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Preference numbers and funnel dimension

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Abstract. The concept of *funnel dimension* of a topological space is introduced and relations with existing notions of dimension are established. The concept of funnel dimension is related to a problem in mathematical economics.

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1. Funnel dimension.

1.1. Considering a set X and a family $\mathcal{L}$ of real-valued functions on X we define the *lower level topology* as the coarsest topology on X such that all functions of $\mathcal{L}$ are lower semi-continuous. We shall be concerned with the question how to relate the dimension of such a space with the number of functions in $\mathcal{L}$.

Definition 1.2. A *funnel* in a topological space X is a collection $\mathcal{F} = \{F_t | t \in D\}$ of closed subsets, indexed by a dense subset D of the real interval [0,1], satisfying

(i) $t < s$ implies $F_t \subset F_s$
(ii) $\cup\{F_t | t \in D\} = X$
(iii) $F_t = \cap\{F_s | s > t,\ s \in D\}$ for each $t \in D$.

Remark. In the above, the restriction of D being a dense subset of [0,1] rather than a dense subset of $\mathbf{R}$ is not essential.

Definition 1.3. The *funnel dimension* of a topological space X is the least number $n \geq 0$ for which there are $n + 1$ funnels $\mathcal{F}_1, \dots, \mathcal{F}_{n+1}$ which together constitute a subbase for the closed subsets of X. We shall write $f\text{-}\dim X = n$. If no such number exists we say that $f\text{-}\dim X = \infty$.

It is clear that the funnel dimension is a topological invariant.

Examples 1.4.

a. A singleton has funnel dimension 0.

b. If X is a T_1-space with more than one point, then $f\text{-}\dim X \geq 1$.

Proof : If p and q are distinct points of X, there must be subbase elements S and T separating p from q and q from p respectively (that is $p \in S$, $q \notin S$ and $q \in T$, $p \notin T$). Because each funnel is linearly ordered by inclusion, S and T cannot belong to the same funnel. ■

c. If X is a finite T_1-space with more than one point, then $f\text{-}\dim X = 1$.

d. If I is a non-degenerate interval, $f\text{-}\dim I = 1$. Taking the interval [0,1] as an example and choosing $\mathcal{F}_1 = \{[0,\alpha] | \alpha \in (0,1]\}$ and $\mathcal{F}_2 = \{[1-\alpha, 1] | \alpha \in (0,1]\}$ we see that the funnel dimension of [0,1] is at most 1.

Theorem 1.5. *The function f-dim is monotone, i.e., if Y is a non-empty subspace of a space X, then $f\text{-dim}\, Y \leq f\text{-dim}\, X$.*

PROOF : The trace of a funnel in X on the subspace Y is a funnel in Y. ∎

1.6. With a funnel $\mathcal{F}$ on a space X we may associate a function $f : X \to [0,1]$ defined by

$$f(x) = \inf\{t \in D | x \in F_t\}.$$

We shall say that f is the *level function* of $\mathcal{F}$. Clearly, f is lower semi-continuous. Conversely, if $g : X \to [0,1]$ is lower semi-continuous and E is a dense subset of [0,1], the collection

$$\{x | g(x) \leq \alpha\} \qquad (\alpha \in E)$$

is a funnel with level function g.

The proof of the above statements is quite standard (cf. [**5**, §19]) and is left to the reader. From the above it is clear that the funnel dimension of a space X is the least number n satisfying the following condition: there exists a set $\mathcal{L}$ of $n+1$ real-valued functions for which the lower level topology and the given topology coincide.

Theorem 1.7. *For each $k \in \mathbb{N}$, we have $f\text{-dim}\, \mathbb{R}^k = k$ and $f\text{-dim}\, \Delta^k = k$, where Δ^k denotes the k-dimensional simplex.*

PROOF : $\mathbb{R}^k$ can be embedded in Δ^k and Δ^k can be embedded in $\mathbb{R}^k$. So f-dim($\mathbb{R}^k$) = f-dim(Δ^k) by Theorem 1.5. Using the $k+1$ barycentric coordinate functions as level functions, we see that $f\text{-dim}(\Delta^k) \leq k$.

