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ON THE BOUNDEDNESS OF A SOLUTION OF A SYSTEM OF NON-LINEAR DIFFERENTIAL EQUATIONS

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Theorem 2 in [1] gives sufficient conditions for the component $x(t)$ or $y(t)$ of the solution (x, y) of a system

$$x' + f_0(t) f_1(x) f_2(y) = 0$$
$$y' + g_0(t) g_1(x) g_2(y) = 0$$

to be bounded on $< a, \infty)$; this theorem is then generalized to give Theorem 3, which deals with boundedness conditions for $x(t)$ or $y(t)$ where (x, y) is a solution of the system

$$x' + f_0(t) f_1(x) f_2(y) + f(t, x, y) = 0$$
$$y' + g_0(t) g_1(x) g_2(y) + g(t, x, y) = 0.$$

The purpose of the present paper is the investigation of the boundedness of solutions of a system having the form

$$\begin{aligned} x' + f_1(t, x) f_2(y) + f(t, x, y) &= 0 \\ y' + g_1(t, x) g_2(y) + g(t, x, y) &= 0, \end{aligned} \tag{1}$$

where $f_1(t, x)$, $f_2(y)$, $f(t, x, y)$, $g_1(t, x)$, $g_2(y)$ and $g(t, x, y)$ are continuous for every $t \geqq t_0$, $x \in (-\infty, \infty)$, $y \in (-\infty, \infty)$ with $t_0 \in (-\infty, \infty)$.

Consider first the following system of non-linear differential equations

$$\begin{aligned} x' + f_1(t, x) f_2(y) &= 0 \\ y' + g_1(t, x) g_2(y) &= 0. \end{aligned} \tag{2}$$

Let $H_1(t, x) = \int_0^x h_1(t, s)\, ds$, $H_2(y) = \int_0^y h_2(s)\, ds$ where

$$h_1(t, x) = \frac{g_1(t, x)}{f_1(t, x)}, \qquad h_2(y) = \frac{f_2(y)}{g_2(y)}$$

and

$$\frac{\partial h_1(t, x)}{\partial t}$$

are continuous functions for every $t \geqq t_0$, $x \in (-\infty, \infty)$, $y \in (-\infty, \infty)$.

We have

Theorem 1. *Suppose that for all* $y \in (-\infty, \infty)$

$$H_2(y) \leqq k_2 < \infty$$

and suppose that for every continuously differentiable function $u(t)$ *on* $\langle t_0, \bar{t})$, $\bar{t} \leqq +\infty$ *which is unbounded as* $t \to \bar{t}_-$ *there exists a sequence* $\{t_i\}_{i=1}^{\infty}$, *such that* $t_i \to \bar{t}_-$ *and*

$$\frac{\partial H_1(t_1, u(t))}{\partial t} \leqq \frac{\partial H_1(t_1, u(t_i))}{\partial t} \qquad t_0 \leqq t \leqq t_i. \tag{3}$$

Moreover, let

$$\lim_{|x| \to \infty} H_1(t_0, x) = H_1 \leqq +\infty. \tag{4}$$

Then for every solution $(x(t), y(t))$ *of* (2) *such that*

$$K_0 = H_1(t_0, x(t_0)) - H_2(y(t_0)) + k_2 < H_1, \tag{5}$$

$x(t)$ *is bounded for* $t \geqq t_0$.

Proof. Let the solution $(x(t), y(t))$ of (2) exist on $\langle t_0, \bar{t})$, $\bar{t} \leqq +\infty$; suppose that it satisfies the condition (5) and that $\limsup_{t \to \bar{t}_-} |x(t)| = +\infty$. Then there exists a sequence $\{t_i\}_{i=1}^{\infty}$, $t_i \to \bar{t}_-$ for $i \to \infty$ such that $\lim_{i \to \infty} |x(t_i)| = +\infty$. From (2) we see that

$$h_1(t, x(t))\, x'(t) = h_2(y(t))\, y'(t) \qquad \text{for} \quad t \in \langle t_0, \bar{t}).$$

