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Some connections between measure, indiscernibility and representation of cuts

MARTIN KALINA, PAVOL ZLATOŠ

Abstract. Every indiscernibility equivalence R on an infinite set u has both restrictions and extensions to indiscernibility equivalences on sets of arbitrary infinite cardinalities preserving the counting measure R-figures. For every ring $\mathfrak{N}$ of real classes closed with respect to the Cartesian product and containing all σ-semisets, whenever a class $X \in \mathfrak{N}$ occurs, such that the pair of cuts $\langle \underline{X}, \overline{X} \rangle$ is not Borel representable, then already each really representable pair of proper cuts $\langle A, B \rangle$ is $\mathfrak{N}$-representable, i.e. $A = \underline{Y}, B = \overline{Y}$ for some $Y \in \mathfrak{N}$.

Keywords: Alternative set theory, indiscernibility equivalence, measure, cuts of classes, real class, normal ring

Classification: Primary 03E70, 28C15, 28E05; Secondary 04A15, 28A05, 54J05

This paper is a direct continuation of our preceding works [**K-Z 1988**], [**K-Z 1989a**] and [**K-Z 1989b**], contributing to the problematics announced in the title. In mathematics based on the classical set theory measures are often studied in connection with some topological structure on their underlying sets. Also in the alternative set theory where, in the most substantial cases, the topological structure can be represented by an indiscernibility equivalence R on an infinite set u (see [**V**]) and the measure is the counting measure μ_a where $a = |u|$ (see [**K-Z 1989b**] and the References there), such investigations are of considerable importance. In such a situation we are primarily interested in measurability of R-figures, i.e. of classes $X \subseteq u$ such that $X = R'' X$. One aim of this note is to show that, roughly speaking, the situation just sketched can be modelled in any infinite set u, independently of the number of its elements. Such a feeling is evoked by the Loeb's construction of measure in nonstandard analysis (see [**L 1975**], cf. [**R 1981**]) which starts with an arbitrary infinite natural number, and is intrinsically connected with the topology of the unit real interval. This feeling, however, at least as far as we know, has not been yet formulated rigorously in a general setting. As a consequence, under fairly weak conditions imposed on the system of classes $\mathfrak{N}$, either for each $b \in N \setminus FN$, every class $X \in \mathfrak{N}$, which is not disqualified for being too large, is b-measurable, or for each $b \in N \setminus FN$ there is a class $X \in \mathfrak{N}$ such that $\overline{X} \leq b$ but X is not b-measurable.

In the final section a far-reaching refinement of the lastly mentioned result is established. Namely, for "normal rings" of classes $\mathfrak{N}$ and pairs of proper cuts, the $\mathfrak{N}$-representability coincides either with the real representability or with the Borel one described in [**K-Z 1989a**], [**K-Z 1989b**], respectively.

1. Preliminaries.

Concerning the basic notions and results of AST the reader is referred to [**V**]. We also assume that he is acquainted with the notion of real class from [**Č-V 1979**] and with the notions, notation, conventions and results of [**K-Z 1988**], [**K-Z 1989a**] and [**K-Z 1989b**].

If R is an equivalence on a class K, then the (codable) quotient class of K with respect to R is denoted by K/R. Particularly, if e is a set equivalence on a set u, then the quotient set u/e has not to be confused with quotients of form $\frac{x}{y}$ denoting rational numbers.

In [**K-Z 1989b**] the relation "X is a class of bounded a-measure", denoted by $M(a,X)$ as well as σ-ring $\mathfrak{M}_a$ of a-measurable classes and the measure μ_a on $\mathfrak{M}_a$ were defined for each $a \neq 0$. For completeness' sake we put $\mathfrak{M}_0 = \{\emptyset\}$. For the purpose of the present article it is convenient to introduce also the concepts of inner and outer a-measure of a class X, which will be denoted by $\underline{\mu}_a(X)$ and $\overline{\mu}_a(X)$ respectively. Both of them take as values nonnegative real numbers or the sign of infinity ∞ and are defined for each class X by

$$\underline{\mu}_a(X) = \sup\{\frac{|v|}{a}; v \subseteq X\},$$

$$\overline{\mu}_a(X) = \inf\{\frac{|w|}{a}; X \subseteq w\}.$$

Obviously, $\underline{\mu}_a(X)$ depends only on the inner cut $\underline{X}$ and similarly $\overline{\mu}_a(X)$ depends only on the outer cut $\overline{X}$. Also the following observation, which can partly be regarded as a definition for the purpose of the present paper, is trivial. Namely, for each X it holds

$$M(a,X) \iff \underline{\mu}_a(X) = \overline{\mu}_a(X) < \infty.$$

Then, of course, $\mu_a(X)$ equals to the common value of the inner and outer measure. However, $\underline{\mu}_a(X) = \overline{\mu}_a(X) = \infty$ and $X \notin \mathfrak{M}_a$ may well happen.

Now, we will record some technical results which will be needed in the future.

