Archivum Mathematicum

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Archivum Mathematicum, Vol. 12 (1976), No. 2, 63--66

Persistent URL: http://dml.cz/dmlcz/106929

ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XII: 63—66, 1976

ON SOME PROPERTIES OF PROXIMITY IN METRIC SPACES

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(Received October 10, 1974)

The task of this article is to prove a lemma concerning proximity in metric spaces and then prove that a metric space is completely bounded if and only if the δ-space constructed by its metric is a completely bounded δ-space.

1. DEFINITION AND SOME PROPERTIES OF PROXIMITY SPACES

Concepts and lemmas given in this chapter are taken from articles (2) and (3).

Proximity space (or δ-space) is a non-empty set P together with a mapping δ (called proximity) of the set $2^P \times 2^P$ into the set $\{0, 1\}$ which fulfils following axioms:

$B1$: $\delta(A, B) = \delta(B, A)$

$B2$: $\delta(A \cup B, C) = \delta(A, C) . \delta(B, C)$

$B3$: $\delta(\{x\}, \{y\}) = 0 \Leftrightarrow x = y$

$B4$: $\delta(A, \emptyset) = 1$

$B5$: if $\delta(A, B) = 1$, then there exist sets $C, D \subseteqq P$ such that $C \cup D = P$ and $\delta(A, C) = \delta(B, D) = 1$.

Instead of $\delta(A, \{x\})$ we write $\delta(A, x)$.

We construct a topology $\mathscr{T}_\delta$ in the δ-space (P, δ) in this way: a set $A \subseteqq P$ is closed iff $\delta(A, x) = 0 \Leftrightarrow x \in A$.

A set $A \subseteqq P$ is a δ-neighbourhood of a set $B \subseteqq P$ iff $\delta(B, P \setminus A) = 1$. In this case we write $B \Subset A$.

A covering γ of the set P is called a δ-covering of the δ-space (P, δ) iff for any $A, B \subseteqq P$ such that $\delta(A, B) = 0$ there is a set $\Gamma \in \gamma$ such that $A \cap \Gamma \neq \emptyset \neq B \cap \Gamma$.

Let γ be a covering of the set P, we put $U_\gamma C = U\{\Gamma \mid \Gamma \in \gamma \text{ and } \Gamma \cap C \neq \emptyset\}$ for any $C \subseteqq P$.

If $C = \{x\}$ then we write only $U_\gamma x$. Evidently if γ is a δ-covering of (P, δ) and $B \subseteqq P$ then $B \Subset U_\gamma B$. Let α, β be δ-coverings of (P, δ). We write $\alpha <^* \beta$ iff for any $x \in P$ there is a set $B \in \beta$ such that $U_\alpha x \subseteqq B$.

A δ-covering γ is uniform iff there exists a sequence $\{\gamma_n\}_{n=1}^\infty$ of δ-coverings such that $\gamma = \gamma_1 >^* \gamma_2 >^* \ldots$ We denote $\mathscr{S}(P, \delta)$ the set of all uniform δ-coverings of (P, δ).

A δ-space is completely bounded iff for any its uniform δ-covering γ there exists a finite subcovering γ_0 of γ.

If (P, ϱ) is a metric space then we can define on P a proximity δ_ϱ in a natural way: $\delta(A, B) = \operatorname{sgn} \varrho(A, B)$.

The two following lemmas, which are taken from article (4), page 276, are used in the sequel.

Lamma 1: *Let (P, δ) be a δ-space, α, β, γ be its δ-coverings such that $\alpha >^* \beta >^* \gamma$, $\Gamma_0 \in \gamma$. Then there exists a set $A \in \alpha$ such that $A \supseteqq U_\gamma \Gamma_0$.*

Lemma 2: *If $\gamma \in \mathscr{S}(P, \delta)$, then $\gamma^0 = \{\Gamma^0 \mid \Gamma \in \gamma\}$ is an open uniform δ-covering of the δ-space (P, δ).*

2. SOME PROPERTIES OF PROXIMITY IN METRIC SPACES

Lemma 3. *Let (M, ϱ) be a metric space, (M, δ_ϱ) be a δ-space constructed from the metric ϱ. Let $\{x_n\}_1^\infty \subseteqq M$ be a cauchy sequence, $\gamma \in \mathscr{S}(M, \delta_\varrho)$. Then there exists a set $\Gamma_0 \in \gamma$ and a positive integer N such that $\{x_n\}_N^\infty \Subset \Gamma_0$.*

Proof: a) Let γ be an open uniform δ-covering of (M, δ_ϱ). Then there is a sequence of open (uniform) δ-coverings $\gamma = \gamma_1 >^* \gamma_2 >^* \ldots$ We shall distinguish two cases:

