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On entire solutions of elliptic equations with a singular nonlinearity

J. CHABROWSKI, M. KÖNIG

Abstract. The paper deals with positive solutions in R_n of the equation $Lu = f(x)u^{-\gamma}$, where L is a uniformly elliptic operator of second order, f is a positive function and $0 < \gamma < \infty$.

Keywords: Positive weak solution, uniformly elliptic, singular nonlinearity

Classification: 35J15, 35B99

1. Introduction.

In this paper we are concerned with the solvability in R_n of the problem

$$(P)\begin{cases} Lu = -\sum_{i=1}^{n} D_i(a_{ij}(x)D_j u) + c(x)u = f(x)u^{-\gamma} \text{ in } R_n \\ u(x) > 0 \text{ on } R_n, \end{cases}$$

where $0 < \gamma < \infty$ is a constant and $n \geq 3$. Problems of this nature arise in the boundary layer theory of viscous fluids (see [1], [2] and [10]). The singular equation appearing in the problem (P) is called the Lane–Emden–Fowler equation. This problem has been recently studied by Edelson [4], Kusano and Swanson [7] in the case $L = \Delta$ and $0 < \gamma < 1$. Under a suitable decay condition on f, they proved the existence of a positive solution in $C^2(R_n)$ using the Schauder fixed point theorem. The article [6] contains some extensions of this result to the exterior Dirichlet problem. The main purpose of this paper is to investigate the existence of weak solutions. Our approach, based mainly on the Sobolev imbedding theorem and some approximation argument, allows us to cover a wider range for the parameter γ. We distinguish two cases: $0 < \gamma \leq 1$ and $1 < \gamma < \infty$. In the case $0 < \gamma \leq 1$ we first solve the Dirichlet problem in a bounded domain with zero boundary data. A solution to the problem (P) is then obtained as a limit of solutions u_m of the Dirichlet problems on Ω_m, with Ω_m exhausting R_n. In the case $1 < \gamma < \infty$ we were unable to solve the Dirichlet problem; we can only prove the existence of local solutions. However, this is sufficient to apply the approximation argument from the previous case $0 < \gamma \leq 1$. In both cases the solution u belongs to $W^{1,2}_{loc}(R_n)$ with $Du \in L^2(R_n)$ and $u \in L^{\frac{2n}{n-2}}(R_n)$ in the case $0 < \gamma \leq 1$, and $u \in L^{\frac{n(\gamma+1)}{n-2}}(R_n)$ and $Du^{\frac{\gamma+1}{2}} \in L^2(R_n)$ in the case $1 < \gamma < \infty$, and obviously u satisfies the equation in the distributional sense. In the final Sections 4 and 5 we briefly discuss the existence of positive solutions with exponential decay, under the additional assumption that $c(x) \geq c_0 > 0$ on R_n for some constant c_0. In particular, in Section 5 we derive pointwise estimate for solutions of the problem (P) with smooth coefficients. We use here a very simple argument based on a classical maximum principle.

2. Case $0 < \gamma \leq 1$.

Throughout this paper we assume that

(A) L is uniformly elliptic in R_n, that is

$$\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2$$

for all $\xi \in R_n$ and $x \in R_n$ and some constant $\lambda > 0$.

(B) The coefficients $a_{ij}(i,j = 1,\ldots,n)$ and c are in $L^\infty(R_n)$, with $c(x) \geq 0$ on R_n.

The assumption on f will be specified later.

We need the following result on the solvability of the Dirichlet problem

$$Lu = f(x)u^{-\gamma} \text{ in } \Omega, \tag{1}$$

$$u(x) = 0 \text{ on } \partial\Omega, \tag{2}$$

in a bounded domain $\Omega \subset R_n$.

Lemma 1. *Let $f \in L^2(\Omega)$, with $f(x) > 0$ on Ω. Then the Dirichlet problem* (1), (2) *admits a unique positive solution $u \in \overset{\circ}{W}{}^{1,2}(\Omega)$.*

PROOF : Uniqueness can be obtained by a straightforward argument: let u_1 and u_2 be two solutions in $\overset{\circ}{W}{}^{1,2}(\Omega)$ of the problem (1), (2). It follows from Lemma 1.2 in [8] that $(u_1-u_2)^+ \in \overset{\circ}{W}{}^{1,2}(\Omega)$. Consequently, taking $(u_1-u_2)^+$ as a test function, we obtain

$$\int_\Omega \Big[\sum_{i,j=1}^{n} a_{ij}D_i(u_1-u_2)^+D_j(u_1-u_2)^+ + c(u_1-u_2)^+(u_1-u_2)^+\Big]\,dx =$$
$$= \int_\Omega f(u_1^{-\gamma} - u_2^{-\gamma})(u_1-u_2)^+\,dx.$$