Lemma 1.1. *Let u be a set, $|u| = a > 0$, and $X \subseteq u$. Then*

$$\underline{\mu}_a(X) + \overline{\mu}_a(u \setminus X) = 1.$$

PROOF :

$$\underline{\mu}_a(X) = \sup\{\frac{|v|}{a}; v \subseteq X\} = \sup\{\frac{|u \setminus w|}{a}; u \setminus w \subseteq X\} =$$

$$= 1 - \inf\{\frac{|w|}{a}; u \setminus X \subseteq w\} = 1 - \overline{\mu}_a(u \setminus X).$$

■

Lemma 1.2. *Let u, v be sets, $|u| = a > 0, |v| = b > 0$, and $X \subseteq v$. Then*

$$\underline{\mu}_{ab}(u \times X) = \underline{\mu}_b(X)$$

and

$$\overline{\mu}_{ab}(u \times X) = \overline{\mu}_b(X).$$

PROOF : is trivial. ■

Lemma 1.3. *Let b be an infinite natural number and X be a real class such that $\overline{X} = b.FN$ and $X \notin \mathfrak{M}_b$. Then for each $c \in b.FN \setminus b/FN$ there is a set u such that $c \leq |u| < b.FN$ and $\neg M(b, X \cap u)$.*

PROOF : By 1.9 from **[K-Z 1989a]** there is a σ-class Z such that $X \subseteq Z$ and $|Z| = b.FN$. For each $c \in b.FN \setminus b/FN$ one can find a sequence $\{u_n; n \in FN\}$ such that $c \leq |u_0|, u_n \subseteq u_{n+1}$ for each n and $Z = \bigcup\{u_n; n \in FN\}$. Since $X = \bigcup\{X \cap u_n; n \in FN\}$ and $X \notin \mathfrak{M}_b$, there has to be an n such that $\neg M(b, X \cap u_n)$. ∎

Lemma 1.4. *For every class X and any real class Y it holds*

$$\underline{X \cup Y} \leq \overline{X} \dot{+} \overline{Y}.$$

PROOF : Let $u \subseteq X \cup Y$. Denote $S = u \cap Y$. By 3.1.7 from **[K-Z 1988]**, $\underline{u \setminus Y} = \underline{u \setminus S} = |u| - \overline{S}$. By a theorem of Sochor **[S 1988]** (cf. 2.3.9 in **[K-Z 1988]**) there is an $a \in N$ and an additive cut A such that either $\overline{S} = a + A$ or $\overline{S} = a - A$. In the first case

$$\underline{u \setminus S} = |u| - a - A,$$

hence

$$|u| \leq |u| + A = (a + A) \dot{+} (|u| - a - A) = \overline{S} \dot{+} \underline{u \setminus S}.$$

Otherwise

$$\underline{u \setminus S} = u - a + A,$$

and

$$|u| \leq |u| + A = (a - A) \dot{+} (|u| - a + A) = \overline{S} \dot{+} \underline{u \setminus S}.$$

Obviously $S \subseteq Y$ and $u \setminus S \subseteq X$. That's why $\overline{S} \leq \overline{Y}$ and $\underline{u \setminus S} \subseteq \underline{X}$, hence $|u| \leq \underline{X} \dot{+} \overline{Y}$. ∎

Before stating our last lemma, let us recall from **[S 1988]** that for each cut A,

$$A^+ = A \dot{-} A = \{a; (\forall b \in A)(a + b \in A)\}$$

is an additive cut, $A \dot{-} A^+ = A = A \dot{+} A^+, a^+ = 0$ for $a \in N, A^+ > 0$ if $A \notin N$, and $A^+ = A$ provided A is additive. Moreover, in its original formulation the Sochor's theorem **[S 1988]** just mentioned says that for each real cut A there is an $a \in N$ such that either

$$A = a + A^+ \quad \text{or} \quad A = a - A^+.$$

Lemma 1.5. *Let X, Z be classes and u be a set such that $X \subseteq Z, u \subseteq Z$, $\overline{X}$ is a real cut, $\overline{X}^+ \leq |u|$ and $\overline{Z} \leq \overline{X} \dot{+} \overline{X}^+$. Then*

$$|u| - \overline{X}^+ \leq \overline{X \cap u}.$$

PROOF : Let us denote $\overline{X} = B, |u| = a$, and compute using 3.1.6 and 3.1.7 from **[K-Z 1988]**

$$\begin{aligned} B = \overline{X} &\leq \overline{X \cap u} \dot{+} \overline{X \setminus u} \leq \overline{X \cap u} \dot{+} \overline{Z \setminus u} = \\ &= \overline{X \cap u} \dot{+} (\overline{Z} - a) \leq \overline{X \cap u} \dot{+} ((B \dot{+} B^+) - a). \end{aligned}$$

Putting $(B \dot{+} B^+) - a = C$, one obtains

$$B \leq \overline{X \cap u} \dot{+} C,$$

hence

$$B \dot{-} C \leq (\overline{X \cap u} \dot{+} C) \dot{-} C \leq \overline{X \cap u},$$

where the last inequality can be found in [**K-Z 1988**, p. 440]. Except of the trivial case $u = \emptyset$, the cut B cannot be additive (this would imply $|u| \leq \overline{Z} \leq B = B^+ \leq |u|$), hence there is a $b \geq a$ such that either $B = b - B^+$ or $B = b + B^+$. In the former case

$$B \dot{-} C = (b - B^+) \dot{-} ((b + B^+) - a) = a - B^+$$

while in the later one

$$B \dot{-} C = (b + B^+) \dot{-} ((b + B^+) - a) = a - B^+.$$

Anyway,

$$|u| - \overline{X}^+ = a - B^+ \leq \overline{X \cap u}.$$

■

2. Proportional restrictions and extensions.

Let u, v be nonempty sets, $a = |u|, b = |v|$, and R, S be equivalences on u, v, respectively. Then the pair $\langle u, R\rangle$ is called a proportional restriction of the pair $\langle v, S\rangle$, and $\langle v, S\rangle$ is called a proportional extension of $\langle u, R\rangle$ if $u \subseteq v = S''u, R = S \cap u^2$ and for each R-figure $X \subseteq u$ and each S-figure $Y \subseteq v$ it holds

$$\underline{\mu}_a(X) = \underline{\mu}_b(S''X), \quad \overline{\mu}_a(X) = \overline{\mu}_b(S''X)$$

and

$$\underline{\mu}_a(Y \cap u) = \underline{\mu}_b(Y), \quad \overline{\mu}_a(Y \cap u) = \overline{\mu}_b(Y).$$

Note that given $\langle v, S\rangle$, a proportional restriction $\langle u, R\rangle$ of $\langle v, S\rangle$ is uniquely determined by the set $u \subseteq v$. On the other hand, given $\langle u, R\rangle$, there can be various equivalences S on a set $v \supseteq u$ forming a proportional extension $\langle v, S\rangle$ of $\langle u, R\rangle$.

