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Abstract. Suppose T is a finite lattice of types and A is a completely decomposable finite rank torsion-free abelian group such that the type of each summand of A is an element of T. If G is a strongly indecomposable group of the form A/X, where X is a rank-1 pure subgroup of A, a sharp upper bound is determined for the rank of G in terms of lattice-theoretic properties of T.

Keywords: Butler groups, strongly indecomposable groups

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0. Introduction.

In the recent sequence of papers [**AV2**], [**AV3**] and [**AV4**], D. Arnold and C. Vinsonhaler were successful in determining a complete set of numerical isomorphism invariants for certain classes of Butler groups. Recall that a Butler group is a torsion-free homomorphic image of a finite rank completely decomposable group. Let $A_1, \dots, A_n$ be subgroups of the rational numbers Q, each of which contains the integers Z. The groups classified by Arnold and Vinsonhaler are the strongly indecomposable groups of the form $G[A_1, \dots, A_n] = A_1 \oplus \cdots \oplus A_n/X$, where X is the pure subgroup generated by $(1, \dots, 1)$. As such, there is justifiable interest in the groups $G[A_1, \dots, A_n]$ and, in particular, the strongly indecomposable groups of this form.

In the first section of this paper, we give a complete description of the typeset of $G[A_1, \dots, A_n]$ in terms of the types of the A_i's. This in turn is used to give a new characterization of when such groups are strongly indecomposable. In the second section, strongly indecomposable T-Butler groups are considered for a finite lattice of types T. Recall that a T-Butler group is a torsion-free homomorphic image of a completely decomposable group $A_1 \oplus \cdots \oplus A_n$, where type $A_i \in T$ for each i. In this last section our main result is Theorem 2.3, which gives a bound on the rank of strongly indecomposable T-Butler groups of the form $G[A_1, \dots, A_n]$ in terms of lattice-theoretic properties of T. In a certain sense, this will be seen to generalize a result of M.C.R. Butler in [**B**].

In the sequel, all groups considered will be torsion-free abelian groups of finite rank. For abelian group notation and terminology not explicitly defined here, we refer the reader to [**F**], [**A**] and [**AV1**]. As general references for lattice theory, we use [**CD**] and [**G**].

1. Typesets and strong indecomposability.

We begin by setting some notation which will remain in force throughout. If n is a positive integer ≥ 2, we write $\bar{n}$ for the set $\{1, 2, \dots, n\}$. For each $i \in \bar{n}$, A_i will always denote a subgroup of Q with $Z \subseteq A_i$. By $G[A_1, \dots, A_n]$ we mean the group

$A_1 \oplus \cdots \oplus A_n/X$, where X is the pure subgroup of $A_1 \oplus \cdots \oplus A_n$ generated by $(1,\ldots,1)$. If $I \subseteq \bar{n}$ is nonempty, , we sometimes write $G[A_I]$ for $\oplus\{A_i : i \in I\}/X_I$, where $X_I = \langle(1,\ldots,1)\rangle * \subseteq \oplus\{A_i : i \in I\}$.

Suppose τ_i is a type (or, more generally, an element of some lattice) for each $i \in \bar{n}$. If $I \subseteq \bar{n}$ is not empty, we write τ^I or $\bigvee_{i\in I} \tau_i$ (respectively, τ_I or $\bigwedge_{i\in I} \tau_i$) for the supremum (respectively, the infimum) of the τ_i with $i \in I$. If $I = \{i, j\}$ we often write τ^{ij} or $\tau_i \vee \tau_j$ (respectively, τ_{ij} or $\tau_i \wedge \tau_j$) for τ^I (respectively, τ_I).

It was recently brought to the authors' attention that Proposition 1.1 and Theorem 1.2 below are proved independently by L. Fuchs and C. Metelli in the preprint [**FM**]. However the authors feel that the formulations and proofs given below are sufficiently different as to merit inclusion. Moreover, the author's methods of proof lead to Corollaries 1.3–1.5.

