Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 4, 685--691

Persistent URL: http://dml.cz/dmlcz/106903

Comment.Math.Univ.Carolinae 31,4 (1990)685–691

On the Hammerstein integral equations in Banach spaces

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Abstract. This paper contains some existence theorems for L_r^p -solutions of nonlinear integral equations in Banach spaces. In our assumptions and proofs we employ measures of noncompactness.

Keywords: Integral equations, measures of noncompactness

Classification: 45N05

1. Introduction.

Assume that E, F are Banach spaces and D is a compact domain in the Euclidean space R^ν. Denote by $L_r^p = L_r^p(D,E)$ $(p > 1)$ the space of all strongly measurable functions $u : D \to E$ with $\int_D r(t)\|u(t)\|^p\,dt < \infty$, provided with the norm $\|u\|_{p,r} = (\int_D r(t)\|u(t)\|^p\,dt)^{1/p}$, where $r : D \to R$ is a nonnegative, bounded and integrable function such that $\operatorname{mes}\{t \in D : r(t) = 0\} = 0$.

In this paper we give sufficient conditions for the existence of a solution $x \in L_r^p$ of the integral equation

$$x(t) = g(t) + \lambda \int_D r(s)K(t,s)f(s,x(s))\,ds \tag{1}$$

or

$$x(t) = g(t) + \int_0^t r(s)K(t,s)f(s,x(s))\,ds. \tag{2}$$

Our results extend some theorems from the papers [8], [9], concerning L^p-solutions.

Throughout this paper we shall assume that:

1° p, q are real numbers such that $p, q > 1$ and $p \geq \min(q,2)$; let $l = \frac{q}{q-1}, m = \max(p,l)$ and let k be a number such that $1 < k \leq \infty$ and $\frac{1}{k} + \frac{1}{m} + \frac{1}{p} = 1$;

2° $g \in L_r^p$;

3° $(s,x) \to f(s,x)$ is a function from $D \times E$ into F such that
(i) f is strongly measurable in s and continuous in x;
(ii) $\|f(s,x)\| \leq a(s) + b\|x\|^{p/q}$ for $s \in D$ and $x \in E$, where $a \in L_r^q(D,R)$ and $b \geq 0$;

4° K is a strongly measurable function from $D \times D$ into the space of continuous linear mappings $F \to E$ and

$$\iint_{D\times D} r(s)\,r(t)\|K(t,s)\|^m\,ds\,dt < \infty.$$

Denote by α and α_p the Kuratowski measures of noncompactness in E and L^p, respectively. For any set V of functions from D into E denote by v the function defined by $v(t) = \alpha(V(t))$ for $t \in D$ (under the convention that $\alpha(A) = \infty$ if A is unbounded), where $V(t) = \{x(t) : x \in V\}$.

Without loss of generality we shall always assume that all functions from $L^1(D, E)$ or $L^1_r(D, E)$ are extended to R^ν by putting $u(t) = 0$ outside D.

Before passing to further considerations we shall quote two lemmas.

Lemma 1 (Heinz [5]). *Let V be a countable set of strongly measurable functions $D \to E$ such that there exists $M \in L^1(D, R)$ such that $\|x(t)\| \leq M(t)$ for all $x \in V$ and $t \in D$. Then the corresponding function v is integrable and $\alpha(\{\int_D x(t)\,dt : x \in V\}) \leq 2\int_D v(t)\,dt$.*

Lemma 2 (Szufla [8]). *Let V be a countable set of strongly measurable functions $D \to E$ such that*

(i) *there exists $M \in L^p(D, R)$ such that $\|x(t)\| \leq M(t)$ for all $x \in V$ and $t \in D$;*
(ii) $\lim_{h\to 0} \sup_{x\in V} \int_D \|x(t+h) - x(t)\|^p\,dt = 0$.

Then

$$\alpha_p(v) \leq 2\left(\int_D v^p(t)\,dt\right)^{1/p}$$

Let us recall that in the last twenty years the measure of noncompactness has been employed for differential and integral equations by many authors (see [1], [3], [4], [6], [8], [9]).

2. The existence of L^p_r-solutions.

Theorem 1. *Let h be a nonnegative function belonging to $L^k_r(D, R)$. If*

$$\alpha(f(t, X)) \leq h(t)\,\alpha(X) \quad \text{for } t \in D \tag{3}$$

and for each bounded subset X of E, then there exists a positive number ϱ such that for any $\lambda \in R$ with $|\lambda| < \varrho$, the equation (1) has at least one solution $x \in L^p_r$.