Since $u_1^{-\gamma} < u_2^{-\gamma}$ on $\{x \in \Omega; u_1(x) > u_2(x)\}$, the right hand side of this identity is nonpositive. Therefore (A) yields

$$\int_\Omega |D(u_1-u_2)^+|^2\,dx \leq 0$$

and consequently $(u_1-u_2)^+ = 0$ a.e. on Ω, that is $u_1(x) \leq u_2(x)$ a.e. on Ω. Changing roles of u_1 and u_2 we get $u_2(x) \leq u_1(x)$ a.e. on Ω and the uniqueness follows. In the proof of the existence of the solution we use some ideas from the paper [3]. To establish the existence of the solution we consider for every $\varepsilon > 0$ the Dirichlet problem for the equation

$$Lu = f(x)\frac{1}{(\varepsilon+|u|)^\gamma} \text{ in } \Omega, \tag{3}$$

with zero boundary condition (2). We now observe that $f(x)\frac{1}{(\varepsilon+|u|)^\gamma} \le \frac{1}{\varepsilon}f(x)$ on Ω for all u. Therefore a standard application of compact imbedding of $\overset{\circ}{W}{}^{1,2}(\Omega)$ in $L^2(\Omega)$ and the Schauder fixed point theorem give the existence of a solution $u_\varepsilon \in \overset{\circ}{W}{}^{1,2}(\Omega)$ of the problem (3), (2), which by the maximum principle is positive a.e. on Ω. Since any solution of (3), (2) in $\overset{\circ}{W}{}^{1,2}(\Omega)$ must be positive, we show as in the previous step, that the solution u_ε is unique. We now check that the sequence $\{u_\varepsilon, \varepsilon > 0\}$ has the following properties: (i) $\{u_\varepsilon\}$ is increasing as $\varepsilon \searrow 0$, (ii) $\{u_\varepsilon + \varepsilon\}$ is decreasing as $\varepsilon \searrow 0$ and (iii) $\{u_\varepsilon\}$ is bounded in $W^{1,2}(\Omega)$.

To establish (i) we take $(u_{\varepsilon_1} - u_{\varepsilon_2})^+$, with $\varepsilon_1 > \varepsilon_2$, as a test function and we obtain

$$\int_\Omega \Big[\sum_{i,j=1}^n a_{ij} D_i(u_{\varepsilon_1} - u_{\varepsilon_2})^+ D_j(u_{\varepsilon_1} - u_{\varepsilon_2})^+ + c((u_{\varepsilon_1} - u_{\varepsilon_2})^+)^2 \Big]\, dx =$$

$$= \int_\Omega f \frac{(\varepsilon_2 + u_{\varepsilon_2})^\gamma - (\varepsilon_1 + u_{\varepsilon_1})^\gamma}{(\varepsilon_1 + u_{\varepsilon_1})^\gamma (\varepsilon_2 + u_{\varepsilon_2})^\gamma} (u_{\varepsilon_1} - u_{\varepsilon_2})^+\, dx \le 0$$

and consequently

$$\int_\Omega |D(u_{\varepsilon_1} - u_{\varepsilon_2})^+|^2\, dx \le 0,$$

that is, $u_{\varepsilon_1} - u_{\varepsilon_2} \le 0$ a.e. on Ω.

We now show that $\{u_\varepsilon + \varepsilon\}$ is decreasing as $\varepsilon \searrow 0$. Let $\varepsilon_1 > \varepsilon_2$ and since $u_{\varepsilon_1} - u_{\varepsilon_2} = 0$ on $\partial\Omega$, $(u_{\varepsilon_1} + \varepsilon_1 - u_{\varepsilon_2} - \varepsilon_2)^- \in \overset{\circ}{W}{}^{1,2}(\Omega)$ and on substitution we obtain

$$- \int_\Omega \Big[\sum_{i,j=1}^n a_{ij} D_i(u_{\varepsilon_1} + \varepsilon_1 - u_{\varepsilon_2} - \varepsilon_2)^- D_j(u_{\varepsilon_1} + \varepsilon_1 - u_{\varepsilon_2} - \varepsilon_2)^- +$$

$$+ c((u_{\varepsilon_1} + \varepsilon_1 - u_{\varepsilon_2} - \varepsilon_2)^-)^2 \Big]\, dx =$$

$$= \int_\Omega f \frac{(u_{\varepsilon_2} + \varepsilon_2)^\gamma - (u_{\varepsilon_1} + \varepsilon_1)^\gamma}{(u_{\varepsilon_1} + \varepsilon_1)^\gamma (u_{\varepsilon_2} + \varepsilon_2)^\gamma} (u_{\varepsilon_1} + \varepsilon_1 - u_{\varepsilon_2} - \varepsilon_2)^-\, dx +$$

$$+ \int_\Omega c(\varepsilon_1 - \varepsilon_2)(u_{\varepsilon_1} + \varepsilon_1 - u_{\varepsilon_2} - \varepsilon_2)^-\, dx.$$