Also, it can be seen that the definition of proportionality is superfluously strong. By Lemma 1.2, the preservation of the inner measure is equivalent to that of the outer one, and, since the following implications hold

$$R = S \cap u^2 \Longrightarrow S''X \cap u = X \quad \text{for each } R\text{-figure } X,$$
$$v = S''u \Longrightarrow S''(Y \cap u) = Y \quad \text{for each } S\text{-figure } Y,$$

the preservation of measures of R-figures is equivalent to that of measures of S-figures. Thus we have proved the following lemma.

Lemma 2.1. *Let u, v be nonempty sets, $|u| = a, |v| = b$, and R, S be equivalences on u, v, respectively, such that $u \subseteq v \subseteq S''u$ and $R = S \cap u^2$. Then the following five conditions are equivalent:*

(a) *$\langle u, R\rangle$ is a proportional restriction of $\langle v, S\rangle$;*
(b) *$\underline{\mu}_a(X) = \underline{\mu}_b(S''X)$ for each R-figure $X \subseteq u$;*
(c) *$\overline{\mu}_a(X) = \overline{\mu}_b(S''X)$ for each R-figure $X \subseteq u$;*
(d) *$\underline{\mu}_a(Y \cap u) = \underline{\mu}_b(Y)$ for each S-figure $Y \subseteq v$;*
(e) *$\overline{\mu}_a(Y \cap u) = \overline{\mu}_b(Y)$ for each S-figure $Y \subseteq v$.*

To handle proportional restrictions and extensions of indiscernibility equivalences on sets, proportional restrictions and extensions of set of equivalences will be used.

Lemma 2.2. *Let u, v be nonempty sets, $|u| = a, |v| = b$, and e, f be equivalences on u, v, respectively, such that $u \subseteq v \subseteq f''u$ and $e = f \cap u^2$. Then the following three conditions are equivalent:*

(a) *$\langle u, e\rangle$ is a proportional restriction of $\langle v, f\rangle$;*
(b) *$\frac{|s|}{a} \doteq \frac{|f''s|}{b}$ for each e-figure $s \subseteq u$;*
(c) *$\frac{|t\cap u|}{a} \doteq \frac{|t|}{b}$ for each f-figure $t \subseteq v$.*

PROOF : (a) $\Rightarrow$ (b) & (c) is trivial, (b) $\Leftrightarrow$ (c) was in fact shown during the proof of Lemma 2.1. To prove (b) & (c) $\Rightarrow$ (a), it suffices to note that for each R-figure X

$$\underline{\mu}_a(X) = \sup\{\frac{|s|}{a}; s \subseteq X \ \&\ e''s = s\},$$
$$\underline{\mu}_b(f''X) = \sup\{\frac{|t|}{b}; t \subseteq f''X \ \&\ f''t = t\}$$

and the classes of rationals occurring in these two formulas have the same figure with respect to the equivalence $\doteq$. Hence $\underline{\mu}_a(X) = \underline{\mu}_b(f''X)$, and by Lemma 2.1 $\langle u, e\rangle$ is a proportional restriction of $\langle v, f\rangle$. ∎

Lemma 2.3. *Let $\langle u, e\rangle$ be a proportional restriction of $\langle v, f\rangle$ and R, S be equivalences on u, v, respectively, such that $R = S \cap u^2$ and $f \subseteq S$. Then $\langle u, R\rangle$ is a proportional restriction of $\langle v, S\rangle$.*

PROOF : Obviously, $e = f \cap u^2 \subseteq S \cap u^2 = R$ and $v = f''u \subseteq S''u = v$. It suffices to show that $\underline{\mu}_a(X) = \underline{\mu}_b(S''X)$ for each R-figure X. This can be done exactly in the same way as in the proof of Lemma 2.2. ∎

Lemma 2.4. *Let e be an equivalence on an infinite set $u, |u| = a$, and $b \le a$ be an infinite natural number such that $\frac{|u/e|}{b} \doteq 0$. Then there is a set $v \subseteq u$ such that $|v| = b$ and $\langle v, e \cap v^2\rangle$ is a proportional restriction of $\langle u, e\rangle$.*

PROOF : Let z be a selector from e on u. For each $x \in z$ let v_x be a set such that $x \in v_x \subseteq e''\{x\}$ and

$$|v_x| - 1 < \frac{b}{a}|e''\{x\}| \le |v_x|.$$

Obviously, the sets v_x can be chosen in such a way that the function $\{\langle v_x, x\rangle; x \in z\}$ is a set. Then for the set $v' = \bigcup\{v_x; x \in z\}$ the summation of the above inequalities over $x \in z$ gives

$$|v'| - |z| < b \leq |v'|.$$

Hence $b \leq v'$, and, since $\frac{|z|}{b} = \frac{|u/e|}{b} \doteq 0$, also $b \simeq |v|$. Then $c = |v'| - b$ satisfies $0 \leq c < |z| < |v' \setminus |z|$. Let $w \subseteq v' \setminus z$ be a set such that $|w| = c$. We put $v = v' \setminus w$. Then obviously $|v| = b, z \subseteq v \subseteq u = e'' v$ and for each e-figure $s \subseteq u$ it holds

$$|s \cap v| - |s \cap z| < \frac{b}{a}|s| \leq |s \cap v| + c,$$

which implies $\frac{|s \cap v|}{b} \doteq \frac{|s|}{a}$, concluding the proof. ■