Our first result describes the typeset of $G[A_1,\ldots,A_n]$ in terms of the types of the $A_i's$.

Proposition 1.1. *Suppose $n \geq 2, G = G[A_1,\ldots,A_n]$, and $\tau_i = \text{type}\, A_i$ for each $i \in \bar{n}$.*

(i) *Let $0 \neq \bar{a} = (a_1,\ldots,a_n) + X \in G$ and partition $\bar{n}$ into nonempty disjoint subsets $I_1,\ldots,I_k$ such that $a_r = a_s$ if and only if there exists $i \in \bar{k}$ with $r, s \in I_i$. Then,*

$$\text{type}\, \bar{a} = \bigwedge_{1\leq i<j\leq k} (\tau_{I_i} \vee \tau_{I_j}).$$

(ii) *If τ is a type, then $\tau \in \text{typeset}\, G$ if and only if there exists a partition of $\bar{n}$ into nonempty disjoint subsets $I_1,\ldots,I_k$ such that $k \geq 2$ and*

$$\tau = \bigwedge_{1\leq i<j\leq k} (\tau_{I_i} \vee \tau_{I_j}).$$

PROOF : Let $f : A_1 \oplus \cdots \oplus A_n \to G$ be the natural map. If G/K is rank-1 and torsion-free, define the *cosupport* of K by $\text{cosupp}\, K = \{i \in \bar{n} : f(A_i) \subseteq K\}$. By [**AV1**, Theorem 1.4], there is a cobalanced embedding

$$\delta : G \to \oplus\{G/K : \text{cosupp}\, K \text{ is maximal with respect to inclusion }\},$$

with δ induced by the various natural maps $G \to G/K$. In the present context, for each distinct pair $r, s \in \bar{n}$, select $K_{rs} \leq A = A_1 \oplus \cdots \oplus A_n$ such that $A/K_{rs} \cong A_r + A_s$. Then, $\bar{K}_{rs} = K_{rs}/X$ is the unique subgroup of G with maximal cosupport $\bar{n} - \{r, s\}$. Thus, by the above mentioned result of [**AV1**], the induced map

$$\delta : G \to \oplus\{A_r + A_s : 1 \leq r < s \leq n\}$$

is a cobalanced (and hence pure) embedding such that the component of $\delta(\bar{a})$ in $A_r + A_s$ can be taken to be $a_r - a_s$.

Now, to see (i), note that $\text{type}\, \bar{a} = \text{type}\, \delta(\bar{a}) = (\bigwedge\{\tau^{rs} : r \in I_1, s \in \bar{n} - I_1\}) \wedge (\bigwedge\{\tau^{rs} : r \in I_2, s \in \bar{n}-(I_1 \cup I_2)\}) \wedge \cdots \wedge (\bigwedge\{\tau^{rs} : r \in I_{k-1}, s \in \bar{n}-(I_1 \cup \cdots \cup I_{k-1}) = I_k\}) = \bigwedge_{1\leq i<j\leq k}(\tau_{I_i} \vee \tau_{I_j})$, as desired.

That $\tau \in \operatorname{typeset} G$ has the described form is an immediate consequence of (i). Conversely, if $\bar{n} = I_1 \dot{\cup} \ldots \dot{\cup} I_k$ is a nontrivial partition of $\bar{n}$ with $k \geq 2$, define $\bar{a} = (a_1, \ldots, a_n) + X \in G$ by $a_r = i$ if and only if $r \in I_i$. Then $\bar{a} \neq 0$ and $\operatorname{type} \bar{a} = \bigwedge_{1 \leq i < j \leq k} (\tau_{I_i} \vee \tau_{I_j})$ by (i). Thus (ii) is proved. ■

For reasons described in the introduction, it is useful to have a readily computable means of determining whether a group of the form $G[A_1, \ldots, A_n]$ is strongly indecomposable. Moreover, if $G[A_1, \ldots, A_n]$ is not strongly indecomposable, it is of interest to know how it splits into strongly indecomposable quasi-summands. These issues are addressed by the following theorem and its corollaries.

