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All linear and bilinear natural concomitants of vector valued differential forms

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Abstract. We give an explicit description of all linear and bilinear operators, which map tangent bundle valued differential forms to themselves and are natural under local diffeomorphism.

Keywords: Natural concomitants, natural operators, vector valued differential forms

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The aim of this paper is to determine explicitly all linear and bilinear natural operators between vector valued differential forms. So for a smooth manifold M we consider the space of vector valued differential forms, $\Omega(M;TM) = \bigoplus \Omega^p(M;TM)$, where $\Omega^p(M;TM)$ is defined to be the space of sections of the vector bundle $\Lambda^p T^*M \otimes TM$, the p-th exterior power of the cotangent bundle tensorized with the tangent bundle of M. As this is a natural bundle a local diffeomorphism $f : M \to N$ induces a pullback operator $f^* : \Omega^p(N;TN) \to \Omega^p(M;TM)$. Now a k-linear natural concomitant of vector valued differential forms is defined to be a family of k-linear operators

$$A_M : \prod_{i=1}^{k} \Omega^{p_i}(M;TM) \to \Omega^r(M;TM)$$

one for each smooth manifold M of dimension m, which satisfies the naturality condition: For any two manifolds M and N, each local diffeomorphism $f : M \to N$ and all $P_i \in \Omega^{p_i}(N;TN)$ we have:

$$A_M(f^*P_1, \dots, f^*P_k) = f^*(A_N(P_1, \dots, P_k)).$$

1. Determination of multilinear natural operators

In this section we describe a method which can be used to determine all multilinear natural concomitants of vector valued differential forms, and (with minor changes) of sections of several other natural vector bundles.

1.1. First of all each operator A_M is easily seen to be local, and thus by a multilinear version of the Peetre theorem (see e.g. [C–dW–G], [Sl] or [K–M–S]) it is of finite order over each compact subset. Since charts on M are just local diffeomorphisms from open subsets of $\mathbb{R}^m$ to M naturality implies that all operators A_M are uniquely determined by the operator $A_{\mathbb{R}^m}$ and that the order of A_M in an arbitrary point $x \in M$ is equal to the order of $A_{\mathbb{R}^m}$ in $0 \in \mathbb{R}^m$. So we may restrict the consideration to some fixed finite order n.

1.2. Now an operator $A_M : \prod_{i=1}^k \Omega^{p_i}(M;TM) \to \Omega^r(M;TM)$ of order n is induced by a vector bundle homomorphism $\bar{A}_M : \prod_{i=1}^k J^n(\Lambda^{p_i}T^*M\otimes TM) \to \Lambda^r T^*M\otimes TM$, where J^n denotes the n–th jet prolongation. In particular the operator $A_{\mathbf{R}^m}$ induces a k–linear map $A_0 : \prod_{i=1}^k V_i \to V$, where V_i denotes the fiber over $0 \in \mathbf{R}^m$ of the bundle $J^n(\Lambda^{p_i}T^*M\otimes TM)$ and V denotes the fiber over zero of $\Lambda^r T^*M\otimes TM$. We identify the fibers over zero with the standard fibers.

Now $J^n(\Lambda^{p_i}T^*M \otimes TM)$ is a natural vector bundle, and thus it determines a representation of the jet group $G_m^{n+1} := inv J_0^n(\mathbf{R}^m, \mathbf{R}^m)$, the Lie group of invertible n–jets from $\mathbf{R}^m$ to $\mathbf{R}^m$ which map 0 to 0, on its standard fiber. Clearly this determines representations of G_m^{n+1} on the standard fibers of all lower jet prolongations, too. This representations are described in detail later on.

Theorem 1.3. *There is a bijective correspondence between the set of all k-linear natural operators $A_M : \prod_{i=1}^k \Omega^{p_i}(M;TM) \to \Omega^r(M;TM)$ of order n and the set of all G_m^{n+1}-equivariant k-linear maps $A_0 : \prod_{i=1}^k V_i \to V$.*

PROOF : This is a special case of a general theorem which describes the set of natural operators between natural bundles. A proof of this theorem can be found in [K–M–S]. ∎

1.4. The first step to apply this theorem is to determine the possible orders of multilinear natural operators between vector valued differential forms.