It is easy to see that the right hand side is nonnegative, as before, we conclude that $|D(u_{\varepsilon_1} + \varepsilon_1 - u_{\varepsilon_2} - \varepsilon_2)^-| = 0$ a.e. on Ω, that is, $u_{\varepsilon_1} + \varepsilon_1 \ge u_{\varepsilon_2} + \varepsilon_2$ a.e. on Ω.

Finally, taking u_ε as a test function and applying the Hölder inequality we obtain, in the case $0 < \gamma < 1$,

$$\lambda^{-1} \int_\Omega |Du_\varepsilon|^2\, dx \le \int_\Omega f u_\varepsilon^{1-\gamma}\, dx \le C(\eta, \gamma) \int_\Omega f^{\frac{2}{1+\gamma}}\, dx + \frac{1}{2}(1-\gamma)\eta \int_\Omega u_\varepsilon^2\, dx$$

for each $\eta > 0$, where the constant $C(\eta, \gamma) > 0$ is independent of ε. On the other hand, by Poincaré's inequality we have

$$\int_\Omega u_\varepsilon^2\, dx \le P \int_\Omega |Du_\varepsilon(x)|^2\, dx$$

for some constant $P > 0$ independent of ε. Hence choosing η so that $\frac{1}{2}(1-\gamma)\eta P < \lambda^{-1}$ we obtain

$$\int_\Omega |Du_\varepsilon(x)|^2\,dx \le C\int_\Omega f(x)^{\frac{2}{1+\gamma}}\,dx,$$

where $C > 0$ is independent of ε. In the case $\gamma = 1$ we obtain

$$\lambda^{-1}\int_\Omega |Du_\varepsilon(x)|^2\,dx \le \int_\Omega f(x)\,dx$$

and this completes the proof of the claim (iii). By Sobolev's imbedding theorem there exists a decreasing sequence $\varepsilon_m \searrow 0$, as $m \to \infty$, such that $u_{\varepsilon_m} \to u$ weakly in $W^{1,2}(\Omega)$, strongly in $L^2(\Omega)$ and a.e. on Ω. To complete the proof we show that u satisfies (1). For every $v \in \overset{\circ}{W}{}^{1,2}(\Omega)$ we have

$$\int_\Omega\left[\sum_{i,j=1}^n a_{ij}D_iu_{\varepsilon_m}D_jv + cu_{\varepsilon_m}v\right]dx = \int_\Omega fv\frac{1}{(\varepsilon_m + u_{\varepsilon_m})^\gamma}\,dx.$$

The left hand side converges to

$$\int_\Omega\left[\sum_{i,j=1}^n a_{ij}D_iuD_jv + cuv\right]dx < \infty\,.$$

On the other hand by the Monotone Convergence Theorem (we may assume that $v \ge 0$ on Ω) we have

$$\int_\Omega vf\frac{1}{u^\gamma}\,dx = \int_\Omega vf\lim_{m\to\infty}\frac{1}{\varepsilon_m + u_{\varepsilon_m}^\gamma}\,dx = \int_\Omega\left[\sum_{i,j=1}^n a_{ij}D_iuD_jv + cuv\right]dx < \infty$$

and this completes the proof. ∎

We point out here that (i) and (ii) imply that

$$0 < u_\varepsilon - u_\delta < \delta - \varepsilon \text{ a.e. on } \Omega \tag{4}$$

for $\varepsilon < \delta$. Consequently, u_ε converges uniformly to u on Ω.

We are now in a position to establish the existence result for the problem (P).

Theorem 1. *Suppose that $f \in L^2_{loc}(R_n) \cap L^{\frac{2n}{n+2+\gamma(n-2)}}(R_n)$ with $0 < \gamma \le 1$ and $f(x) > 0$ on R_n. Then the problem (P) has a positive solution $u \in W^{1,2}_{loc}(R_n)$ with $Du \in L^2(R_n)$ and $u \in L^{\frac{2n}{n-2}}(R_n)$.*