Theorem 1.2. *Suppose $n \geq 3$ and $G = G[A_1, \ldots, A_n]$ where each A_i has type τ_i. Then G is strongly indecomposable if and only if for every $k \in \bar{n}$ and partition $\bar{n} - \{k\} = I \dot{\cup} J$ into nonempty disjoint sets I and J, $\tau_k \nleq \tau_I \vee \tau_J$.*

PROOF : $\Rightarrow$: Suppose $\tau_k \leq \tau_I \vee \tau_J$ for some $k \in \bar{n}$ and nontrivial partition $\bar{n} - \{k\} = I \dot{\cup} J$. Since $\tau_k = \tau_k \wedge (\tau_I \vee \tau_J) = \tau_{I \cup \{k\}} \vee \tau_{j \cup \{k\}}$, after changing notation we may assume that $\tau_k = \tau_I \vee \tau_J$, where $I = \{1, 2, \ldots, k\}$, $J = \{k, k+1, \ldots, n\}$ are such that $|I| \geq 2$, $|J| \geq 2$ and $I \cap J = \{k\}$. Define a mapping $\varphi : G \to G[A_I] \oplus G[A_J]$ by

$$\varphi((a_1, \ldots, a_n) + X) = ((a_1, \ldots, a_k) + X_I, \quad (a_k, \ldots, a_n) + X_J).$$

Clearly φ is a well-defined monomorphism. We claim that φ is in fact a quasi-isomorphism, thereby showing that G is not strongly indecomposable. Since $\tau_k = \tau_I \vee \tau_J$, there exists an integer $m \neq 0$ with $mA_k \subseteq (\bigcap\{A_i : i \in I\}) + (\bigcap\{A_j : j \in J\})$. Now suppose that $((b_1, \ldots, b_k) + X_I, (c_1, \ldots, c_n) + X_J)$ is an arbitrary element of $G[A_I] \oplus G[A_J]$. Then, $m(b_k - c_k) = b' - c'$ for some $b' \in \cap\{A_i : i \in I\}$ and $c' \in \cap\{A_j : j \in J\}$. Thus, $mb_i - b' \in A_i$ for all $i \in I$ and $mc_j - c' \in A_j$ for all $j \in J$. Moreover, $\varphi(mb, -b', \ldots, mb_k - b', mc_{k+1} - c', \ldots, mc_n - c') = m((b_1, \ldots, b_k) + X_I, (c_k, \ldots, c_n) + X_J)$, since $mb_k - b' = mc_k - c'$. Therefore, $m(G[A_I] \oplus G[A_J]) \subseteq \operatorname{Im} \varphi$, and φ is a quasi-isomorphism as claimed.

$\Leftarrow$: Suppose now that $\tau_k \nleq \tau_I \vee \tau_J$ for every $k \in \bar{n}$ and nontrivial partition $I \dot{\cup} J$ of $\bar{n} - \{k\}$. Set $G' = G[A_1', \ldots, A_n']$, where $A_i' = A_i + (\bigcap\{A_j : j \in \bar{n} - \{i\}\})$. It is easily seen that $G \cong G'$ (see [Le] or [GU]) and that if $\tau_i' = \operatorname{type} A_i'$, then $\tau_i' = \tau_i \vee (\bigwedge\{\tau_j : j \in \bar{n} - \{i\}\})$. Moreover, it is easily seen that the natural image $\overline{A_i}'$ of A_i' in G' is pure.