Proposition. *If $A_M : \prod_{i=1}^k \Omega^{p_i}(M;TM) \to \Omega^r(M;TM)$ is a k-linear natural operator, then A_M is a differential operator homogeneous of total order $r - \sum_i p_i + k - 1$. In particular there is no nonzero such operator for $r \leq \sum p_i - k$.*

PROOF : This is an easy generalization of a result of [Mi]. ∎

1.5. The standard fiber of the bundle $J^n(\Lambda^{p_i}T^*M \otimes TM)$ is the vector space $\prod_{j=0}^n(\Lambda^{p_i}\mathbf{R}^{m*} \otimes \mathbf{R}^m \otimes S^j\mathbf{R}^{m*})$, where S^j denotes the j–th symmetric power, and thus the associated map to a k–linear natural operator of order n is:

$$A_0 : \prod_{i=1}^k(\prod_{j=0}^n(\Lambda^{p_i}\mathbf{R}^{m*} \otimes \mathbf{R}^m \otimes S^j\mathbf{R}^{m*})) \to \Lambda^r\mathbf{R}^{m*} \otimes \mathbf{R}^m.$$

From homogeneity of A it follows that not all factors in this large product must be considered, but A_0 splits into a sum $A_0 = \sum_{j_1+\dots+j_k=n} A_{j_1\dots j_k}$, for k–linear maps

$$A_{j_1\dots j_k} : \prod_{i=1}^k(\Lambda^{p_i}\mathbf{R}^{m*} \otimes \mathbf{R}^m \otimes S^{j_i}\mathbf{R}^{m*})) \to \Lambda^r\mathbf{R}^{m*} \otimes \mathbf{R}^m.$$

1.6. The action of the jet group.

We now describe the representations of the jet group G_m^{n+1} on the standard fibers of the bundles $J^\ell(\Lambda^p T^*M \otimes TM)$ for $\ell \leq n$, which come from the naturality of these bundles. The action of an element of G_m^{n+1} can be described as follows:

For $X \in G_m^{n+1}$ pick a representative, i.e. a diffeomorphism φ of $\mathbf{R}^m$ which maps 0 to 0 and has $n+1$–jet X at 0. By naturality of the bundle $J^\ell(\Lambda^p T^*M \otimes TM)$ the diffeomorphism φ induces a vector bundle automorphism of $J^\ell(\Lambda^p T^*\mathbf{R}^m \otimes T\mathbf{R}^m)$ which covers φ and thus maps the fiber over 0 to itself. Now since the induced map on the standard fiber of $\Lambda^p T^*\mathbf{R}^m \otimes T\mathbf{R}^m$ depends only on the 1–jet of φ at 0, the induced maps on the standard fibers of $J^\ell(\Lambda^p T^*\mathbf{R}^m \otimes T\mathbf{R}^m)$ depend only on the $\ell+1$–jet of φ at 0 and are thus independent of the choice of φ for $\ell \leq n$.

One easily verifies that this indeed defines representations of G_m^{n+1}.

This description of the representations shows two important facts:

(1) The induced representations of the subgroup $GL(m, \mathbf{R})$ are just the usual ones.

(2) Considering the elements of the standard fibers as partial derivatives at 0 of the coordinate functions of vector valued differential forms on $\mathbf{R}^m$, equivariancy under the actions of G_m^{n+1} is equivalent to equivariancy under the usual transformation laws for partial derivatives.

1.7. For the actual determination of equivariant maps we use a method which was developed by I. Kolář ([Ko]): Since $GL(m, \mathbf{R})$ is a subgroup of G_m^{n+1} each G_m^{n+1}–equivariant map must in particular be $GL(m, \mathbf{R})$–equivariant. So we take as an ansatz all $GL(m, \mathbf{R})$–equivariant maps and then check which of them are G_m^{n+1}–equivariant, too.

For the determination of all $GL(m, \mathbf{R})$–equivariant maps consider the following diagram:

$$\begin{array}{ccc}
\bigotimes_{i=1}^k(\Lambda^{p_i}\mathbf{R}^{m*} \otimes \mathbf{R}^m \otimes S^{j_i}\mathbf{R}^{m*}) & \xrightarrow{A_{j_1\dots j_k}} & \Lambda^r\mathbf{R}^{m*} \otimes \mathbf{R}^m \\
{\scriptstyle \bigotimes_{i=1}^k(Alt_{p_i}\otimes Id\otimes Symm_{j_i})}\Big\uparrow & & \Big\uparrow{\scriptstyle Alt_r\otimes Id} \\
\bigotimes_{i=1}^k(\bigotimes^{p_i}\mathbf{R}^{m*} \otimes \mathbf{R}^m \otimes \bigotimes^{j_i}\mathbf{R}^{m*}) & \xrightarrow{\varphi_{j_1\dots j_k}} & \bigotimes^r\mathbf{R}^{m*} \otimes \mathbf{R}^m.
\end{array}$$

Here *Alt* and *Symm* denote the alternator and the symmetrizer, i.e. the canonical projections from tensor powers to exterior and symmetric powers, respectively.