From the result in the next section the reverse inequality follows. There we shall show that $ind(\Delta^k) \leq f\text{-dim}(\Delta^k)$, where ind denotes the small inductive dimension. It is a well-known fact that $ind(\Delta^k) = k$ (see [4]). ∎

1.8. Now we relate the above notions to preference relations.
A preference relation $\leq$ on X is a transitive, reflexive and complete relation. Any real valued function f on X defines a preference relation by

$$(*) \qquad x \leq y \quad \text{iff} \quad f(x) \leq f(y)$$

A multiply ordered space X is a set supplied with a (finite) number of preference relations $\leq_1, \ldots, \leq_N$. The (lower) - preference topology on X is the coarsest topology for which the sets

$$\{x | x \leq_i a\} \qquad (i \leq N, a \in X)$$

are closed. One should notice that the lower level topology is generally *finer* than the corresponding (by means of (*)) lower preference topology.
This is because of the fact that the images of the level functions might contain certain gaps. Therefore, the lower preference number of a topological space, as defined in [**6**], is generally larger than its funnel dimension. Also it should be emphasized that the preference number does not behave monotonically. That is, subspaces of a given space X might have larger (even infinite) preference numbers.

1.9. There is also a certain relation of our notions with some concepts of general convexity theory, in particular the concepts of *generating degree* and *directional degree* which are discussed in [8].

2. Relations with the small inductive dimension.

2.1. In this section we investigate the relation between the funnel dimension and the small inductive dimension ind.

Theorem. *For a separable metric space X we have:*

$$\operatorname{ind} X \leq f\text{-}\dim X \leq 2 \operatorname{ind} X + 1$$

PROOF : We assume f-$\dim X = k$, $\operatorname{ind} X = n$ and $k < n$. Let n be the smallest number for which this is possible. Because of Example 1.4 b, we have $n \geq 2$. For this number n we select a space X with $\operatorname{ind} X = n$ and k minimal. Let $\mathcal{F}_1, \ldots, \mathcal{F}_{k+1}$ be funnels such that $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{k+1}$ is a subbase for the closed sets. As $\operatorname{ind} X = n$, there is a closed set G and a point $p \notin G$ such that for every closed set S with $G \subset S$ and $p \notin S$, we have $\operatorname{ind} \partial(S) \geq n-1$, where ∂ denotes the topological boundary. Now let $\mathcal{F}' \subset \mathcal{F}$ be a finite collection satisfying $p \notin \cup\mathcal{F}'$ and $G \subset \cup\mathcal{F}'$.
We must have $\operatorname{ind} \partial(\cup\mathcal{F}') \geq n-1$. By the (finite) sum theorem and the subset theorem of dimension theory (see [4]) we conclude $\operatorname{ind} \partial H \geq n-1$ for some $H \in \mathcal{F}'$. Assume $H \in \mathcal{F}_1$. We shall show that $\mathcal{F}_2 \cup \cdots \cup \mathcal{F}_{k+1}$, when intersected with ∂H, is a subbase for ∂H.
To this end let B be any closed subset of ∂H and $q \in \partial H \setminus B$. Let U and V be disjoint neighborhoods of q and B in X respectively, with $cl\, U \cap cl\, V = \emptyset$. There are finitely many elements $A_1, \ldots, A_m$ of $\mathcal{F}$ such that $V \subset A_1 \cup \cdots \cup A_m$ and $q \notin A_1 \cup \cdots \cup A_m$. Define $C = \cup\{A_j | 1 \leq j \leq m \text{ and } A_j \not\subset H\}$. Because $C \supset V \setminus H$ and C is closed we must have $B \subset C$ and also $q \notin C$.
It follows that none of the A_j used to build C is an element of $\mathcal{F}_1$. Thus we have proved that f-$\dim \partial H \leq k-1$, whence it follows that $\operatorname{ind} \partial H = n-1$. This contradicts the minimality of n. To prove the second inequality observe that a space X with $\operatorname{ind} X = n$ can be embedded in $\mathbf{R}^{2n+1}$. It follows that f-$\dim X \leq f$-$\dim(\mathbf{R}^{2n+1}) \leq 2n+1$. ■