Lemma 2.5. *Let e be an equivalence on an infinite set $u, |u| = a$, and $b \geq a$ be a natural number such that $\frac{|u/e|}{b} \doteq 0$. Then there is a set $v \supseteq u$ and an equivalence f on v such that $|v| = b$ and $\langle v, f\rangle$ is a proportional extension of $\langle u, e\rangle$.*

PROOF : Let z be a selector from e on u. For each $x \in z$ let v_x be a set such that $e''\{x\} \subseteq v_x$ and

$$|v_x| \leq \frac{b}{a}|e''\{x\}| < |v_x| + 1.$$

Obviously, the sets v_x can be chosen in such a way that the function $\{\langle v_x, x\rangle; x \in z\}$ is a set and $v_x \cap v_y = \emptyset$ for $x \neq y$. Then for the set $v' = \bigcup\{v_x; x \in z\}$ it holds

$$|v'| \leq b < |v'| + |z|.$$

Hence $b \geq |v'|$, and, since $\frac{|z|}{b} = \frac{|u/e|}{b} \doteq 0$, also $b \simeq |v'|$. Then $c = b - |v'|$ satisfies $0 \leq c < |z|$. Let w be an arbitrary set such that $|w| = c$ and $v \cap w = \emptyset$. We put $v = v \cup w$. Ten obviously $|v| = b$ and $u \subseteq v$. For f any set equivalence on w such that $w \subseteq f'' z$ and $f \cap (v')^2 = \bigcup\{v_x^2; x \in z\}$ can be chosen, and $v = f'' u, e = f \cap u^2$ will hold. Now, for each e-figure $s \subseteq u$ it holds

$$|f'' s| - c \leq \frac{b}{a} \leq |f'' s| + |s \cap z|,$$

which implies $\frac{|f'' s|}{b} \doteq \frac{|s|}{a}$, concluding the proof. ■

Lemma 2.6. *Let R be a π-equivalence on a nonempty set u. Let c be a natural number such that there is a set $w \subseteq u$ satisfying $2^{|w|} \leq c$ and $R'' w = u$. Then there is an equivalence e on u such that $e \subseteq R$ and $|u/e| \leq c$.*

PROOF : Let w be such a set. Then there is even a reflexive and symmetric relation $r \subseteq R$ on u such that $r'' w = u$. It suffices to put $e = \{\langle x, y\rangle \in u^2; (\forall z \in w)(\langle x, z\rangle \in r \Leftrightarrow \langle y, z\rangle \in r)\}$. ■

Corollary 2.7. *Let R be an indiscernibility equivalence on an infinite set u. Then for each infinite natural number $c \leq |u|$ there is an equivalence e on u such that $e \subseteq R$ and $|u/e| \leq c$.*

Now, we are able to prove the announced results.

Theorem 2.8. (On restrictions) *Let R be an indiscernibility equivalence on a set u and $b \leq |u|$ be an infinite natural number. Then there is a set $v \subseteq u$ such that $|v| = b$ and $\langle v, R \cap v^2 \rangle$ is a proportional restriction of $\langle u, R \rangle$.*

PROOF : Let c be an infinite natural number, such that $\frac{c}{b} \doteq 0$, and e be an equivalence on u such that $e \subseteq R, |u/e| \leq c$, guaranteed by the Corollary. By Lemma 2.4, there is a $v \subseteq u$, such that $|v| = b$ and $\langle v, e \cap v^2 \rangle$ is a proportional restriction of $\langle u, e \rangle$. By Lemma 2.3, $\langle v, R \cap v^2 \rangle$ is a proportional restriction of $\langle u, R \rangle$. ∎

Theorem 2.9. (On extensions) *Let R be an indiscernibility equivalence on an infinite set u and $b \geq |u|$ be a natural number. Then there is a set $v \subseteq u$ and an indiscernibility equivalence S on v, such that $|v| = b$ and $\langle v, S \rangle$ is a proportional extension of $\langle u, R \rangle$.*

PROOF : Let c and e be as in the proof of the previous theorem. By Lemma 2.5, there is a set $v \subseteq u$ and an equivalence f on v such that $|v| = b$ and $\langle v, f \rangle$ is a proportional extension of $\langle u, e \rangle$. We put

$$S = \{\langle x, y \rangle \in v^2; (f'' \{x\} \times f'' \{y\}) \cap u^2 \subseteq R\}.$$

Obviously, S is an indiscernibility equivalence on v, $R = S \cap u^2$ and $f \subseteq S$. Then $\langle u, S \rangle$ is a proportional extension of $\langle u, R \rangle$ by Lemma 2.3. ∎

For every system of classes $\mathfrak{N}$ we introduce the class of natural numbers called the regularity of $\mathfrak{N}$

$$\operatorname{Reg}(\mathfrak{N}) = \{b; (\forall X \in \mathfrak{N}) (\overline{X} \leq b.FN \Longrightarrow X \in \mathfrak{M}_b)\}.$$

Thus $b \in \operatorname{Reg}(\mathfrak{N})$ iff each class $X \in \mathfrak{N}$ which is not "too large" with respect to b already is b-measurable.