Since the alternator and the symmetrizer are $GL(m, \mathbf{R})$–equivariant maps one easily sees that each $GL(m, \mathbf{R})$–equivariant map $A_{j_1\dots j_k}$ is given by applying a $GL(m, \mathbf{R})$–equivariant map $\varphi_{j_1\dots j_k}$ and taking the alternator of the result.

1.8. By the classical theory of invariant tensors (see e.g. [D–C]) the vector space of all $GL(m, \mathbf{R})$–equivariant maps: $\bigotimes^{r+k-1}\mathbf{R}^{m*} \otimes \bigotimes^k \mathbf{R}^m \to \bigotimes^r \mathbf{R}^{m*} \otimes \mathbf{R}^m$ is generated by all kinds of permutations of the indices, all contractions and tensorizing with the identity $\mathbf{I} \in \mathbf{R}^{m*} \otimes \mathbf{R}^m \cong L(\mathbf{R}^m, \mathbf{R}^m)$. So by (1.7) a generating system for all $GL(m, \mathbf{R})$–equivariant maps $A_{j_1\dots j_k}$ is given by alternating these generators.

But we do not have to consider all these generators since by antisymmetry of the alternator permutations of the indices give rise to linearly dependent maps. Moreover if we perform contractions it makes no difference to contract an index into different indices of one "group" $\otimes^{p_i}\mathbf{R}^{m*}$ or $\otimes^{j_i}\mathbf{R}^{m*}$, since there the expression is antisymmetric respectively symmetric. Finally as there is just one factor $\mathbf{R}^m$ on the right hand side we have just two possibilities for the number of contractions: Either we perform $k-1$ contractions, or we perform k contractions and tensorize with the identity.

1.9. So we proceed as follows: As an ansatz for A_0 we take a linear combination of all generators for $GL(m,\mathbb{R})$–equivariant maps $A_{j_1\dots j_k}$ as described above. This can be viewed as the coordinate expression at $0\in\mathbb{R}^m$ of a k–linear operator between vector valued differential forms. Now we compute the action of a general element of G_m^{n+1} on this expression and get a system of linear equations in the coefficients of the generators which is equivalent to equivariancy of A_0.

1.10. Using the description of the generators in (1.8) we can now compute an upper bound for the possible order of k–linear natural operators between vector valued differential forms. This is simply due to the fact that if in one of the terms $S^{j_i}\mathbb{R}^{m*}$ at least two indices remain free (i.e. uncontracted), then the whole expression is symmetric in these two indices and thus becomes zero after application of the alternator. So to get a nonzero $GL(m,\mathbb{R})$–equivariant map there may only survive k indices from these terms (one for each of them). On the other hand we cannot contract more than k indices and thus the number of indices in all terms $S^{j_i}\mathbb{R}^{m*}$ which is equal to the order of the operator must be less or equal to $2k$.

2. The transformation laws for partial derivatives of vector valued differential forms

2.1. As we will see later on we will need at most 3–jets of vector valued differential forms, and thus at most the action of G_m^4. To get the usual form for this transformation laws we interpret a diffeomorphism representing an element of G_m^4 as a change of coordinates from a system $\{x^i\}$ to a system $\{\bar{x}^i\}$. For coordinate expressions we will always use Einstein sum convention.