PROOF : Let Ω_m be an increasing sequence of bounded domains with smooth boundaries and such that $R_n = \bigcup_{m\ge1}\Omega_m$. By Lemma 1, the Dirichlet problem for the equation

$$Lu = fu^{-\gamma} \text{ in } \Omega_m$$

with zero boundary data on $\partial\Omega_m$ has a unique positive solution $u \in \overset{\circ}{W}{}^{1,2}(\Omega_m)$. We extend u_m by 0 outside Ω_m. The resulting function is in $W^{1,2}(R_n)$. Taking u_m as a test function and using the Sobolev inequality we obtain, in the case $0 < \gamma < 1$,

$$(5)\quad \lambda^{-1}C(n)\Big(\int_{\Omega_m} u_m^{\frac{2n}{n-2}}\,dx\Big)^{\frac{n-2}{n}} \le \lambda^{-1}\int_{\Omega_m} |Du_m|^2\,dx \le \int_{\Omega_m} f u_m^{1-\gamma}\,dx \le$$
$$\le \Big(\int_{\Omega_m} u_m^{\frac{2n}{n-2}}\,dx\Big)^{\frac{(1-\gamma)(n-2)}{2n}} \Big(\int_{\Omega_m} f^{\frac{2n}{n+2+\gamma(n-2)}}\,dx\Big)^{\frac{n+2+\gamma(n-2)}{2n}}$$

where $C(n) > 0$ is a constant independent of m. Since

$$\frac{n-2}{n} - \frac{(n-2)(1-\gamma)}{2n} = \frac{(n-2)(1+\gamma)}{2n} > 0,$$

the inequality (5) implies that

$$(6)\qquad \int_{\Omega_m} u_m^{\frac{2n}{n-2}}\,dx \le C\Big(\int_{R_n} f^{\frac{2n}{n+2+\gamma(n-2)}}\,dx\Big)^{\frac{n+2+\gamma(n-2)}{(n-2)(1+\gamma)}}$$

Obviously, the estimate (6) continues to hold also for $\gamma = 1$. The estimates (5) and (6) imply that

$$(7)\qquad \int_{\Omega_m} |Du_m|^2\,dx \le C_1$$

for all $m \geq 1$ and some constant $C_1 > 0$ independent of m. We now show that the sequence $\{u_m\}$ is increasing. Let $m < l$, then $\Omega_m \subset \Omega_l$ and

$$(u_m - u_l)^+ = \begin{cases} (u_m - u_l)^+ & \text{on } \Omega_m\,, \\ 0 & \text{on } \Omega_l - \Omega_m\,, \end{cases}$$

that is $(u_m - u_l)^+ \in \overset{\circ}{W}{}^{1,2}(\Omega_m)$. Therefore taking $(u_m - u_l)^+$ as a test function we obtain on substitution

$$\int_\Omega \Big[\sum_{i,j=1}^n a_{ij} D_i(u_m - u_l)^+ D_j(u_m - u_l)^+ + c((u_m - u_l)^+)^2\Big]\,dx =$$
$$= \int_{\Omega_m} f\Big(\frac{1}{u_m^\gamma} - \frac{1}{u_l^\gamma}\Big)(u_m - u_l)^+\,dx \le 0\,.$$

Hence $(u_m - u_l)^+ = 0$ om Ω_m, that is $u_m \le u_l$ on Ω_m. This inequality continues to hold on $\Omega_l - \Omega_m$ because $u_m = 0$ on $\Omega_l - \Omega_m$ and $u_l > 0$ on $\Omega_l - \Omega_m$. The estimates (6) and (7) together with the diagonal method imply that we may assume that there exists $u \in L^{\frac{2n}{n-2}}(R_n)$ with $Du \in L^2(R_n)$ such that $u_m \to u$ weakly in $W^{1,2}(K)$, strongly in $L^2(K)$ for each bounded domain $K \subset R_n$, moreover $u_m \to u$ a.e. on R_n. By monotonicity of $\{u_m\}$ the limit u is positive on R_n. The proof of the fact that u is a solution of (P) is similar to the corresponding part of the proof of Lemma 1. ■

Remark 1. If $c(x) \geq c_0 > 0$ on R_n for some constant c_0 and if in addition $f \in L^{\frac{2}{1+\gamma}}(R_n)$, then the solution constructed in Theorem 1 belongs to $W^{1,2}(R_n)$.

Indeed, taking u_m as a test function we obtain

$$\lambda^{-1} \int_{\Omega_m} |Du_m|^2\,dx + c_0 \int_{\Omega} u_m^2\,dx \leq C(\eta,\gamma) \int_{\Omega_m} f^{\frac{2}{1+\gamma}}\,dx + \eta \int_{\Omega_m} u_m^2\,dx,$$

and choosing $\eta < c_0$ the result follows.

3. Case $1 < \gamma < \infty$.

The existence of positive solutions of (P) in the case $1 < \gamma < \infty$ can also be obtained by approximation method of Section 2. However, for bounded domains we only prove the existence of local solutions. We need the following auxiliary results.