By integrating this, we get, for all $t \in \langle t_0, \bar{t})$

$$H_1(t, x(t)) = H_1(t_0, x(t_0)) - H_2(y(t_0)) + H_2(y(t)) + + \int_{t_0}^{t} \frac{\partial H_1(s, x(s))}{\partial s}\, ds, \tag{6}$$

and therefore

$$H_1(t, x(t)) \leqq K_0 + \int_{t_0}^{t} \frac{\partial H_1(s, x(s))}{ds}\, ds. \tag{7}$$

For a given sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \to \bar{t}_-$ for $i \to \infty$ (7) yields, with the help of (3) (putting $u(t) = x(t)$)

$$H_1(t_i, x(t_i)) \leqq K_0 + \int_{t_0}^{t_i} \frac{\partial H_1(s, x(t_i))}{\partial s}\, ds =$$

$$= K_0 + H_1(t_i, x(t_i)) - H_1(t_0, x(t_i)),$$

or

$$\cdot\ H_1(t_0, x(t_i)) \leqq K_0 .$$

For $i \to \infty$ we can use this, together with (4), to obtain a contradiction to (5).

Theorem 2. *Suppose that, for every $t \geqq t_0$ and $x \in (-\infty, \infty)$,*

$$(8) \qquad -\infty < k_1 \leqq H_1(t, x), \qquad \frac{\partial H_1(t, x)}{\partial t} \leqq \alpha(t)$$

and let

$$(9) \qquad \lim_{|y| \to \infty} H_2(y) = -H_2 \geqq -\infty .$$

If

$$(10) \qquad \int_{t_0}^{\infty} \alpha(t)\, \mathrm{d}t = A < \infty ,$$

then for any solution $(x(t), y(t))$ of (2) such that

$$(11) \qquad K_0^* = H_1(t_0, x(t_0)) - H_2(y(t_0)) + A - k_1 < H_2 ,$$

$y(t)$ is bounded for $t \geqq t_0$.

Proof. Suppose that the solution $(x(t), y(t))$ of (2) is defined on $\langle t_0, \bar{t})$, $\bar{t} \leqq +\infty$ and that (11) holds. We shall prove that in that case $y(t)$ is bounded on $\langle t_0, \bar{t})$. Let $\limsup_{t \to \bar{t}-} | y(t) | = +\infty$. Owing to (8) and (10), (6) yields:

$$-H_2(y(t)) \leqq K_0^* .$$

Consider a sequence $\{t_i\}_{i=1}$ such that $t_i \to \bar{t}_-$ for $i \to \infty$ and $\lim_{t \to \infty} | y(t_i) | = +\infty$. Now if we put $t = t_i$ and let $i \to \infty$, we can use (9) to obtain a contradiction to the assumption (11).

Remark 1. If $H_1 = +\infty$ or $H_2 = +\infty$ in (4) or (9) respectively, then evidently for any solution $(x(t), y(t))$ of (2) $x(t)$ or $y(t)$ is bounded for all $t \geqq t_0$ from the domain of the solution.

Theorem 3. *Under the assumptions of Theorem 2, let $H_2 = +\infty$, $\alpha(t) \leqq 0$ and suppose that for all $y \in (-\infty, \infty)$*

$$H_2(y) \leqq k_2 < +\infty .$$

If for any sequences $\{t_i\}_{i=1}^{\infty}$, $\{x_i\}_{i=1}^{\infty}$ such that for $i \to \infty$ $t_i \to \infty$ and $| x_i | \to \infty$

$$(12) \qquad \lim_{i \to \infty} H_1(t_i, x_i) = +\infty ,$$

then, for any solution $(x(t), y(t))$ of (2), $| x(t) | + | y(t) |$ is bounded for $t \geqq t_0$.