An element of G_m^4 has a canonical representative, the Taylor polynomial up to order 4. In coordinates this is just given by:

$$\begin{aligned}\bar{x}^i &= \tfrac{\partial\bar{x}^i}{\partial x^j}(0)x^j + \tfrac{1}{2!}\tfrac{\partial^2\bar{x}^i}{\partial x^j\partial x^k}(0)x^jx^k + \tfrac{1}{3!}\tfrac{\partial^3\bar{x}^i}{\partial x^j\partial x^k\partial x^\ell}(0)x^jx^kx^\ell + \\ &\quad + \tfrac{1}{4!}\tfrac{\partial^4\bar{x}^i}{\partial x^j\partial x^k\partial x^\ell\partial x^m}(0)x^jx^kx^\ell x^m = \\ &=: A^i_jx^j + \tfrac{1}{2!}B^i_{jk}x^jx^k + \tfrac{1}{3!}C^i_{jk\ell}x^jx^kx^\ell + \tfrac{1}{4!}D^i_{jk\ell m}x^jx^kx^\ell x^m.\end{aligned}$$

We will denote this jet by $(A^i_j, \frac{1}{2!}B^i_{jk}, \frac{1}{3!}C^i_{jk\ell}, \frac{1}{4!}D^i_{jk\ell m})$. We will also need the inverse of such a jet, which will be denoted by $(\bar{A}^i_j, \frac{1}{2!}\bar{B}^i_{jk}, \frac{1}{3!}\bar{C}^i_{jk\ell}, \frac{1}{4!}\bar{D}^i_{jk\ell m})$.

2.2. First we need the transformation laws for vector valued differential forms. We write lower case greek letters for groups of free form indices. Then using the notation introduced above the transformation law reads as:

$$\bar{P}^i_\alpha = P^j_\beta A^i_j\bar{A}^\beta_\alpha, \quad \text{where } \bar{A}^\beta_\alpha := \bar{A}^{\beta_1}_{\alpha_1}\dots\bar{A}^{\beta_p}_{\alpha_p}.$$

To get the transformation laws for partial derivatives one has to differentiate this equation partially. For first order partial derivatives this gives:

$$\bar{P}^i_{\alpha,k} = P^j_{\beta,t}\bar{A}^t_k A^i_j \bar{A}^\beta_\alpha + P^j_\beta \bar{A}^t_k B^i_{tj}\bar{A}^\beta_\alpha + P^j_\beta A^i_j \bar{A}^\beta_{\alpha,k}, \quad \text{where}$$

$$\bar{A}^\beta_{\alpha,k} := \tfrac{\partial}{\partial \bar{x}^k}(\bar{A}^\beta_\alpha) = \sum_{u=1}^{p} \bar{A}^{\beta_1}_{\alpha_1}\dots\bar{A}^{\beta_{u-1}}_{\alpha_{u-1}}\bar{B}^{\beta_u}_{k\alpha_u}\bar{A}^{\beta_{u+1}}_{\alpha_{u+1}}\dots\bar{A}^{\beta_p}_{\alpha_p}.$$

For second order partial derivatives we get:

$$\begin{aligned}\bar{P}^i_{\alpha,k\ell} ={}& P^j_{\beta,st}\bar{A}^s_\ell\bar{A}^t_k A^i_j\bar{A}^\beta_\alpha + P^j_{\beta,t}\bar{B}^t_{k\ell}A^i_j\bar{A}^\beta_\alpha + P^j_{\beta,t}\bar{A}^t_k\bar{A}^s_\ell B^i_{js}\bar{A}^\beta_\alpha + P^j_{\beta,t}\bar{A}^t_k A^i_j\bar{A}^\beta_{\alpha,\ell}+\\ &+P^j_{\beta,s}\bar{A}^s_\ell\bar{A}^t_k B^i_{tj}\bar{A}^\beta_\alpha + P^j_\beta\bar{B}^t_{k\ell}B^i_{tj}\bar{A}^\beta_\alpha + P^j_\beta\bar{A}^t_k\bar{A}^s_\ell C^i_{stj}\bar{A}^\beta_\alpha + P^j_\beta\bar{A}^t_k B^i_{tj}\bar{A}^\beta_{\alpha,\ell}+\\ &+P^j_{\beta,t}\bar{A}^t_\ell A^i_j\bar{A}^\beta_{\alpha,k} + P^j_\beta\bar{A}^t_\ell B^i_{tj}\bar{A}^\beta_{\alpha,k} + P^j_\beta A^i_j\bar{A}^\beta_{\alpha,k\ell}, \quad \text{with}\end{aligned}$$

$$\begin{aligned}\bar{A}^\beta_{\alpha,k\ell} := \tfrac{\partial}{\partial\bar{x}^\ell}(\bar{A}^\beta_{\alpha,k}) ={}& \sum_{u=1}^{p}\bar{A}^{\beta_1}_{\alpha_1}\dots\bar{C}^{\beta_u}_{k\ell\alpha_u}\dots\bar{A}^{\beta_p}_{\alpha_p}+\\ &+\sum_{u\neq v}\bar{A}^{\beta_1}_{\alpha_1}\dots\bar{B}^{\beta_u}_{k\alpha_u}\dots\bar{B}^{\beta_v}_{\ell\alpha_v}\dots\bar{A}^{\beta_p}_{\alpha_p}.\end{aligned}$$