We claim that each $\overline{A_i}'$ is fully invariant in G'. Suppose to the contrary that $\overline{A_k}'$ is not fully invariant for some $k \in \bar{n}$. Then, there exists a nonzero f in the endomorphism ring $E(G')$ such that $B = \langle f(\overline{A_k}') \rangle * \nsubseteq \overline{A_k}'$. Say $B = \langle \bar{b} \rangle *$ where $\bar{b} = (b_1, \ldots, b_n) + X' \in G'$, with each $b_i \in A_i'$ and where X' is the pure subgroup of $A_1' \oplus \cdots \oplus A_n'$ generated by $(1, \ldots, 1)$. Let $I' = \{r \in \bar{n} : b_r \neq 0\}$ and $J = \bar{n} - I'$. Without loss we may assume $k \in I'$ and $J \neq \phi$. Set $I = I' - \{k\}$ and note that $I \neq \phi$ (since otherwise, $B \subseteq \overline{A_k}'$). Moreover, $I \dot{\cup} J = n - \{k\}$. With an application of Proposition 1.1, we obtain the contradiction $\tau_k \leq \tau_k' \leq \operatorname{type} B = \operatorname{type} \bar{b} \leq (\bigwedge\{\tau_r' : r \in I'\}) \vee (\bigwedge\{\tau_s' : s \in J\}) \leq (\bigwedge\{\tau_r' : r \in I\}) \vee (\bigwedge\{\tau_s' : s \in J\}) = \tau_I \vee \tau_J$. Therefore, each $\overline{A_i}'$ is fully invariant, as claimed.

Now suppose $g \in E(G')$. Then, for each i, $g \mid \overline{A_i}'$ is multiplication by some $r_i \in Q$. But $0 = g((1,\cdots,1) + X') = (r_1 \ldots, r_n) + X'$ and so $r_1 = \cdots = r_n$ and we conclude $E(G') \subseteq Q$. Therefore, $QE(G) \cong QE(G') \cong Q$ and G is strongly indecomposable by a well-known result of J.D. Reid [**Re**]. ∎

From the first part of the proof of Theorem 1.2, we obtain some information concerning the quasi-decomposition of $G[A_1, \ldots, A_n]$.

Corollary 1.3. *Suppose $n \geq 2$ and $G = G[A_1, \ldots, A_n]$. Then, there exist non-empty subsets $I_1, \ldots, I_k (k \geq 1)$ of $\bar{n}$ such that $\bar{n} = I_1 \cup \cdots \cup I_k$, $|I_i \cap I_j| \leq 1$ whenever $i \neq j$, and G is quasi-isomorphic to $G[A_{I_1}] \oplus \cdots \oplus G[A_{I_k}]$ with each $G[A_{I_i}]$ strongly indecomposable.*

In [**Le**] it was shown that $G[A_1, \ldots, A_n]$ is strongly indecomposable if the types $\tau^{ij} = \text{type}(A_i + A_j)$ are pairwise incomparable for all $i \neq j$ in $\bar{n}$. Using Theorem 1.2, this result can be sharpened as follows.

Corollary 1.4. *Suppose that for every three element subset $\{i, j, k\}$ of $\bar{n}$, the types τ^{ij}, τ^{ik} and τ^{jk} are pairwise incomparable. Then $G[A_1, \ldots, A_n]$ is strongly indecomposable.*

Proof : We may assume $n \geq 3$. Suppose that $G[A_1, \ldots, A_n]$ is not strongly indecomposable. Then, by Theorem 1.2, there exists $k \in \bar{n}$ and a nontrivial partition $I \dot{\cup} J = \bar{n} - \{k\}$ with $\tau_k \leq \tau_I \vee \tau_j$. Thus, $\tau_k \leq \tau^{ij}$ with $i \in I$ and $j \in J$. Consequently, $\tau^{ij}, \tau^{ik}, \tau^{jk}$ are not pairwise incomparable. ∎

In [**GU**, Proposition 3], it was shown that if $G[A_1, A_2, A_3]$ is strongly indecomposable, then the types $\tau^{12}, \tau^{13}, \tau^{23}$ are pairwise incomparable. Combining this with Corollary 1.4 the following result is obtained.