Proof : For simplicity put

$$Q = \left(\int_D r(t)\left(\int_D r(s)\|K(t,s)\|^l\,ds\right)^{p/l} dt\right)^{1/p},$$

$$S = \left(\int_D r(t)\left(\int_D r(s)\|K(t,s)\|^m\,ds\right)^{p/m} dt\right)^{1/p}$$

and $k(t) = \|K(t,\cdot)\|_{l,r}$. It follows from 1°and 4°that $Q, S < \infty$ and $k \in L^p_r$.

Choose a positive number ϱ such that

$$\varrho \leq \min\left(\sup_{\eta>0} \frac{\eta - \|g\|_{p,r}}{Q(\|a\|_{q,r} + b\eta^{p/q})}, \frac{1}{2S\|h\|_{k,r}}\right).$$

Fix $\lambda \in R$ with $|\lambda| < \varrho$ and choose $c > 0$ satisfying the inequality $\|g\|_{p,r} + |\lambda| Q(\|a\|_{q,r} + bc^{p/q}) \leq c$. Let $B = \{x \in L_r^p : \|x\|_{p,r} \leq c\}$. We define a mapping F by

$$F(x)(t) = g(t) + \lambda \int_D r(s) K(t,s) f(s, x(s)) \, ds \tag{4}$$

for $x \in B, t \in D$.

By the Fubini theorem for vector functions, the Hölder inequality for L_r^p spaces and 1°– 4°, for any $x \in B$ the function $F(x)$ is strongly measurable on D and

$$\|F(x)(t)\| \leq M(t) \quad \text{for } t \in D, \tag{5}$$

where $M(t) = \|g(t)\| + |\lambda| k(t) (\|a\|_{q,r} + bc^{p/q})$. It is clear that F is a mapping $B \to B$. Using the standard argument it can be shown that 1°– 4°imply the continuity of F.

Furthermore, from the conditions 1°– 4°and the Hölder inequality it follows that

$$\|r(t+h) F(x)(t+h) - r(t) F(x)(t)\| \leq d(t,h) \quad \text{for all } x \in B,$$

where

$$d(t,h) = \begin{cases} r(t) M(t) & \text{if } t \in D \text{ and } t+h \notin D \\ \|r(t+h) g(t+h) - r(t) g(t)\| + & \\ + |\lambda| (\|a\|_{q,r} + bc^{p/q}) (\int_D r(s) \|r(t+h) K(t+h,s) - & \\ - r(t) K(t,s)\|^l \, ds)^{1/l} & \text{if } t, t+h \in D. \end{cases}$$

Let $r(t) \leq R$ for $t \in D$. Since

$$\lim_{h \to 0} \int_D \|r(t+h) K(t+h,s) - r(t) K(t,s)\|^m \, dt = 0 \text{ for a.e. } s \in D,$$

$$\lim_{h \to 0} \int_D \Big[\int_D r(s) \|r(t+h) K(t+h,s) - r(t) K(t,s)\|^l \, ds \Big]^{m/l} dt \leq$$
$$\leq \lim_{h \to 0} \int_D \Big[(\operatorname{mes} D)^{\frac{1}{l} - \frac{1}{m}} \int_D r(s) \|r(t+h) K(t+h,s) -$$
$$- r(t) K(t,s)\|^m \, ds \Big] dt =$$
$$= (\operatorname{mes} D)^{\frac{1}{l} - \frac{1}{m}} \lim_{h \to 0} \int_D r(s) \Big[\int_D \|r(t+h) K(t+h,s) -$$
$$- r(t) K(t,s)\|^m \, dt \Big] ds = 0$$

and $p \leq m$, we see that

$$\lim_{h \to 0} \int_D d^p(t,h) \, dt = 0. \tag{6}$$

Moreover

$$\iint\limits_{D\times\Omega_\eta} d^p(t,s)\,dt\,ds \tag{7}$$

where Ω_η means the closed ball in R^ν with center 0 and radius η. This implies that

$$\lim_{h\to 0}\sup_{x\in B}\int_D \|r(t+h)\,F(x)\,(t+h) - r(t)\,F(x)\,(t)\|^p\,dt = 0. \tag{8}$$