We do not need the transformation law for third order partial derivatives in full generality, but only in the special case when $B^i_{jk} = \bar{B}^i_{jk} = 0$. In this case we get:

$$\begin{aligned}\bar{P}^i_{\alpha,k\ell m} ={}& P^j_{\beta,rst}\bar{A}^r_m\bar{A}^s_\ell\bar{A}^t_k A^i_j\bar{A}^\beta_\alpha + P^j_{\beta,t}\bar{C}^t_{k\ell m}A^i_j\bar{A}^\beta_\alpha + P^j_{\beta,t}\bar{A}^t_k\bar{A}^s_\ell\bar{A}^r_m C^i_{rsj}\bar{A}^\beta_\alpha+\\ &+P^j_{\beta,t}\bar{A}^t_k A^i_j\bar{A}^\beta_{\alpha,\ell m} + P^j_{\beta,s}\bar{A}^s_\ell\bar{A}^t_k\bar{A}^r_m C^i_{rtj}\bar{A}^\beta_\alpha + P^j_{\beta,r}\bar{A}^r_m\bar{A}^t_k\bar{A}^s_\ell C^i_{stj}\bar{A}^\beta_\alpha+\\ &+P^j_\beta\bar{A}^t_k\bar{A}^s_\ell\bar{A}^r_m D^i_{rstj}\bar{A}^\beta_\alpha + P^j_{\beta,t}\bar{A}^t_\ell A^i_j\bar{A}^\beta_{\alpha,km} + P^j_{\beta,t}\bar{A}^t_m A^i_j\bar{A}^\beta_{\alpha,k\ell}+\\ &+P^j_\beta A^i_j\bar{A}^\beta_{\alpha,k\ell m}, \quad \text{where}\end{aligned}$$

$$\bar{A}^\beta_{\alpha,k\ell m} := \tfrac{\partial}{\partial\bar{x}^m}(\bar{A}^\beta_{\alpha,k\ell}) = \sum_{u=1}^{p}\bar{A}^{\beta_1}_{\alpha_1}\dots\bar{D}^{\beta_u}_{k\ell m\alpha_u}\dots\bar{A}^{\beta_p}_{\alpha_p}.$$

2.3. Next we have to express the inverse jet $(\bar{A},\bar{B},\bar{C},\bar{D})$ in terms of the jet (A,B,C,D). Clearly $\bar{A}$ is the inverse matrix to A and thus $\bar{A}^i_t A^t_j = \delta^i_j$, the Kronecker delta. Iterated partial differentiation of this equation gives

$$\begin{aligned}0 &= \bar{B}^i_{tk}A^t_j + \bar{A}^i_t\bar{A}^s_k B^t_{sj}\\ 0 &= \bar{C}^i_{tk\ell}A^t_j + \bar{B}^i_{tk}\bar{A}^s_\ell B^t_{sj} + \bar{B}^i_{t\ell}\bar{A}^s_k B^t_{sj} + \bar{A}^i_t\bar{B}^s_{k\ell}B^t_{sj} + \bar{A}^i_t\bar{A}^s_k\bar{A}^r_\ell C^t_{rsj}.\end{aligned}$$

The formula for $\bar{D}$ is again only interesting for us in the special case $B^i_{jk} = \bar{B}^i_{jk} = 0$. In this case we get:

$$0 = \bar{D}^i_{t\ell km}A^t_j + \bar{A}^i_t\bar{A}^s_k\bar{A}^r_\ell\bar{A}^u_m D^t_{jrsu}.$$