3. Relations with the directional dimension.

3.1. We now establish a relation between f-dim and d-dim, introduced by Deak [1]. Recall that d-$\dim X$ is the smallest n for which there exists a set $\mathcal{L}$ of n real-valued functions with the property that the coarsest topology in which all of the functions of $\mathcal{L}$ become continuous, coincides with the given topology.
Our main result is

Theorem 3.2. *If X is separable metric, then f-$\dim X \leq d$-$\dim X$.*

PROOF : Assume that d-$\dim X = n$.
As X can be embedded in $\mathbf{R}^n$ (see [1]), it follows that f-$\dim X \leq n$. ■

We have not been able to decide whether there exists a separable metric space X with f-$\dim X < d$-$\dim X$.

4. On embeddability into Euclidean space.

The question of embeddability in Euclidean space cannot be answered in a uniform manner. Here we show that T_1-spaces X with f-dim $X = 1$ can be embedded into $\mathbf{R}$ and by a counterexample we show that a similar result fails for higher funnel dimension. In view of the discussion in 1.8 it would be profitable to have a definite answer for the compact case. Sofar, however, this is open. First we prove a lemma.

Lemma 4.1. *Let X be a T_1-space where the topology is generated by the funnels $\mathcal{F}_1, \dots, \mathcal{F}_{n+1}$, with level functions $f_1, \dots, f_{n+1}$. Let $f(x) = (f_1(x), \dots, f_{n+1}(x))$; $\sigma(x) = f_1(x) + \cdots + f_{n+1}(x)$ and $\pi(x) = \frac{f(x)}{\sigma(x)}$, for $x \in X$.*
Then π is a closed injection to the space $\pi(X)$.

PROOF : We first show that π is injective. Notice that $\sigma(x) = 0$ is impossible, if X consists of more than one point. Indeed, if $\sigma(x) = 0$, then $f_i(x) = 0$ for all i. It follows that x is a member of all closed sets. If $\pi(x) = \pi(y)$, then $f(x) = \frac{\sigma(x)}{\sigma(y)} f(y)$ where we may assume that $\sigma(x) \leq \sigma(y)$.
Consequently $f_i(x) \leq f_i(y)$ for all $i \leq n+1$, whence $x \in cl\{y\}$, implying $x = y$.
Next we show that π is a closed mapping from X to $\Delta^n \cap \pi(X)$.
To this end it is sufficient to demonstrate that the sets $\pi(\{x | f_i(x) \leq \alpha\})$ are closed subsets of $\Delta^n \cap \pi(X)$, since π has already been shown to be injective.
Let $T = \{x | f_1(x) \leq \alpha\}$ and suppose y^m is a converging sequence in $\pi(T)$ with limit y. Thus $y = \pi(x) = \lim y^m = \lim \pi(x^m)$. We have to show $x \in T$.
By passing to subsequences (using the compactness of Δ^n) we may assume that all sequences $f_i(x^m)$ converge to a number γ_i and that $\sigma(x^m)$ converges to a number γ.
Thus we have $\pi(x^m)$ converging to $y = \frac{1}{\gamma}(\gamma_1, \dots, \gamma_{n+1})$ which means $f_1(x) = \frac{\sigma(x)}{\gamma}\gamma_1$ and since $f_1(x^m) \leq \alpha$ we conclude $\gamma_1 \leq \alpha$, whence $f_1(x) \leq \frac{\sigma(x)}{\gamma}\alpha$.
Now we show that $\sigma(x) \leq \gamma$ which clearly implies that $x \in T$.
Assume $\sigma(x) > \gamma$.
Since $\lim \pi(x^m) = \pi(x)$, we have $\lim f_i(x^m) = \frac{\gamma}{\sigma(x)} f_i(x)$, for all $i \leq n+1$.
Thus, for m large enough, all $f_i(x^m) < f_i(x)$, implying that $x^m \neq x$ and $x^m \in cl\{x\}$ which is a contradiction. ∎

Remark. If, in the above, X is assumed to be compact, one can show that the mapping $F(x) = (f_1(x), \dots, f_n(x), 1 - (f_1(x) + \cdots + f_n(x)))$ is also a closed injection to the space $F(X)$.