Obviously, $FN \subseteq \operatorname{Reg}(\mathfrak{N})$ for every $\mathfrak{N}$, and $\mathfrak{N}_1 \subseteq \mathfrak{N}_2$ implies $\operatorname{Reg}(\mathfrak{N}_2) \subseteq \operatorname{Reg}(\mathfrak{N}_1)$. Also, if $X \in \mathfrak{N}_1 \Leftrightarrow X \in \mathfrak{N}_2$ for each semiset X, then $\operatorname{Reg}(\mathfrak{N}_1) = \operatorname{Reg}(\mathfrak{N}_2)$.

A system of classes $\mathfrak{N}$ will be called a ring of classes if for all $X, Y \in \mathfrak{N}$ it holds $X \cup Y \in \mathfrak{N}, X \setminus Y \in \mathfrak{N}$ (hence also $X \cap Y \in \mathfrak{N}$). A ring of classes $\mathfrak{N}$ will be called a Cartesian ring of classes if for all $X, Y \in \mathfrak{N}$ it follows $X \times Y \in \mathfrak{N}$.

Proposition 2.10. *Let $\mathfrak{N}$ be a ring of real classes containing all sets. Then*

$$\operatorname{Reg}(\mathfrak{N}) = \{b; (\forall X \in \mathfrak{N}) (\overline{X} < b.FN) \Longrightarrow M(b, X)\} =$$
$$= \{b; (\forall X \in \mathfrak{N}) (\overline{X} < b) \Longrightarrow M(b, X)\}.$$

PROOF : We will prove that for each infinite b the following three conditions are equivalent (since all the three classes considered contain FN as a subclass, this will be enough):

(1) $(\exists X \in \mathfrak{N})(\overline{X} \leq b.FN \;\&\; X \notin \mathfrak{M}_b)$,
(2) $(\exists X \in \mathfrak{N})(\overline{X} < b.FN \;\&\; \neg M(b, X))$,
(3) $(\exists X \in \mathfrak{N})(\overline{X} \leq b \;\&\; \neg M(b, X))$.

(1) $\Longrightarrow$ (2) Let $X \in \mathfrak{N} \setminus \mathfrak{M}_b, \overline{X} \leq b.FN$. If $\overline{X} < b.FN$, there is nothing to prove. If $\overline{X} = b.FN$, then by Lemma 1.3 there is a set u such that $\overline{X \cap u} \leq |u| < b.FN$ and $\neg M(b, X \cap u)$ though $X \cap u \in \mathfrak{M}$.
(2) $\Longrightarrow$ (3) It suffices to consider an $X \in \mathfrak{N}$ such that $\neg M(b, X)$ and $b < \overline{X} < b.FN$. Then there is a c such that $\overline{X} \leq c < b.FN$ and a set u satisfying $|u| = c, X \subseteq u$. Obviously, $\neg M(b, X)$ holds. As X is a real class, by Theorem 2.8 there is a set $v \subseteq u$ such that $|v| = b$ and $\underline{\mu}_b(X \cap v) = \underline{\mu}_c(X) < \overline{\mu}_c(X) = \overline{\mu}_b(X \cap v)$. At the same time $X \cap v \in \mathfrak{N}$.
(3) $\Longrightarrow$ (1) is trivial. ∎

Theorem 2.11. *Let $\mathfrak{N}$ be a Cartesian ring of real classes containing all sets. Then either* $\mathrm{Reg}(\mathfrak{N}) = N$ *or* $\mathrm{Reg}(\mathfrak{N}) = FN$.

PROOF : As $FN \subseteq \mathrm{Reg}(\mathfrak{N})$, it suffices to show that whenever there is an infinite $b \notin \mathrm{Reg}(\mathfrak{N})$ then already $c \notin \mathrm{Reg}(\mathfrak{N})$ for any infinite c. Let $X \in (\mathfrak{N}), X \subseteq u, |u| = b$ and $\neg M(b, X)$. If $c \leq b$, then by Theorem 2.8 there is a $v \subseteq u, |v| = c$, such that

$$\underline{\mu}_c(X \cap v) = \underline{\mu}_b(X) < \overline{\mu}_b(X) = \overline{\mu}_c(X \cap v).$$

As $X \cap v$ and $\neg M(c, X \cap v), c \notin \mathrm{Reg}(\mathfrak{N})$. If $b \leq c$, let us fix any a such that $c \leq ab$. Then $|a \times u| = ab$ and, by Lemma 1.2,

$$\underline{\mu}_{ab}(a \times X) = \underline{\mu}_b(X) < \overline{\mu}_b(X) = \overline{\mu}_{ab}(a \times X).$$

As $a \times X \in \mathfrak{N}, a \times X \leq ab$ and $\neg M(ab, a \times X), ab \notin \mathrm{Reg}(\mathfrak{N})$. Then $c \notin \mathrm{Reg}(\mathfrak{N})$ by the first part of the proof. ∎

Let us denote $\mathcal{B}, \mathcal{R}$ and $\mathcal{P}$ the systems of all Borel, real and projective classes, respectively. ($\mathcal{P}$ is the least system of classes containing all Sd-classes and closed with respect to countable unions and definitions by normal formulas of the language FL_V.) Obviously, $\mathcal{B} \subseteq \mathcal{P} \subseteq \mathcal{R}$. As shown in **[K-Z 1989b]** and **[K-Z 1989a]**, respectively,

$$\mathrm{Reg}(\mathcal{B}) = N \quad \text{and} \quad \mathrm{Reg}(\mathcal{R}) = FN.$$

In **[Č 1986]** it is proved that the equality $\mathrm{Reg}(\mathcal{P}) = N$ is consistent with the basic axioms of the Alternative Set Theory. According to an unpublished result of K. Čuda announced in **[Č 1989]**, the equality $\mathrm{Reg}(\mathcal{P}) = FN$ is consistent, too (the theory AST $+ \mathrm{Reg}(\mathcal{P}) = FN$ even is interpretable in AST). According to our last Theorem 2.11, there is no other possibility.