2.4. In (1.9) we saw that we need the transformation laws to decide whether a $GL(m,\mathbb{R})$–equivariant map is G^4_m–equivariant, too. So without loss of generality we may always assume that A is the identity matrix. This simplifies the transformation laws. We will simplify the situation more by checking equivariancy for certain subsets of G^4_m separately. Therefore we define the following subsets of G^4_m:

$$\tilde{G}^2_m := \{(A,B,C,D) \in G^4_m : A^i_j = \delta^i_j, C^i_{jk\ell} = D^i_{jk\ell m} = 0 \quad \forall i,j,k,\ell,m\}$$
$$\tilde{G}^3_m := \{(A,B,C,D) \in G^4_m : A^i_j = \delta^i_j, B^i_{jk} = D^i_{jk\ell m} = 0 \quad \forall i,j,k,\ell,m\}$$
$$\tilde{G}^4_m := \{(A,B,C,D) \in G^4_m : A^i_j = \delta^i_j, B^i_{jk} = C^i_{jk\ell} = 0 \quad \forall i,j,k,\ell\}.$$

Easy computations show that for elements of these subsets the transformation laws are given as follows:

Proposition 2.5. *For $(\delta^i_j, B^i_{jk}, 0, 0) \in \tilde{G}^2_m$ we have $\bar{B}^i_{jk} = -B^i_{jk}$ and $\bar{C}^i_{jk\ell} = B^i_{jt}B^t_{k\ell} + B^i_{kt}B^t_{j\ell} + B^i_{\ell t}B^t_{jk}$, and the transformation laws are:*

$$\bar{P}^i_\alpha = P^i_\alpha \tag{1}$$

$$\bar{P}^i_{\alpha,k} = P^i_{\alpha,k} + P^j_\alpha B^i_{jk} - \sum_{u=1}^{p} P^i_{\alpha_1\dots\alpha_{u-1}t\alpha_{u+1}\dots\alpha_p} B^t_{k\alpha_u} \tag{2}$$

$$\begin{aligned}\bar{P}^i_{\alpha,k\ell} =& P^i_{\alpha,k\ell} - P^i_{\alpha,t}B^t_{k\ell} + P^j_{\alpha,k}B^i_{j\ell} + P^j_{\alpha,\ell}B^i_{jk} - P^j_\alpha B^i_{jt}B^t_{k\ell} - \\ &- \sum_{u=1}^{p} P^i_{\alpha_1\dots\alpha_{u-1}t\alpha_{u+1}\dots\alpha_p,k} B^t_{\ell\alpha_u} - \sum_{u=1}^{p} P^i_{\alpha_1\dots\alpha_{u-1}t\alpha_{u+1}\dots\alpha_p,\ell} B^t_{k\alpha_u} - \\ &- \sum_{u=1}^{p} P^j_{\alpha_1\dots\alpha_{u-1}t\alpha_{u+1}\dots\alpha_p} B^i_{jk}B^t_{\ell\alpha_u} - \sum_{u=1}^{p} P^j_{\alpha_1\dots\alpha_{u-1}t\alpha_{u+1}\dots\alpha_p} B^i_{j\ell}B^t_{k\alpha_u} + \\ &+ \sum_{u=1}^{p} P^i_{\alpha_1\dots\alpha_{u-1}t\alpha_{u+1}\dots\alpha_p} B^t_{ks}B^s_{\ell\alpha_u} + \sum_{u=1}^{p} P^i_{\alpha_1\dots\alpha_{u-1}t\alpha_{u+1}\dots\alpha_p} B^t_{\ell s}B^s_{k\alpha_u} + \\ &+ \sum_{u=1}^{p} P^i_{\alpha_1\dots\alpha_{u-1}t\alpha_{u+1}\dots\alpha_p} B^t_{s\alpha_u}B^s_{k\ell} + \sum_{u\neq v} P^i_{\alpha_1\dots t\dots s\dots\alpha_p} B^t_{k\alpha_u}B^s_{\ell\alpha_v}.\end{aligned} \tag{3}$$

Proposition 2.6. *For $(\delta^i_j, 0, C^i_{jk\ell}, 0) \in \tilde{G}^3_m$ we have $\bar{B}^i_{jk} = 0$, $\bar{C}^i_{jk\ell} = -C^i_{jk\ell}$ and $\bar{D}^i_{jk\ell m} = 0$, and the transformation laws are:*

$$\bar{P}^i_\alpha = P^i_\alpha \tag{1}$$

$$\bar{P}^i_{\alpha,k} = P^i_{\alpha,k} \tag{2}$$

$$\bar{P}^i_{\alpha,k\ell} = P^i_{\alpha,k\ell} + P^j_\alpha C^i_{jk\ell} - \sum_{u=1}^{p} P^i_{\alpha_1\dots\alpha_{u-1}t\alpha_{u+1}\dots\alpha_p} C^t_{k\ell\alpha_u} \tag{3}$$