Lemma 2. *Let Ω be a bounded domain in R_n and suppose that $f \in L^p(\Omega)$ with $f > 0$ on Ω and $p > n$. Then there exists a positive function $u \in W^{1,2}_{loc}(\Omega) \cap L^{\frac{n(1+\gamma)}{n-2}}(\Omega) \cap L^\infty(\Omega)$ satisfying*

$$Lu = f(x)u^{-\gamma} \quad in\ \Omega,$$

moreover $u^{\frac{1+\gamma}{2}} \in \overset{\circ}{W}{}^{1,2}(\Omega)$.

PROOF : Let $\{u_\varepsilon\}$ be the solution in $\overset{\circ}{W}{}^{1,2}(\Omega)$ of (3) with zero boundary condition (2). Taking u_ε^γ as a test function, we obtain on substitution,

$$\gamma \int_\Omega \sum_{i,j=1}^n a_{ij} D_i u_\varepsilon D_j u_\varepsilon u_\varepsilon^\alpha\,dx + \int_\Omega c u^{1+\gamma}\,dx = \int_\Omega f \frac{u_\varepsilon^\gamma}{(\varepsilon + u_\varepsilon)^\gamma}\,dx \leq \int_\Omega f\,dx,$$

$\alpha = \gamma - 1$, and consequently

$$\int_\Omega |D(u_\varepsilon^{\frac{1+\gamma}{2}})|^2\,dx \quad \text{and} \quad \int_\Omega u_\varepsilon^\alpha |Du_\varepsilon|^2\,dx \tag{8}$$

are bounded as $\varepsilon \searrow 0$. By the Sobolev inequality we also have

$$\int_\Omega u_\varepsilon^{\frac{n(1+\gamma)}{n-2}}\,dx \leq C \tag{9}$$

for some $C > 0$ independent of $\varepsilon > 0$. As in the proof of Lemma 1 we show that u_ε is increasing and $u_\varepsilon + \varepsilon$ is decreasing as $\varepsilon \searrow 0$. Let v be a positive solution in $\overset{\circ}{W}{}^{1,2}(\Omega)$ of the Dirichlet problem

$$Lv = f(x)\frac{1}{1+v^\gamma} \quad \text{in } \Omega,$$
$$v(x) = 0 \quad \text{on } \partial\Omega.$$

By a standard regularity theorem $v \in L^\infty(\Omega) \cap C(\Omega)$ (see Theorems 8.16 and 8.22 in [5]). We also have $v(x) \le u_\varepsilon(x)$ on Ω for each $0 < \varepsilon < 1$. This combined with the second estimate (9) gives the following property of the sequence $\{u_\varepsilon\}$:
for each compact set $K \subset \Omega$, there exists a constant $C(K) > 0$ independent of $\varepsilon > 0$ such that

$$\int_K |Du_\varepsilon(x)|^2\, dx \le C(K).$$

Consequently, using the diagonal method we can select a sequence $\varepsilon_m \searrow 0$ such that $u_{\varepsilon_m} \to u$ weakly in $W^{1,2}(K)$ and strongly in $L^2(K)$ for each compact set $K \subset \Omega$, moreover $u_\varepsilon \to u$ a.e. on Ω. By virtue of the first estimate (9) we may assume that $u_{\varepsilon_m}^{\frac{\gamma+1}{2}} \to u$ weakly in $W^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$. According to the estimate (10) $u \in L^{\frac{n(\gamma+1)}{n-2}}(\Omega)$. It is also obvious that $u^{\frac{1+\gamma}{2}} \in \overset{\circ}{W}{}^{1,2}(\Omega)$ and $v(x) \le u(x)$ on Ω. It remains to show that u is a solution of our equation. Let $w \in W^{1,2}(\Omega)$ with compact support in Ω, then for each m we have

$$\int_\Omega \left[\sum_{i,j=1}^n a_{ij} D_i u_{\varepsilon_m} D_j w + c u_{\varepsilon_m} w\right] dx = \int_\Omega f \frac{w}{\varepsilon_m + u_{\varepsilon_m}^\gamma}\, dx.$$