Lemma 4.2. *If X is a T_1-space where the topology is generated by two funnels $\mathcal{F}_1, \mathcal{F}_2$ with level functions f_1, f_2 we have for all $x, y \in X$:*

$$f_1(x) \leq f_1(y) \quad \textit{iff} \quad f_2(x) \geq f_2(y)$$

PROOF : If $f_1(x) \leq f_1(y)$ and $f_2(x) < f_2(y)$ we have $x \neq y$ and $x \in cl\{y\}$, which is impossible. ∎

4.3. Under the conditions of Lemma 4.2 we now consider a set of the form $T = \{x \in X | f_1(x) \geq \alpha\}$.

If $\alpha = f_1(y)$ for some y we have, by the above lemma, $T = \{x \in X | f_2(x) \leq f_2(y)\}$ which is closed. Suppose that $\alpha \notin \text{ im } (f_1)$.
Put $\beta = \sup\{f_1(x) | f_1(x) < \alpha, x \in X\}$ and $\gamma = \inf\{f_1(x) | f_1(x) > \alpha, x \in X\}$. If $\beta \notin \text{ im } (f_1)$ there exists a sequence $f_1(y_n)$ approaching β. We conclude that $T = \bigcap_{n \in \mathbf{N}} \{x \in X | f_1(x) \geq f_1(y_n)\}$, which is closed by the previous lemma.
If $\gamma \in \text{ im } (f_1)$ we see that $T = \{x | f_1(x) \geq \gamma, x \in X\}$, again a closed set by the same arguments.
The case left to consider is the case that $\beta = f_1(p)$, for some p and $\gamma \notin \text{ im } (f_1)$, where γ can be approximated by a sequence $f_1(z_n)$ from the above.
Now consider the funnel $\mathcal{F}_1$. By applying the defining properties of a funnel we see that $F_\gamma = \bigcap_{n \in \mathbf{N}} F_{f_1(z_n)}$. If F_β is a proper subset of F_γ there exists $x \in X$ with $x \in F_\gamma$ and $x \notin F_\beta$ which means $f_1(x) \leq \gamma$ and $f_1(x) > \beta$, a contradiction. Thus $F_{f_1(p)} = \bigcap_{n \in \mathbf{N}} F_{f_1(z_n)}$, whence by filling the gap $(\beta, \gamma]$ by means of a function f_1' defined by

$$f_1'(x) = \begin{cases} f_1(x) & \text{if } f_1(x) > \alpha \\ f_1(x) + \gamma - \beta & \text{if } f_1(x) < \alpha \end{cases}$$

we obtain a level function f_1' of $\mathcal{F}_1$, where only the indexing differs from that of the original funnel. The closed sets defined by $\mathcal{F}$ remain unchanged. This observation, which can be found in a similar form in [2], enables us to prove the following result.

Theorem 4.4. *For a T_1-space X, f-dim $X \leq 1$ iff X is embeddable in* $\mathbf{R}$.

PROOF : Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the funnels on the T_1-space X which generate the topology and f_1 and f_2 the level functions. We assume that the indexing of both $\mathcal{F}_1$ and $\mathcal{F}_2$ is such, that no gaps of the form $(\beta, \gamma]$ occur in the images of f_1 and f_2. See [2] for a detailed proof. By the argument of 4.3 it is clear that f_1 and f_2 are in fact both upper semi-continuous, and hence continuous. Therefore, the mapping π becomes continuous and hence an embedding of X into the one-dimensional simplex. ∎

4.5. The next example shows that for higher funnel dimensions embeddability into Euclidean space is not generally possible. Let Q denote the set of rationals. Let $X = [0,1], f_1(x) = x, f_2(x) = 1 - x$ and f_3 be defined by

$$f_3(x) = \begin{cases} 1 & \text{if } x \in Q, \\ 0 & \text{else }, \end{cases}$$

for $x \in [0,1]$.

Now the lower level topology generated by this set of functions is the topology generated by the Euclidean topology and the extra open set Q. This topology is a non regular Hausdorff topology, which is not embeddable into Euclidean space. Clearly, by the above theorem, it has funnel dimension 2. Notice that the topology discussed here is in fact a lower preference topology as well.

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