$$\begin{aligned}\bar{P}^i_{\alpha,k\ell m} =& P^i_{\alpha,k\ell m} - P^i_{\alpha,t}C^t_{k\ell m} + P^j_{\alpha,k}C^i_{j\ell m} + P^j_{\alpha,\ell}C^i_{jkm} + P^j_{\alpha,m}C^i_{jk\ell} - \\ &- \sum_{u=1}^{p} P^i_{\alpha_1\dots\alpha_{u-1}t\alpha_{u+1}\dots\alpha_p,k} C^t_{\ell m\alpha_u} - \sum_{u=1}^{p} P^i_{\alpha_1\dots\alpha_{u-1}t\alpha_{u+1}\dots\alpha_p,\ell} C^t_{km\alpha_u} - \\ &- \sum_{u=1}^{p} P^i_{\alpha_1\dots\alpha_{u-1}t\alpha_{u+1}\dots\alpha_p,m} C^t_{k\ell\alpha_u}.\end{aligned} \tag{4}$$

Proposition 2.7. *For $(\delta^i_j, 0, 0, D^i_{jk\ell m}) \in \tilde{G}^4_m$ we have $\bar{B}^i_{jk} = 0$, $\bar{C}^i_{jk\ell} = 0$ and $\bar{D}^i_{jk\ell m} = -D^i_{jk\ell m}$, and the transformation laws are:*

$$\bar{P}^i_\alpha = P^i_\alpha \tag{1}$$

$$\bar{P}^i_{\alpha,k} = P^i_{\alpha,k} \tag{2}$$

$$\bar{P}^i_{\alpha,k\ell} = P^i_{\alpha,k\ell} \tag{3}$$

$$\bar{P}^i_{\alpha,k\ell m} = P^i_{\alpha,k\ell m} + P^j_\alpha D^i_{jk\ell m} - \sum_{u=1}^{p} P^i_{\alpha_1\dots\alpha_{u-1}t\alpha_{u+1}\dots\alpha_p} D^t_{k\ell m\alpha_u}. \tag{4}$$

2.8. Until now we neither used the fact, that the terms we consider have antisymmetric as well as symmetric indices, nor that we will apply the alternator, which acts on all free lower indices. In several situations these facts simplify the transformation laws again. As an example take formula (2.7)(4): From antisymmetry of the alternator and symmetry in the lower indices of D one easily concludes that all terms in the sum for which α_u is free become equal and if moreover one of the derivation indices k, ℓ, m is free they all vanish.

We will use all simplifications later on without reference.

3. Linear and bilinear natural concomitants of vector valued differential forms

3.1. To formulate our results we have to define several operators which act on vector valued differential forms. Since all operators we are going to use are local and each form $P \in \Omega^p(M;TM)$ can be written locally as a finite sum of decomposable forms $\varphi \otimes X$ where $\varphi \in \Omega^p(M)$ is an ordinary p-form and $X \in \mathcal{X}(M)$ is a vector field we define all operators only on decomposable forms and require that they are linearly respectively bilinearly extended.

Definition 3.2.

(1) The Frölicher–Nijenhuis bracket:
$[\ ,\ ] : \Omega^p(M;TM) \times \Omega^q(M;TM) \to \Omega^{p+q}(M;TM)$.
For $\varphi \in \Omega^p(M)$, $\psi \in \Omega(M)$ and $X, Y \in \mathcal{X}(M)$ it is given by:

$$\begin{aligned}[\varphi \otimes X, \psi \otimes Y] =& \varphi \wedge \psi \otimes [X,Y] + \varphi \wedge \mathcal{L}(X)(\psi) \otimes Y - \mathcal{L}(Y)(\varphi) \wedge \psi \otimes X + \\ &+ (-1)^p (d\varphi \wedge i(X)(\psi) \otimes Y + i(Y)(\varphi) \wedge d\psi \otimes X).\end{aligned}$$

For a more algebraic definition and the properties of this bracket see [Mi].