Since $u_{\varepsilon_m} \ge \inf_{\operatorname{supp} w} v > 0$, the result follows from the weak convergence u_{ε_m} in $W^{1,2}(\operatorname{supp} w)$ and the Monotone Convergence Theorem. As in Lemma 1 we have

$$0 < u_\varepsilon - u_\delta < \delta - \varepsilon \ \text{ a.e. on } \Omega \tag{10}$$

for all $\delta > \varepsilon$. ■

Lemma 3. *Let $f \in L^1(R_n) \cap L^p(R_n)$ with $f(x) > 0$ on R_n and $p > n$. Then there exists a positive solution $v \in W^{1,2}_{loc}(R_n) \cap L^{\frac{2n}{n-2}}(R_n)$ with $Dv \in L^2(R_n)$ of the equation*

$$Lu = f(x)\frac{1}{(1+u)^\gamma} \quad \text{in } R_n. \tag{11}$$

PROOF : Let $\{\Omega_m\}$ be an increasing sequence with smooth boundaries such that $R_n = \bigcup_{m\ge 1} \Omega_m$. For each m there exists a unique positive solution $v_m \in \overset{\circ}{W}{}^{1,2}(\Omega_m)$ of the Dirichlet problem

$$Lu = f(x)\frac{1}{1+u^\gamma} \quad \text{in } \Omega_m,$$
$$u(x) = 0 \quad \text{on } \partial\Omega_m.$$

We extend v_m by 0 outside Ω_m. As in Theorem 1 we check that $\{v_m\}$ is an increasing sequence. Taking v_m as a test function we obtain

$$\int_{\Omega_m}\left[\sum_{i,j=1}^n a_{ij} D_i v_m D_j v_m + c v_m^2\right] dx = \int_{\Omega_m} f \frac{v_m}{(1+v_m)^\gamma}\, dx \le \int_\Omega f\, dx.$$

Consequently the sequences of integrals $\int_{\Omega_m} |Dv_m|^2\,dx$ and $\int_{\Omega_m} v_m^{\frac{2n}{n-2}}\,dx$ are bounded independently of m. Applying the diagonal method we may assume that there exists $v \in W^{1,2}_{loc}(R_n) \cap L^{\frac{2n}{n-2}}(R_n)$ with $Dv \in L^2(R_n)$ such that $v_m \to v$ weakly in $W^{1,2}(K)$ and strongly in $L^2(K)$ for each bounded domain $K \subset R_n$. Also, $v_m \to v$ a.e. on R_n. It is easy to see that v is a solution of the equation (11). ■

Remark 2. If $c(x) \geq c_0 > 0$ on R_n for some constant c_0, then $v \in W^{1,2}(R_n)$.

Theorem 2. *Let $f \in L^1(R_n) \cap L^p(R_n)$ with $p > n$ and $f(x) > 0$ on R_n. Then the problem (P) has a solution u in $W^{1,2}_{loc}(R_n) \cap L^{\frac{n(\gamma+1)}{n-2}}(R_n)$ with $Du^{\frac{\gamma+1}{2}} \in L^2(R_n)$.*

PROOF : Let $\{\Omega_m\}$ be an increasing sequence of domains from Lemma 3. By Lemma 2 for each m there is a positive function $u_m \in W^{1,2}_{loc}(\Omega_m) \cap L^{\frac{n(\gamma+1)}{n-2}}(R_n)$ with $u_m^{\frac{\gamma+1}{2}} \in \overset{\circ}{W}{}^{1,2}(\Omega_m)$ satisfying the equation

$$Lu_m = f(x)u_m^{-\gamma} \text{ in } \Omega_m .$$

It follows from the proof of Lemma 2 that

$$v_m(x) \leq u_m(x) \text{ on } \Omega_m , \tag{12}$$

where v_m is the positive solution in $\overset{\circ}{W}{}^{1,2}(\Omega_m)$ of the problem

$$Lv_m = f(x)\frac{1}{1+v_m^\gamma} \text{ in } \Omega_m ,$$
$$v_m(x) = 0 \text{ on } \partial\Omega_m .$$

According to (9) and (10)

$$\int_{\Omega_m} |D(u_m^{\frac{\gamma+1}{2}})|^2\,dx \leq C_1 \text{ and } \int_{\Omega_m} u_m^\alpha |Du_m|^2\,dx \leq C_1 \tag{13}$$

and

$$\int_{\Omega_m} u_m^{\frac{n(\gamma+1)}{n-2}}\,dx \leq C_1 \tag{14}$$

for some constant $C_1 > 0$ independent of m. Since the sequence $\{u_m\}$ is increasing it follows from the second estimate (13) that

$$\int_{\Omega_p} u_p^\alpha |Du_m|^2\,dx \leq C_1$$

for $p < m$. We may also assume that $v_m \in C(\Omega)$ (see Theorems 8.16 and 8.22 in [5]). Therefore for each compact set $K \subset R_n$ there exists $\Omega_p \supset K$ with $\inf_K u_p > 0$. Consequently, by the diagonal method we may assume that $u_m \to u$ weakly in $W^{1,2}(K)$, strongly in $L^2(K)$ for each compact set $K \subset R_n$, also $u_m \to u$ a.e. on R_n. According to Lemma 3, $v_m \to v$ a.e. on R_n, where v is a positive solution of the equation (11). Consequently, we have by (12) $0 < v(x) \leq u(x)$ on R_n. It is now a routine to show that u is a solution of the problem (P) with required properties. ■

Remark 3. If $c(x) \geq c_0 > 0$ for some constant c_0, then $u^{\frac{\gamma+1}{2}} \in W^{1,2}(R_n)$.