Proof. Suppose that a solution $(x(t), y(t))$ exists on $\langle t_0, \bar{t})$, $\bar{t} \leqq +\infty$. The boundedness of $y(t)$ for $t \in \langle t_0, \bar{t})$ is ensured by Theorem 2. Suppose now that $x(t)$ is unbounded for $t \to \bar{t}_-$, i.e. that there exists a sequence $\{t_i\}_{i=1}^{\infty}$, $t_i \to \bar{t}_-$ for $i \to \infty$, such that $\lim_{i\to\infty} | x(t_i) | = +\infty$. Further let $\{\tilde{t}_i\}_{i=1}$ be an arbitrary sequence such that $\tilde{t}_i \to \infty$ for $i \to \infty$ and for all i, $t_i \leqq \tilde{t}_i$. Since $\alpha(t) \leqq 0$, we have

$$H_1(\tilde{t}_i, x(t_i)) \leqq H_1(t_i, x(t_i)),$$

and we can use this and the relation (7) to get

$$H_1(\tilde{t}_i, x(t_i)) \leqq H_1(t_i, x(t_i)) \leqq K_0,$$

For $i \to \infty$, this contradicts the hypothesis (12). Thus for $t \in \langle t_0, \bar{t})$ $|x(t)| + |y(t)|$ is bounded.

Remark 2. The equation

$$x'' + f(t, x)\, g(x') = 0$$

is a special case of (2). Theorems 18 and 19 of [12] deal with the boundedness of solutions of this equation.

Now let us consider the system (1). If $(x(t), y(t))$ is a solution of (1) which exists on $\langle t_0, \bar{t})$, $\bar{t} \leqq +\infty$, then for $t \in (t_0, \bar{t})$ (1) yields:

$$h_1(t, x(t))\, x'(t) = h_2(y(t))\, y'(t) + \\ + g(t, x(t), y(t))\, h_2(y(t)) - f(t, x(t), y(t))\, h_1(t, x(t))$$

which means that

$$H_1(t, x(t)) = H_2(y(t)) + H_1(t_0, x(t_0)) + \\ + \int_{t_0}^{t} \frac{\partial H_1(s, x(s))}{\partial s}\, \mathrm{d}s + \int_{t_0}^{t} [g(s, x(s), y(s))\, h_2(y(s)) - f(s, x(s), y(s))\, h_1(s, x(s))]\, \mathrm{d}s.$$

It is easy to see from the proofs of Theorems 1 to 3 that the following theorems hold:

Theorem 1′. *Suppose that for all* $t \geqq t_0$, $x \in (-\infty, \infty)$, $y \in (-\infty, \infty)$

$$g(t, x, y)\, h_2(y) - f(t, x, y)\, h_1(t, x) \leqq \beta(t)$$

and let

$$\int_{t_0}^{\infty} \beta(t)\, \mathrm{d}t = B < +\infty.$$

If the hypotheses of Theorem 1 *hold, then for any solution* $(x(t), y(t))$ *of* (1) *such that*

$$H_1(t_0, x(t_0)) - H_2(y(t_0)) + k_2 + B < H_1$$

$x(t)$ *is bounded for* $t \geqq t_0$.

Theorem 2'. *Suppose that the hypotheses of Theorem* 2 *hold, with* $\frac{\partial H_1(t, x)}{\partial t} \leqq \alpha(t)$ *and the assumption* (10) *replaced by the assumptions*

$$\frac{\partial H_1(t, x)}{\partial t} + g(t, x, y) h_2(y) - f(t, x, y) h_1(t, x) \leqq \gamma(t)$$

and

$$\int_{t_0}^{\infty} \gamma(t) \, dt = C < +\infty$$

respectively.

Then for any solution $(x(t), y(t))$ *of* (1) *such that*

$$H_1(t_0, x(t_0)) - H_2(y(t_0)) + C - k_1 < H_2$$

$y(t)$ *is bounded for* $t \geqq t_0$.

Theorem 3'. *Suppose that the hypotheses of Theorem* 3 *hold, with the assumption* $\alpha(t) \leqq 0$ *replaced by* $\gamma(t) \leqq 0$. *Then for any solution* $(x(t), y(t))$ *of* (1) $| x(t) | + | y(t) |$ *is bounded for* $t \geqq t_0$.

REFERENCES

[1] W. J. Coles: *Boundedness of solutions of two-dimensional first order differential systems,* Bolletino U. M. I., S. IV, No 2 (225—231), 1971.

[2] P. Šoltés: *On certain properties of the solutions of a non-linear differential equation of the second order,* Arch. Math. 2, VII, 47—63, 1971.

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