1. There is a set ${}^3\Gamma \in \gamma_3$ and a subsequence $\{x_{n_k}\}_1^\infty$ such that $\{x_{n_k}\}_1^\infty \subseteqq {}^3\Gamma$. According to lemma 1 there is a set $\Gamma_0 \in \gamma$ such that $\Gamma_0 \supseteqq U_{\gamma_3}{}^3\Gamma \supset {}^3\Gamma \supseteqq \{x_{n_k}\}_1^\infty$. Therefore $\varrho(\{x_{n_k}\}_1^\infty, M \setminus \Gamma_0) = \varepsilon > 0$. $\{x_n\}_1^\infty$ is a cauchy sequence which implies that for $\varepsilon > 0$ there is a positive integer N_0 such that $\varrho(x_{n_k}, x_n) < \varepsilon/2$ for any $n \geqq N_0$ and any k such that $n_k \geqq N_0$. Put $N = \max(n_1, N_0)$. Now $\varrho(x_n, M \setminus \Gamma_0) \geqq \varepsilon/2$ for any $n \geqq N$, hence $\varrho(\{x_n\}_N^\infty, M \setminus \Gamma_0) \geqq \varepsilon/2$ and therefore $\{x_n\}_N^\infty \Subset \Gamma_0$.
2. Suppose that any set ${}^3\Gamma \in \gamma_3$ contains a finite number of elements of $\{x_n\}_1^\infty$. We choose such a subsequence $\{x_{n_k}\}_1^\infty$ that any set ${}^3\Gamma \in \gamma_3$ (and therefore evidently also any set ${}^i\Gamma \in \gamma_i$, $i = 4, 5, \ldots$) doesn't contain more than one element of $\{x_{n_k}\}_1^\infty$. Denote ${}^6\Gamma_{n_k}$ a set of γ_6 containing $U_{\gamma_7} x_{n_k}$, for every positive integer k. According to lemma 1 there exists (for any positive integer k) a set ${}^4\Gamma_{n_k} \in \gamma_4$ such that ${}^4\Gamma_{n_k} \supseteqq$ $\supseteqq U_{\gamma_6}{}^6\Gamma_{n_k} \supset {}^6\Gamma_{n_k}$. Suppose that ${}^4\Gamma_{n_i} \cap {}^4\Gamma_{n_j} \neq \emptyset$ for some $i \neq j$. Then there exists

a set ${}^3\Gamma \in \gamma_3$ such that ${}^3\Gamma \supseteqq {}^4\Gamma_{n_i} \cup {}^4\Gamma_{n_j} \supseteqq \{x_{n_i}, x_{n_j}\}$, which contradicts our supposition. Thus ${}^4\Gamma_{n_i} \cap {}^4\Gamma_{n_j} = \emptyset$ for any i, j, $i \neq j$ and thus $\delta_\varrho({}^6\Gamma_{n_i}, {}^6\Gamma_{n_j}) = 1$ if $i \neq j$. $\{x_{n_k}\}_1^\infty$ is a cauchy sequence. Hence for any $\varepsilon > 0$ there exists a positive integer K such that $\varrho(x_{n_j}, x_{n_k}) < \varepsilon$ for any $j, k \geqq K$. Besides $x_{n_j} \in M \setminus {}^6\Gamma_{n_k}$ if $j \neq k$ and thus $\varrho(x_{n_k}, M \setminus {}^6\Gamma_{n_k}) < \varepsilon$ for $k \geqq K$. This implies that there exists a point y_k, of the boundary of ${}^6\Gamma_{n_k}$ for which $\varrho(x_{n_k}, y_k) < \varepsilon$; $y_k \in M \setminus {}^6\Gamma_{n_k}$ as ${}^6\Gamma_{n_k}$ is an open set. As ${}^6\Gamma_{n_j} \cap {}^6\Gamma_{n_i} = \emptyset$ for $i \neq j$, we have $y_k \in M \setminus U\{{}^6\Gamma_{n_j} \mid j = 1, 2, \ldots\}$. Hence $\varrho(\{x_{n_k}\}_1^\infty, M \setminus U\{{}^6\Gamma_{n_j} \mid j = 1, 2, \ldots\}) < \varepsilon$ for any $\varepsilon > 0$, therefore $\delta_\varrho(\{x_{n_k}\}_1^\infty, M \setminus U\{{}^6\Gamma_{n_j} \mid j = 1, 2, \ldots\}) = 0$. Then there exists a set ${}^7\Gamma \in \gamma_7$ such that $\{x_{n_k}\}_1^\infty \cap {}^7\Gamma \neq \emptyset \neq (M \setminus U\{{}^6\Gamma_{n_j} \mid j = 1, 2, \ldots\}) \cap {}^7\Gamma$. This implies that $x_{n_{k_0}} \in {}^7\Gamma$ for an appropriate positive integer k_0, but then ${}^7\Gamma \subseteqq U_{\gamma_7} x_{n_{k_0}} \subseteqq {}^7\Gamma_{n_{k_0}}$ and ${}^7\Gamma \cap (M \setminus U\{{}^6\Gamma_{n_j} \mid j = 1, 2, \ldots\}) \neq \emptyset$, which is a contradiction. Case (2) is then impossible.

b) Let $\gamma \in \mathscr{S}(P, \delta)$, then according to lemma 2 $\gamma^0 = \{\Gamma^0 \mid \Gamma \in \gamma\}$ is an open uniform δ-covering and we have already proved the assertion for γ^0. It is then proved for γ, too.