4. Solutions with exponential decay.

In this section we show that if f has an exponential decay at infinity, then the same is true for a solution of the problem (P).

Theorem 3. *Let $0 < \gamma \leq 1$. Suppose that $c(x) \geq c_0 > 0$ on R_n, where c_0 is a constant and that $f \in L^\infty(R_n)$ with*

$$0 < f(x) \leq K \exp\left(-\alpha \sum_{i=1}^{n} |x_i|\right) \quad \text{on } R_n$$

for some constant $\alpha > 0$. Then the solution u of the problem (P) satisfies

$$\int_{R_n} \left[|Du(x)|^2 + u(x)^2\right] \exp\left(\delta_0 \sum_{i=1}^{n} |x_i|\right) dx < \infty$$

for some $\delta_0 > 0$.

PROOF : Let $\{u_m\}$ be a sequence from Theorem 2. Taking as a test function $v(x) = u_m(x)H(x)^2$ where $H(x) = \prod_{i=1}^n \cosh \delta x_i$, with $\delta > 0$ to be determined, we obtain

$$\int_{\Omega_m} \left[\sum_{i,j=1}^{n} a_{ij} D_i u_m D_j u_m H^2 + 2 \sum_{i,j=1}^{n} a_{ij} D_i u_m u_m D_j H H + c u_m^2 H^2\right] dx =$$
$$= \int_{\Omega_m} f u_m^{1-\gamma} H^2\, dx \leq \frac{c_0}{2} \int_{\Omega_m} u_m^2 H^2\, dx + C(c_0, \gamma) \int_{\Omega} f^{\frac{2}{1+\gamma}} H^2\, dx.$$

Since

$$2 \int_{\Omega_m} \sum_{i,j=1}^{n} a_{ij} D_i u_m u_m D_j H H\, dx \leq \frac{1}{2\lambda} \int_{\Omega_m} |Du_m|^2 H^2\, dx + C(\lambda) \int_{\Omega_m} u_m^2 |DH|^2\, dx,$$

we obtain

$$\frac{1}{2\lambda} \int_{\Omega_m} |Du_m|^2 H^2\, dx + \int_{\Omega_m} \left(\frac{c_0}{2} H^2 - C(\lambda)|DH|^2\right) u_m^2\, dx \leq C(c_0, \gamma) \int_{\Omega_m} f^{\frac{2}{2+\gamma}} H^2\, dx.$$

We now note that there exists $\delta_0 > 0$ such that

$$\frac{c_0}{2} H^2 - C(\lambda)|DH|^2 \geq \frac{c_0}{4} H^2$$

for all $0 < \delta \leq \delta_0$ and all $x \in R_n$, we may also assume that $\delta_0 < \frac{2\alpha}{1+\gamma}$. Hence

$$\frac{1}{2\lambda} \int_{\Omega_m} |Du_m|^2 H^2\, dx + \frac{c_0}{4} \int_{\Omega_m} u_m^2 H^2\, dx \leq C(c_0, \gamma) \int_{\Omega_m} f^{\frac{2}{2+\gamma}} H^2\, dx.$$

Letting $m \to \infty$ the result follows. ∎

By a similar argument using Lemma 2 one can establish the following result.

Theorem 4. *Let $1 < \gamma < \infty$ and suppose that f and c satisfy hypotheses of Theorem 3. Then the solution u of the problem (P) satisfies*

$$\int_{R_n} \left[|D(u(x)^{\frac{\gamma+1}{2}})|^2 + u(x)^{\gamma+1} \right] \exp(\delta_0 \sum_{i=1}^{n} |x_i|)\, dx < \infty$$

for some constant $\delta_0 > 0$.

5. Pointwise estimate.

The estimate of Theorem 3 can be improved in case of the equation with smooth coefficients. Inspection of the proofs of Lemmas 1 and 2 shows that the solution u of (P) satisfies the estimate

$$0 < v(x) \le u(x) \le v(x) + 1, \tag{15}$$

where v is a positive solution of the equation (11). This is an immediate consequence of the inequalities (4) and (10). In this section we additionally assume that $a_{ij}, D_i a_{ij}, c$ and f are locally Hölder continuous. Using standard regularity results the solutions u and v of (P) and (11), respectively, are locally $C^{2+\alpha}$ on R_n. The equation (1) can be written in the form

$$Lu = -\sum_{i,j=1}^{n} a_{ij} D_{ij} u + \sum_{j=1}^{n} b_j D_j u + cu = f u^{-\gamma},$$

where $b_j(x) = -\sum_{i=1}^{n} D_i a_{ij}(x)$. We point out here that some existence results for the Dirichlet problem in bounded domains for the equation with smooth coefficients can be found in [**3**] and [**9**]. To use the classical maximum principle we assume that $c(x) \ge c_0$ on R_n for some constant $c_0 > 0$. We need the following well known result.