Let V be a countable subset of B such that

$$V \subset \overline{\mathrm{conv}}(F(V)\cup\{0\}). \tag{9}$$

Then $rV \subset L^p$ and by (5)

$$\|x(t)\| \leq M(t) \quad \text{for } x\in V \tag{10}$$

and for a.e. $t\in D$. Hence $\|f(s,x(s))\| \leq \eta(s) = a(s) + bM^{p/q}(s)$ for $x\in V, s\in D$ and $\eta \in L_r^q(D,R)$. By the Hölder inequality, from this we deduce that for fixed $t\in D$ the function $s \to \|K(t,s)\|\,\eta(s)$ belongs to $L_r^1(D,E)$. Moreover, from (10) and Lemma 1 it follows that the function $t\to\alpha(rV(t))$ is measurable on D and $\alpha(V(t)) \leq 2M(t)$ for a.e. $t\in D$.

Therefore, by Lemma 1 and (3), we obtain

$$\begin{aligned} v(t) &\leq \alpha(F(V)\,(t)) \leq \alpha(\{\lambda\int_D r(s)\,K(t,s)\,f(s,x(s))\,ds : x\in V\}) \leq \\ &\leq 2|\lambda|\int_D \alpha(\{r(s)\,K(t,s)\,f(s,x(s)) : x\in V\})\,ds \leq \\ &\leq 2|\lambda|\int_D r(s)\|K(t,s)\|\,h(s)\,v(s)\,ds \quad \text{for a.e. } t\in D. \end{aligned}$$

Consequently, by the Hölder inequality, we have

$$v(t) \leq 2|\lambda|\,\|h\|_{k,r}\|v\|_{p,r}(\int_D r(s)\,\|K(t,s)\|^m\,ds)^{1/m}$$

As the above inequality holds for a.e $t\in D$, we get

$$\|v\|_{p,r} \leq 2|\lambda|\,S\|h\|_{k,r}\|v\|_{p,r}\,.$$

Since $|\lambda|2S\|h\|_{k,r} < 1$, from this we infer that $\|v\|_{p,r} = 0$. It follows from (8), (9 and Lemma 2 that $\alpha_p(rV) \leq 2\|rv\|_p \leq 2R^{1-\frac{1}{p}}\|v\|_{p,r} = 0$, i.e. rV is relatively compact in L^p.

We shall show that V is relatively compact in L_r^p. Let $u_n\in V$ for $n=1,2,\ldots$. Then there exists a subsequence (u_{n_k}) of (u_n) such that $\lim_{k\to\infty}\|ru_{n_k} - u_0\|_p = 0$

for some $u_0 \in L^p$. Put $u(t) = u_0(t)/r(t)$ for $t \in D$ such that $r(t) \neq 0$. Without loss of generality (by passing to a subsequence if necessary) we may assume that $\lim_{k\to\infty}(r(t)u_{n_k}(t) - r(t)\,u(t)) = 0$ for a.e. $t \in D$. From (10) it is clear that $\|u_{n_k}(t) - u(t)\| \leq 2M(t)$ for a.e. $t \in D$. Thus, by Lebesgue theorem,

$$\lim_{k\to\infty} \|u_{n_k} - u\|_{p,r}^p = \lim_{k\to\infty} \int_D r(t)\|u_{n_k}(t) - u(t)\|^p\,dt =$$
$$= \int_D r(t) \lim_{k\to\infty} \|u_{n_k}(t) - u(t)\|^p\,dt = 0.$$

Hence, V is relatively compact in L_r^p. Applying now the Mönch fixed point theorem [7, Theorem 2.2]we conclude that there exists $x \in B$ such that $x = F(x)$. Obviously x is a solution of (1). ■