Theorem: *Let (M, ϱ) be a metric space, (M, δ_ϱ) be a δ-space constructed from the metric ϱ. Then the following statements are equivalent.*

a) *(M, ϱ) is a completely bounded metric space,*

b) *(M, δ_ϱ) is a completely bounded δ-space.*

Proof: a) Let (M, ϱ) be a completely bounded metric space, $\gamma \in \mathscr{S}(M, \delta_\varrho)$. Then there are $\gamma_1, \gamma_2, \ldots \in \mathscr{S}(M, \delta_\varrho)$ such that $\gamma = \gamma_1 >^* \gamma_2 >^* \ldots$ Suppose that for any positive integer n there is a point $x \in M$ such that $k(x, 1/n) \setminus \Gamma \neq \emptyset$ for any $\Gamma \in \gamma$. Thus $k(x, 1/n) \setminus {}^i\Gamma \neq \emptyset$ for any ${}^i\Gamma \in \gamma_i$ and any $i = 1, 2, \ldots$ Let $x_1 \in M$ be such a point that $k(x_1, 1) \setminus {}^i\Gamma \neq \emptyset$ for any ${}^i\Gamma \in \gamma_i$, $i = 1, 2, \ldots$ Assume that there exists a set ${}^2\Gamma_x \in \gamma_2$ such that $k(x, 1/2) \subseteqq {}^2\Gamma_x$ for every $x \in k(x_1, 1/2)$. Then $k(x_1, 1) = U\{k(x, 1/2) \mid \varrho(x_1, x) < 1/2, x \in M\} \subseteqq U_{\gamma_2} x_1 \subseteqq \Gamma_1$ for an appropriate $\Gamma_1 \in \gamma$, which is a contradiction. Thus there exists a point $x_2 \in k(x_1, 1/2)$ such that $k(x_2, 1/2) \setminus \Gamma \neq \emptyset$ for any $\Gamma \in \gamma$. We can construct a sequence $\{x_n\}_1^\infty$ such that $x_{n+1} \in k(x_n, 1/2n)$ and $k(x_n, 1/n) \setminus \Gamma \neq \emptyset$ for any $\Gamma \in \gamma$ and any $n = 1, 2, \ldots$ $\{x_n\}_1^\infty$ is evidently a cauchy sequence, therefore (according to lemma 3) there is a set $\Gamma_0 \in \gamma$ and a positive integer N_0 such that $\{x_n\}_{N_0}^\infty \in \Gamma_0$, i.e. $\varrho(\{x_n\}_{N_0}^\infty, M \setminus \Gamma_0) = \varepsilon > 0$.

There exists also a positive integer N_1 such that $\varrho(x_{N_1}, x_n) < \varepsilon$ for $n \geqq N_1$. Let N be a positive integer such that $N \geqq \max(N_0, N_1, 1/\varepsilon)$. Then $k(x_N, 1/N) \subseteqq \Gamma_0$ which contradicts our supposition that $k(x_N, 1/N) \setminus \Gamma \neq \emptyset$ for any $\Gamma \in \gamma$. Hence there exists a positive integer n_0 such that for any point $x \in M$ there is a set $\Gamma \in \gamma$ such that $k(x, 1/n_0) \subseteqq \Gamma$. (M, ϱ) is a completely bounded metric space. Thus there exists a finite set $\{y_1, \ldots, y_m\} \subseteqq M$ such that there exists an index $i_0 \in \{1, \ldots, m\}$ with $\varrho(x, y_{i_0}) < 1/n_0$ for every $x \in M$. Choose any $\Gamma_i \in \gamma$ with $\Gamma_i \supseteqq k(y_i, 1/n_0)$ for $i =$

$= 1, \ldots, m$. Then $\{\Gamma_1, \ldots, \Gamma_m\}$ is a finite covering of M which is a subcovering of γ. Therefore (M, δ_ϱ) is a completely bounded δ-space.

b) Let (M, δ_ϱ) be a completely bounded δ-space, $\varepsilon > 0$. Then there exists a positive integer n such that $2^{-n} \leqq \varepsilon$. As $\gamma_n = \{k(x, 2^{-n}) \mid x \in M\} \in \mathscr{S}(M, \delta_\varrho)$ there exists its finite subcovering $\{k(x_1, 2^{-n}), \ldots, k(x_m, 2^{-n})\}$. Then there exists an index $i_0 \in \in \{1, \ldots, m\}$ with $\varrho(x_{i_0}, x) < 2^{-n}$ for every $x \in M$. Therefore (M, ϱ) is a completely bounded metric space.

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