Lemma 5. *Suppose that u is a bounded function in $C^2(R_n)$ and that $Lu \ge 0$ on R_n. Then $u(x) \ge 0$ on R_n.*

To derive pointwise estimates we compare the solution of (P) with a function H given by

$$H(x, \delta) = \prod_{i=1}^{n} \cosh \delta x_i \quad \text{for } x \in R_n \text{ and } 0 < \delta.$$

It is easy to see that there exist constants $\delta_0 > 0$ and $K > 0$ such that

$$L(H^{-1}) \frac{1}{2} c_0 H^{-1} \quad \text{on } R_n \text{ and for } 0 < \delta \le \delta_0$$

and

$$L(H^{-1}) \le K H^{-1} \quad \text{on } R_n.$$

Moreover, we have

$$e^{-\delta \sum_{i=1}^{n} |x_i|} \le H(x, \delta)^{-1} \le 2^n e^{-\delta \sum_{i=1}^{n} |x_i|}$$

for $x \in R_n$ and $0 < \delta < \infty$.

Lemma 6. *Suppose that* $o < \gamma < \infty$ *and that*

$$C_1 e^{-\delta_1 \sum_{i=1}^n |x_i|} \leq f(x) \leq C_2$$

on R_n *for some constants* $C_1 > 0, C_2 > 0$ *and* $0 < \delta_1 \leq \delta_0$. *Then the solution* v *of the equation* (11) *satisfies the estimate*

$$v(x) \geq C_1 2^{-n} K^{-1} (1 + \frac{C_2}{c_0})^{-\gamma} e^{-\delta_1 \sum_{i=1}^n |x_i|} \quad \text{on } R_n . \tag{16}$$

PROOF : Let v_m be the sequence of the solutions of the Dirichlet problems in Ω_m from the proof of Lemma 3. By a classical maximum principle we obtain that

$$0 < v_m(x) \leq \frac{C_2}{c_0} \quad \text{on } \Omega_m .$$

Letting $m \to \infty$ we see that this estimate continues to hold for v on R_n. This also remains true for $0 < \gamma \leq 1$. Let $d = C_1 2^{-n} K^{-1} (1 + \frac{C_2}{c_0})^{-\gamma}$ and $H_1 = H(x, \delta_1)$, then

$$L(v - dH_1^{-1}) \geq C_1 (1 + \frac{C_2}{c_0})^{-\gamma} e^{-\delta_1 \sum_{i=1}^n |x_i|} - dK H_1^{-1} \geq$$
$$\geq e^{-\delta_1 \sum_{i=1}^n |x_i|} (C_1 (1 + \frac{C_2}{c_0})^{-\gamma} - 2^n dK) = 0$$

in R_n and the estimate (16) follows from Lemma 5. ■

We now establish the lower and upper bound of the solution of the problem (P). To derive this estimates we need some restrictions on γ.

Theorem 5. *Suppose that*

$$C_1 e^{-\delta_1 \sum_{i=1}^n |x_i|} \leq f(x) \leq C_2 e^{-\delta_2 \sum_{i=1}^n |x_i|}$$

on R_n *for some constants* $C_1 > 0, C_2 > 0$ *and* $0 < \delta_2 \leq \delta_1 \leq \delta_0$ *and let* $0 < \gamma \leq \frac{\delta_2}{\delta_1}$. *Then the solution* u *of the problem* (P) *satisfies the estimate*

$$K_1 e^{-\delta_1 \sum_{i=1}^n |x_i|} \leq u(x) \leq K_2 e^{-(\delta_2 - \gamma\delta_1) \sum_{i=1}^n |x_i|},$$

on R_n, *where* $K_1 = C_1 2^{-n} K^{-1} (1 + \frac{C_2}{c_0})^{-\gamma}$ *and* $K_2 = 2^{n+1} C_2 c_0 K_1^{-\gamma}$.

PROOF : The lower bound follows from Lemma 6. We set $w = u - k\bar{H}^{-1}$, where $k = 2^{1+n\gamma} K^{\gamma} C_1^{-\gamma} (\frac{C_2}{c_0})^{-\gamma}$ and $\bar{H} = H(x, \delta_2 - \delta\gamma)$. Using the lower bound we check that $Lw \leq 0$ on R_n and the result follows from Lemma 5. ■

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