3. $\mathfrak{N}$-representable pairs of cuts.

If $\mathfrak{N}$ is any system of classes, then the pair of cuts $\langle A, B\rangle$ will be called $\mathfrak{N}$-representable if there is a class $X \in \mathfrak{N}$ such that $A = \underline{X}$ and $B = \overline{X}$. Theorem 3.3 in [**K-Z 1989a**] gives a full description of really representable (i.e. $\mathcal{R}$-representable) pairs of cuts. According to this description, such pairs split into ten pairwise nonoverlaping types depending on some parameters. As stated in Theorem 3.4 [**K-Z 1989b**], Borel representable (i.e. $\mathcal{B}$-representable) pairs of cuts are exactly those falling under the first two of the ten types. In the present section we are going to show that whenever $\mathfrak{N}$ is a system of real classes closed with respect to the most basic class-theoretical operations and containing besides of all sets also all the immediately following semisets in the Borel hierarchy, then, in the main, $\mathfrak{N}$-representability coincides either with the Borel or with the real one.

A system of classes $\mathfrak{N}$ will be called a normal ring of classes if $\mathfrak{N}$ is a Cartesian ring of real classes and all σ-semisets (hence all π-semisets, too) belong to $\mathfrak{N}$.

Theorem 3.1. *For every normal ring of classes $\mathfrak{N}$ the following alternative is satisfied:*
Either

for every pair of proper cuts $\langle A, B\rangle$ it holds
$\langle A, B\rangle$ is $\mathfrak{N}$-representable iff $\langle A, B\rangle$ is Borel representable

or

for every pair of proper cuts $\langle A, B\rangle$ it holds
$\langle A, B\rangle$ is $\mathfrak{N}$-representable iff $\langle A, B\rangle$ is really representable.

Remark 3.2. According to our definition, a normal ring $\mathfrak{N}$ need not contain classes which are not semisets. That is why the Borel (and the more, really) representable pair $\langle N, N\rangle$ need not be $\mathfrak{N}$-representable. This makes the restriction to proper cuts in the formulation of the Theorem necessary.

Since all $\mathfrak{N}$-representable classes obviously are really representable and, according to Theorem 5 from [**K-Z 1989b**], all Borel representable pairs of proper cuts are $\mathfrak{N}$-representable, to prove Theorem 3.1 it suffices to show that whenever there is an $X \in \mathfrak{N}$ such that the pair $\langle \underline{X}, \overline{X}\rangle$ is not Borel representable, then already all really representable pairs of proper cuts are $\mathfrak{N}$-representable.

To formulate the intermediate results in a uniform way let us introduce some notation

$$D_3(b) = D_4(b) = D_7(b) = N,$$
$$D_5(b) = D_6(b) = D_8(b) = N \setminus b/FN,$$
$$D_9(b) = D_{10}(b) = N \setminus b.FN,$$
$$\varphi_3(a, b, A, B) \equiv (A = a + b/FN \ \&\ B = a + b - b/FN),$$
$$\varphi_4(a, b, A, B) \equiv (A = a + b/FN \ \&\ B = a + b + b/FN),$$
$$\varphi_5(a, b, A, B) \equiv (A = a - b/FN \ \&\ B = a + b - b/FN),$$
$$\varphi_6(a, b, A, B) \equiv (A = a - b/FN \ \&\ B = a + b + b/FN),$$
$$\varphi_7(a, b, A, B) \equiv (A = a + b/FN \ \&\ B = a + b.FN),$$

$$\begin{aligned}\varphi_8(a,b,A,B) &\equiv (A = a - b/FN \ \&\ B = a + b.FN),\\ \varphi_9(a,b,A,B) &\equiv (A = a - b.FN \ \&\ B = a - b/FN),\\ \varphi_{10}(a,b,A,B) &\equiv (A = a - b.FN \ \&\ B = a + b/FN).\end{aligned}$$

The unusual enumeration for $i = 3,\dots,10$ corresponds to the numbers of types of really representable pairs of cuts which are not Borel representable as listed in Theorem 3.3 [**K-Z 1989a**]. Thus, e.g., the cuts of the class X are of i-th type, $3 \leq i \leq 10$ iff it holds

$$(\exists\, b \in N \setminus FN)(\exists\, a \in D_i(b))\varphi_i(a,b,\underline{X},\overline{X}).$$

Also, Theorem 3.3 from [**K-Z 1989a**] can be now restated, taking account to Theorem 3.4 from [**K-Z 1989b**], in the following form.

Theorem 3.3. *A pair of cuts $\langle A,B\rangle$ is really representable if and only if it is Borel representable, or it is not and for exactly one $i, 3 \leq i \leq 10$, it holds*

$$(\exists\, b \in N \setminus FN)(\exists\, a \in D_i(b))\varphi_i(a,b,A,B).$$

In addition, we put for each $i, 3 \leq i \leq 10$, and every system of classes $\mathfrak{M}$

$$\Phi_i(a,b,\mathfrak{M}) \equiv (\exists\, X \in \mathfrak{M})\varphi_i(a,b,\underline{X},\overline{X}).$$

In other words, $\Phi_i(a,b,\mathfrak{M})$ says that the pair of cuts of the i-th type with parameters a, b is $\mathfrak{M}$-representable.