Corollary 1.5. *Suppose $G[A_i, A_j, A_k]$ is strongly indecomposable for each three element subset $\{i, j, k\} \subseteq \bar{n}$. Then $G[A_1, \ldots, A_n]$ is strongly indecomposable.*

2. Strongly indecomposable T-Butler groups.

Let T be a finite lattice of types and suppose $\alpha, \beta \in T$. As usual, call β a *cover* of α if $\beta > \alpha$ and there is no $\gamma \in T$ with $\beta > \gamma > \alpha$. Throughout this section we set

$$k(T) = \max\{m : \exists\, \tau \in T \text{ with } m \text{ distinct covers in } T\}.$$

A theorem of Butler (Theorem 6 in [**B**]) asserts that every strongly indecomposable T-Butler group has rank 1 if and only if $k(T) \leq 2$. Here we will further investigate the relationship between strongly indecomposable T-Butler groups and the number $k(T)$.

Given a positive integer $m \geq 1$, the set of all positive divisors of m forms a distributive lattice $L(m)$ where $d \wedge d' = gcd(d, d')$ and $d \vee d' = lcm(d, d')$. Note that $L(m)$ is Boolean if and only if m is square-free.

Call an element α in a lattice L *join irreducible* in L if $\alpha \neq \bigwedge\{\delta : \delta \in L\}$ and $\alpha = \beta \vee \gamma$ for some $\beta, \gamma \in L$ implies that either $\beta = \alpha$ or $\gamma = \alpha$. Set

$$J(L) = \{\alpha \in L : \alpha \text{ is join irreducible in } L\}.$$

Theorem 2.1. *Let T be a finite lattice of types. There exists a positive integer $m = m(T)$ with $k = k(T)$ distinct prime factors such that T embeds in $L(m)$.*

PROOF : R. Dilworth has shown that $k = k(T) = \max\{n : \exists \gamma \in T$ which covers n distinct elements of $T\}$ (see [**G**, p. 121]). Dualizing the argument used on page 89 of [**CD**] one can then show that any $k+1$ elements of $J(T)$ contain at least one comparable pair. Hence, by a well-known result of Dilworth (1.1 on page 3 in [**CD**]), $J(T)$ is the disjoint union of k nonempty chains $C_1, \ldots, C_k$. Set $\tau_0 = \bigwedge T$ and $S_i = C_i \cup \{\tau_0\}$. Each element $\alpha \in T$ can then be uniquely expressed as $\alpha = \alpha_1 \vee \cdots \vee \alpha_k$ with $\alpha_i \in S_i$. Define $\phi : T \to C = S_1 \times \cdots \times S_k$ by $\phi(\alpha) = (\alpha_1, \ldots, \alpha_k)$. For $c_i = |C_i|$ and $p_1, \ldots, p_k$ distinct primes, clearly $C \cong L(m)$ for $m = p_1^{c_1} \ldots p_k^{c_k}$. ■

Before proving the main result of this section, we show how the theorem of Butler mentioned above can be retrieved from the results obtained thus far.

Corollary 2.2. ([**B**]) *Every strongly indecomposable T-Butler group has rank 1 if and only if $k(T) \leq 2$.*

PROOF : $\Rightarrow$: If $\tau_1, \tau_2, \tau_3 \in T$ are distinct covers of $\alpha \in T$, then $\tau_{\sigma(1)} \not\leq \tau_{\sigma(2)} \vee \tau_{\sigma(3)}$ for any permutation σ of $\bar{3}$. Thus, if we select A_1, A_2, A_3 with type $A_i = \tau_i$, $G[A_1, A_2, A_3]$ is strongly indecomposable by Theorem 1.2. Moreover, rank $G[A_1, A_2, A_3] = 2$.

$\Leftarrow$: Suppose $G = C/K$ is a T-Butler group with C T-completely decomposable and K a pure subgroup of C. Let $x_1, x_2 \ldots, x_\ell$ be a maximal linearly independent subset of K and set $G_i = C/\langle x_1, \ldots, x_i \rangle *$. Note that for $0 \leq i \leq \ell - 1$, G_i maps onto G_{i+1} with rank-1 kernel. Thus, it is enough to show that every strongly indecomposable group $G = G[A_1, \ldots, A_n]$ with type $A_i = \tau_i \in T$ has rank 1; i.e. has $n = 2$.