3. The Volterra–Hammerstein integral equation.

Consider now the equation (2), with $D = [0, d]$. Choose $\eta \in (0, \frac{1}{2})$ and an interval $J = [0, a] \subset D$ in such a way that for each ε, $0 \leq \varepsilon \leq \eta$, the maximal continuous solution z_ε of the integral equation

$$z(t) = \varepsilon + 2^{p-1} \int_0^t r(s)\,(\|g(s)\| + k(s)\,\|a\|_{q,r} + bk(s)z^{1/q}(s))^p\,ds$$

is defined on J and $z_\varepsilon(t) \leq z_0(t) + 1$ for $t \in J$. Let $c = \max_{t\in J}(z_0(t) + 1)^{1/p}$, $L_r^p = L_r^p(J, E)$, $B = \{x \in L_r^p : \|x\|_{p,r} \leq c\}$ and $U = \{x \in L_r^p : \|x\|_{p,r} \leq \eta\}$. Put $F(x)\,(t) = g(t) + \int_0^t r(s)\,K(t,s)\,f(s, x(s))\,ds$ for $x \in L_r^p, t \in J$. Then $\|F(x)\,(t)\| \leq M(t)$ and $\|r(t+h)\,F(x)\,(t+h) - r(t)\,F(x)\,(t)\| \leq d(t, h)$ for $x \in B, t \in J$, where d satisfies (6) and (7). Moreover F is a continuous mapping $L_r^p \to L_r^p$.

Theorem 2. *Let h be a nonnegative function belonging to $L_r^k(D, R)$. If $1°$– $4°$ hold and $\alpha(f(t, X)) \leq h(t)\alpha(X)$ for $t \in D$ and for each bounded subset X of E, then the equation (2) has at least one solution $x \in L_r^p$.*

PROOF : For any positive integer n we define a function $u_n : J \to E$ by

$$u_n(t) = \begin{cases} g(t) & \text{for } 0 \leq t \leq a_n \\ g(t) + \int_0^{t-a_n} r(s)\,K(t,s)\,f(s, u_n(s))\,ds & \text{for } a_n < t \leq a\,, \end{cases}$$

where $a_n = a/n$. From the Hölder inequality and $3°$ it follows that

$$\|u_n(t)\| \leq \|g(t)\| + k(t)\|a\|_{q,r} + bk(t)\,(\int_0^t r(s)\|u_n(s)\|^p\,ds)^{1/q}$$

and

$$\begin{aligned} &\|u_n(t) - g(t) - \int_0^t r(s)\,K(t,s)\,f(s, u_n(s))\,ds\| \leq \\ &\leq k_n(t)\,(\|a\|_{q,r} + b\,(\int_0^t r(s)\,\|u_n(s)\|^p\,ds)^{1/q}) \quad \text{for } t \in J, \end{aligned} \tag{11}$$

where

$$k_n(t) = \begin{cases} k(t) & \text{for } 0 \le t \le a_n \\ \|K(t,\cdot)\chi_{[t-a_n,t]}\|_{l,r} & \text{for } a_n < t \le a. \end{cases}$$

Putting $w_n(t) = \int_0^t r(s)\,\|u_n(s)\|^p\,ds$ we see that

$$w_n(t) \le \int_0^t r(s)\,(\|g(s)\| + k(s)\|a\|_{q,r} + bk(s)w_n^{1/q}(s))^p\,ds.$$

By the theorem on integral inequalities this implies $w_n(t) \le z_0(t)+1 \le c^p$ for $t \in J$. Hence $u_n \in B$ and $\|u_n(t)\| \le \|g(t)\| + k(t)\,(\|a\|_{q,r} + bc^{p/q}) = M(t)$. Moreover $\lim_{n\to\infty} k_n(t) = 0$ and $k_n(t) \le k(t)$ for a.e. $t \in J$. By (11), $\lim_{n\to\infty}(u_n(t) - F(u_n)(t)) = 0$ for a.e. $t \in J$ and $\lim_{n\to\infty} \|u_n - F(u_n)\|_{p,r} = 0$. Arguing similarly as in the proof of Theorem 1, we can show that the set $\{u_n : n = 1, 2, \dots\}$ is relatively compact in L^p_r. Thus we can find a subsequence (u_{n_k}) of (u_n) which converges in L^p_r to a limit u. Consequently

$$\|u - F(u)\|_{p,r} = \lim_{k\to\infty} \|u_{n_k} - F(u_{n_k})\|_{p,r} = 0,$$

which proves that u is a solution of (2). ■

Combining the proofs of Theorem 2 and Theorem from [**9**], we can prove the following Aronszajn-type

Theorem 3. *Under the assumptions of Theorem 2, the set S of all solutions $x \in L^p_r$ of (2) is a compact R_δ, i.e. S is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.*

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(*Received* May 2, 1990)