The following observation is trivial, however, it enables us to simplify the proof as well as to increase the power of the next crucial result.

Lemma 3.4. *Let $\mathfrak{M}$ be an arbitrary ring of classes containing all sets. Then for each $i = 3,\dots,10$ and each $b \in N \setminus FN$ it holds*

$$(\forall\, a_1, a_2 \in D_i(b))\,(\Phi_i(a_1,b,\mathfrak{M}) \Longleftrightarrow \Phi_i(a_2,b,\mathfrak{M})),$$

or equivalently

$$(\exists\, a \in D_i(b))\Phi_i(a,b,\mathfrak{M}) \Longleftrightarrow (\forall\, a \in D_i(b))\Phi_i(a,b,\mathfrak{M}).$$

Proposition 3.5. *If $\mathfrak{N}$ is a normal ring of classes, then for all $i, j, 3 \leq i < j \leq 10$, and each $b \in N \setminus FN$ it holds*

$$(\exists\, a \in D_i(b))\Phi_i(a,b,\mathfrak{N}) \Longleftrightarrow (\exists\, a \in D_j(b))\Phi_j(a,b,\mathfrak{N}).$$

PROOF : We will produce two four-terms cycles which then will be connected, according to the scheme

$$\begin{tikzcd}
 & (4) \arrow[dr] & & & (8) \arrow[dl] & \\
(5) \arrow[ur] & & (3) \arrow[dl] \arrow[r, leftrightarrow] & (7) \arrow[dr] & & (10) \arrow[ul] \\
 & (6) \arrow[ul] & & & (9) \arrow[ur] &
\end{tikzcd}$$

The arrow $(i) \to (j)$ stands for the implication

$$(\exists\, a \in D_i(b))\Phi_i(a, b, \mathfrak{N}) \Longrightarrow (\exists\, a \in D_j(b))\Phi_j(a, b, \mathfrak{N})$$

or for something equivalent to it in view of 3.4.
(3) $\to$ (6) It suffices to start with an $X \in \mathfrak{N}$ such that $\underline{X} = b/FN, \overline{X} = b - b/FN$. Let $a \in D_6(b) = N \backslash b/FN$. Three are sets v, w such that $X \cap v = X \cap w = v \cap w = \emptyset$, and a σ-class $Y \subseteq v$ and a π-class $Z \subseteq w$ such that $|y| = a - b/FN, |Z| = b/FN$. Then $X \cup Y \cup Z \in \mathfrak{N}$ and, by the results of [**K-Z 1988**],

$$\underline{X \cup Y \cup Z} = a - b/FN, \overline{X \cup Y \cup Z} = a + b + b/FN.$$

(6) $\to$ (5) Let $X \in \mathfrak{N}$ satisfy $\underline{X} = a - b/FN, \overline{X} = a + b + b/FN$, where $a > b/FN$. By 1.7 from [**K-Z 1989a**], there is a σ-class Y and a π-class Z such that $Y \subseteq X \subseteq Z$ and $|Y| = a - b/FN, |Z| = a + b + b/FN$. Then one can find a set u such that $Y \subseteq u \subseteq Z, |u| = a + b$. Obviously, $X \cap u \in \mathfrak{N}$ and $\underline{X \cap u} = a - b/FN$. Since $\overline{X}^{+} = b/FN$, by Lemma 1.5 it holds $a + b - b/FN = a + \mathrm{int}(b) \leq \overline{X \cap u} \leq a + b$. Since $X \cap u$ is a real class, $\overline{X \cap u} = a + b - b/FN$ by 3.3.
(5) $\to$ (4) Let $X \in \mathfrak{N}, \underline{X} = a - b/FN, \overline{X} = a + b - b/FN$ where $a > b/FN$. There is a set v such that $X \subseteq v$ and $|v| = a + b$. Then $v \backslash X \in \mathfrak{N}$ and $\underline{v \setminus X} = b/FN, \overline{v \setminus X} = b + b/FN$ by 3.1.7 from [**K-Z 1988**].
(4) $\to$ (3) Let $X \in \mathfrak{N}$ be such that $\underline{X} = b/FN, \overline{X} = b + b/FN$. Then, by 1.7 from [**K-Z 1989a**], there is a π-class Z such that $X \subseteq Z$ and $|Z| = b + b/FN$. Let $u \subseteq Z$ be any set such that $|u| = b$. By Lemma 1.5 it holds $b - b/FN \leq \overline{X \cap u} \leq b$. Obviously, $\underline{X \cap u} \leq b/FN$ and $X \cap u \in \mathfrak{N}$. Then from 3.3 it follows that

$$\underline{X \cap u} = b/FN \quad \text{and} \quad \overline{X \cap u} = b - b/FN.$$