If $k(T) = 1$, then T is linearly ordered and in this case it clearly follows from Theorem 1.2 that $n = 2$. So, we may assume $k(T) = 2$. Thus, $m = m(T)$ has only two distinct prime factors.

Set $A_i' = A_i + (\bigcap_{j \neq i} A_j)$. It is straight forward to show that $G \cong G[A_1', \ldots, A_n']$ (see [**Le**] or [**GU**]) and observe that type $A_i' = \tau_i' \vee (\bigwedge_{j \neq i} \tau_j')$, where $\tau_i =$ type A_i'. But if $d_i \in L(m)$ corresponds to τ_i', then $d_i = d_i \vee (\bigwedge_{j \neq i} d_j)$. From this and the fact that m has only two distinct prime factors, it follows that $n = 2$. ■

The case for $k(T) > 2$ is more complicated.

If $d \in L(m)$ for some positive integer m, there is a sequence $d_1 = 1 < d_2 < \cdots < d_r = d$ of divisors of m such that d_{i+1}/d_i is prime for all i. Note that for a given d, r is independent of the choice of d_i's. We call r the rank of d and write $r = r(d)$. The *Whitney numbers* associated with $L(m)$ are $N_j = |\{d \in L(m) : r(d) = j\}|$ (see [**And**]).

Theorem 2.3. *Let T be a finite lattice of types and $m = m(T)$. Then, the rank of a strongly indecomposable group of the form $G[A_1, \ldots, A_n]$ with* type $A_i \in T$ *is at most N_c-1, where $j = |J(T)|$ and $c = [j/2]$, the greatest integer in $j/2$.*

PROOF : There is a finite lattice T' of types for which $L(m) \cong T'$. To see this, embed $L(m)$ in the power set $P(X)$, where X is a finite set of primes, and note that the correspondence $S \longleftrightarrow \text{type}\, Z_S$ for $S \in P(X)$ is a lattice isomorphism. Theorem 1.2 asserts that the strong indecomposability of $G[A_1, \ldots, A_n]$ is a lattice property, hence we may regard $T \subseteq T'$.

Let $G = G[A_1, \ldots, A_n]$ be strongly indecomposable of maximal rank and set $\tau_i = \text{type}\, A_i$. By Theorem 1.2, the $\tau_i's$ must be pairwise incomparable. By a generalization of Sperner's theorem (Theorem 3.1.3 in [**And**]), $n \leq N_c$, where if $m = p_1^{c_1} \ldots p_1^{c_r}$ is the prime factorization of m and $j = \sum c_i$, then $c = [j/2]$. From the proof of Theorem 2.1, we note that $\sum c_i = |J(T)|$. ∎

We remark that the estimate given in Theorem 2.3 is sharp in the following sense. If T is a Boolean lattice to begin with, then $J(T)$ contains no comparable pair and $|J(T)| = k(T)$. In this case, $T \cong L(m)$ for $m = p_1 \ldots p_k$, and $N = N_c = \binom{k}{c}$ for $c = [k/2]$. If A_i has type $\tau_i = \text{type}\, Z_{S_i}$ for $i = 1, 2, \ldots, N$, where S_i is a c element subset of $\{p_1, \ldots, p_k\}$, then $G = G[A_1, \ldots, A_N]$ is strongly indecomposable by Theorem 1.2 and rank $G = N_c - 1$.

As a final example, let T be any lattice of types isomorphic to $L(60)$, and let τ_1, τ_2, τ_3 and τ_4 correspond to 4, 6, 10 and 15, respectively. For A_i having type τ_i, consider $G = G[A_1, A_2, A_3, A_4]$. Here $k(T) = 3$ and G is strongly indecomposable by Theorem 1.2. So, the obvious conjecture in view of Corollary 2.2 that rank $G \leq k(T) - 1$ is false. In this case $c = 2$ and $\tau_1, \tau_2, \tau_3, \tau_4$ are the $N_c = 4$ elements of rank 2.

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