathscr{L}\mathscr{T}(\mathfrak{A})$, thus, for previous lattice $\mathfrak{A}$, $\mathscr{C}(\mathfrak{A})$ is not a sublattice of $\mathscr{L}\mathscr{T}(\mathfrak{A})$. Hence, as $\mathscr{C}(\mathfrak{A})$ is a subset of $\mathscr{L}\mathscr{T}(\mathfrak{A})$, there exist $R_\gamma \in \mathscr{L}\mathscr{T}(\mathfrak{A})$ such that $\vee_{\mathscr{L}\mathscr{T}}\{R_\gamma;\ \gamma \in \Gamma\} \neq \vee_{\mathscr{C}}\{R_\gamma;\ \gamma \in \Gamma\}$. Then for these R_γ also $R_\gamma \in \mathscr{R}(\mathfrak{A})$, $R_\gamma \in \mathscr{S}(\mathfrak{A})$, $R_\gamma \in \mathsf{Comp}(\mathfrak{A})$ and, by Lemma 3 and Lemma 4,

$$\begin{aligned} \vee_{\mathsf{Comp}}\{R_\gamma; \gamma \in \Gamma\} &= \vee_{\mathscr{S}}\{R_\gamma; \gamma \in \Gamma\} = \vee_{\mathscr{R}}\{R_\gamma; \gamma \in \Gamma\} = \\ &= \vee_{\mathscr{L}\mathscr{T}}\{R_\gamma; \gamma \in \Gamma\} \neq \vee_{\mathscr{C}}\{R_\gamma; \gamma \in \Gamma\} = \vee_{\mathsf{In}}\{R_\gamma; \gamma \in \Gamma\} = \\ &= \vee_{\mathscr{Q}}\{R_\gamma; \gamma \in \Gamma\} = \vee_{\mathscr{T}}\{R_\gamma; \gamma \in \Gamma\} \end{aligned}$$

Hence, the assertion is clear.

Notation. Let the circles on a diagram denote the lattices $\mathscr{P}(\mathfrak{A})$ for $\mathscr{P} \in A$ and fixed $\mathfrak{A}$ and the solid line joins the circle A with B (where A is situated below B) if and only if A is a sublattice of B. Further, if A is a subset of B and A is not a sublattice of B for some algebra $\mathfrak{A}$, A and B are joined by a dashed line.

Now, we can illustrate the situation by



Theorem 10. *Let the circles on a diagram denote the lattices $\mathscr{P}(\mathfrak{A})$ for $\mathscr{P} \in \Lambda$. Then the following diagram shows exactly the relationship "to be a sublattice for every algebra $\mathfrak{A}$".*

The proof follows directly from Lemmas 3, 4, 5.

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