(7) $\to$ (9) Let $X \in \mathfrak{N}$ satisfy $\underline{X} = a + b/FN, \overline{X} = a + b.FN$ where $a > b.FN$. Then there is a set v such that $X \subseteq v$ and $|v| = 2a$. Obviously, $v \setminus X \in \mathfrak{N}$ and, by 3.1.7 from [**K-Z 1988**],

$$\underline{v \setminus X} = a - b.FN, \overline{v \setminus X} = a - b.FN$$

(9) $\to$ (10) Let $X \in \mathfrak{N}$ be such that $\underline{X} = a - b.FN, \overline{X} = a - b/FN$ where $a > b.FN$. There is a set v and a π-class Y such that $X \subseteq v, Y \cap v = \emptyset$ and $|Y| = b/FN$. Then $X \cup Y \in \mathfrak{N}$ and, by the results of [**K-Z 1988**],

$$\underline{X \cup Y} = a - b.FN, \overline{X \cup Y} = a + b/FN.$$

(10) $\to$ (8) Let $X \in \mathfrak{N}, \underline{X} = a - b.FN, \overline{X} = a + b/FN$ where $a > b.FN$. Let v be a set such that $X \subseteq v$ and $v = 2a$. Then $v \backslash X \in \mathfrak{N}$ and $\underline{v \backslash X} = a - b/FN, \overline{v \setminus X} = a + b.FN$ by 3.1.7 from [**K-Z 1988**].
(8) $\to$ (7) Let $X \in \mathfrak{N}, \underline{X} = a - b/FN, \overline{X} = a + b.FN$ where $a > b/FN$. By 1.7 and 1.6 from [**K-Z 1989a**] there is a σ-class Y such that $Y \subseteq X, |Y| = a - b/FN$,

and a π-class Z such that $Y \subseteq Z, |Z| = a + b/FN$. Then $X \cup Z = X \cup (Z \setminus Y) \in \mathfrak{N}$. Lemma 3.1.7 from **[K-Z 1988]** yields the estimation

$$\overline{Z \setminus Y} \leq |Z| \dot{-} Y = (a + b/FN) \dot{-} (a - b/FN) = b/FN.$$

Now, let us compute using Lemma 1.4 in the first and 3.1.6 from **[K-Z1988]** in the second case

$$\begin{aligned} a + b/FN = |Z| \leq \underline{X \cup (Z \setminus Y)} \leq \underline{X} \dot{+} \overline{Z \setminus Y} \leq (a - b/FN) \dot{+} b/FN = \\ = a + b/FN, \\ a + b.FN = \overline{X} \leq \overline{X \cup (Z \setminus Y)} \leq \overline{X} \dot{+} \overline{Z \setminus Y} \leq (a + b/FN) \dot{+} b/FN = \\ = a + b.FN. \end{aligned}$$

Hence $\underline{X \cup Z} = a + b/FN, \overline{X \cup Z} = a + b.FN$.
(3) $\to$ (7) Let $X \in \mathfrak{N}$ be such that $\underline{X} = b/FN, \overline{X} = b - b/FN$. Then obviously $X \times FN \in \mathfrak{N}$ and

$$\underline{X \times FN} = b/FN, \ \overline{X \times FN} = b.FN.$$

(7) $\to$ (3) Again, it suffices to start with an $X \in \mathfrak{N}$ satisfying $\underline{X} = b/FN, \overline{X} = b.FN$. Obviously, $X \notin \mathfrak{M}_b$, hence by Lemma 1.3 there is a set u such that $b/FN < |u| < b.FN$ and $\neg M(b, X \cap u)$. Then $\underline{X \cap u} \leq b/FN$ and $b/FN < \overline{X \cap u} < b.FN$, hence by 3.3 there is a $c \in b.FN \setminus b/FN$ such that $\underline{X \cap u} = a/FN = b/FN$, and either $\overline{X \cap u} = c - c/FN$ or $\overline{X \cap u} = c + c/FN$. Let $k \in FN$ be such that $2b \leq kc$. We put $Y = k \times (X \cap u)$. Then $\underline{Y} = b/FN$, and either $\overline{Y} = kc - kc/FN$ or $\overline{Y} = kc + kc/FN$. By 1.6 and 1.7 from **[K-Z 1989a]** there is a π-class Z such that $Y \subseteq Z$ and $|Z| = kc + kc/FN$. Let $v \subseteq Z$ be any set satisfying $|v| = b$. Then $\overline{Y}^+ = b/FN < |v|, |Z| = \overline{Y} \dot{+} \overline{Y}^+ = kc + kc/FN$, hence, by Lemma 1.5 $b - b/FN \leq \overline{Y \cap v} \leq b$. Since $Y \cap v$ is a real class and $Y \cap v \leq b/FN$, it follows $\underline{Y \cap v} = b/FN, \overline{Y \cap v} = b - b/FN$ by 3.3. Also $Y \cap v \in \mathfrak{N}$ is trivial. ■

Now, Theorem 3.1 is an immediate consequence of the following result.

Theorem 3.6. *Let $\mathfrak{N}$ be a normal ring of classes. Then for each $i = 3, \dots, 10$ it holds*

$$\begin{aligned} \mathrm{Reg}(\mathfrak{N}) &= \{b; \neg(\exists\, a \in D_i(b))\, \Phi_i(a, b, \mathfrak{N})\} \\ &= \{b; (\exists\, a \in D_i(b))\, \neg\Phi_i(a, b, \mathfrak{N})\}. \end{aligned}$$

PROOF : According to 3.4, for given i, both the expressions on the right coincide. Owing to 3.5 they determine the same class for all possible values of i. Thus the desired results follow from 2.10 and the following obvious inclusions

$$\bigcap_{i=3}^{10} \{b; \neg(\exists\, a \in D_i(b))\, \Phi_i(a, b, \mathfrak{N})\} \subseteq \mathrm{Reg}(\mathfrak{N}) \subseteq \{b; \neg\Phi_3(0, b, \mathfrak{N})\}.$$

■

A partial case of this Theorem is the following supplement to Proposition 2.10.

Corollary 3.7. *If $\mathfrak{N}$ is a normal ring of classes, then*

$$\mathrm{Reg}(\mathfrak{N}) = \{b; (\forall X)(\overline{X} = b.FN \Longrightarrow X \in \mathfrak{M}_b)\}.$$

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