(2) The insertion operator:
$i : \Omega^p(M;TM) \times \Omega^q(M;TM) \to \Omega^{p+q-1}(M;TM)$ defined by $i(\varphi \otimes X)(\psi \otimes Y) = \varphi \wedge i(X)(\psi) \otimes Y$. Here $i(X)$ denotes the usual insertion operator for vector fields. Note that this definition makes also sense with an arbitrary vector bundle valued differential form in the second position.

(3) The contraction $C : \Omega^p(M;TM) \to \Omega^{p-1}(M)$, defined by $C(\varphi \otimes X) = i(X)(\varphi)$.

(4) The symmetric contraction $S : \Omega^p(M;TM) \times \Omega^q(M;TM) \to \Omega^{p+q-2}(M)$ defined by $S(\varphi \otimes X, \psi \otimes Y) = i(Y)(\varphi) \wedge i(X)(\psi)$.

3.3. All these operators are natural under local diffeomorphisms. For the Frölicher–Nijenhuis bracket this is shown in [Mi] and for the other operators it is easily verified.

Moreover we use that $\Omega(M;TM)$ is a graded module over the graded commutative algebra $\Omega(M)$ under the action $\varphi \wedge (\psi \otimes X) = (\varphi \wedge \psi) \otimes X$, linearly extended.

By I we denote the identity Id_{TM}, viewed as an element of $\Omega^1(M;TM)$, and by d we denote the exterior derivative of differential forms.

3.4. Let us now start with the determination of all linear natural concomitants of vector valued differential forms. So we are looking for natural operators $A_M : \Omega^p(M;TM) \to \Omega^r(M;TM)$. By (1.4) and (1.10) these operators have to be of order 0, 1 or 2 and this corresponds to the cases $r = p$, $p+1$ and $p+2$, respectively. Let us start with the algebraic case $r = p$.

Theorem 3.5.

(1) *If $dim(M) = 0$ or $m := dim(M) < p$, then there is no nonzero natural linear operator $A : \Omega^p(M;TM) \to \Omega^p(M;TM)$.*

(2) *If $p = 0$ and hence $\Omega^p(M;TM) = \mathcal{X}(M)$ and $m \geq 1$ or if $p \geq 1$ and $m = p$, then each linear natural operator on $\Omega^p(M;TM)$ is a scalar multiple of the identity.*

(3) *If $p \geq 1$ and $dim(M) > p$, then the natural linear operators on $\Omega^p(M;TM)$ form a two dimensional vector space, linearly generated by the identity and $C(P) \wedge \mathsf{I}$.*

PROOF : (1) is clear, since $\Omega^p(M;TM) = 0$ in this case.

Clearly the identity and $P \mapsto C(P) \wedge \mathsf{I}$ are indeed natural, and inserting special forms one immediately sees that for $p \geq 1$, $m > p$ they are linearly independent, while for $p \geq 1$, $m = p$ one has $C(P) \wedge \mathsf{I} = (-1)^{p-1} P$.

Thus we only have to show that there cannot be more linear natural operators. According to (1.3) and (1.5) we can do this by determining all $GL(m,\mathbb{R})$–equivariant maps $A_0 : \Lambda^p\mathbb{R}^{m*} \otimes \mathbb{R}^m \to \Lambda^p\mathbb{R}^{m*} \otimes \mathbb{R}^m$. By (1.8) all these maps are linear combinations of the identity and of maps obtained by performing one contraction, tensorizing with I and then alternating the result. Thus A_0 must be of the form:

$$(A_0(P))^i_{\alpha_1 \dots \alpha_p} = aP^i_{\alpha_1 \dots \alpha_p} + bP^m_{m\alpha_1 \dots \alpha_{p-1}} \delta^i_{\alpha_p}$$

(Alternation is not explicitly indicated.) But these are (up to a sign) the coordinate expressions of P and $C(P) \wedge \mathsf{I}$. Moreover it is clear, that for $p = 0$ only the first term makes sense. ∎

Remark. In the sequel we will write expressions like $P^m_{m\alpha_1 \dots \alpha_{p-1}} \delta^i_{\alpha_p}$ as $P^m_{m\alpha} \delta^i_\ell$, since it is always clear what the values of α and ℓ must be.

3.6. Next we consider the case $r = p + 1$ and hence A homogeneous of order 1.

Theorem.

(1) *If $p = 0$ or $m := dim(M) \leq p$, then there is no nonzero linear natural operator $A : \Omega^p(M;TM) \to \Omega^{p+1}(M;TM)$.*

(2) *If $p \geq 1$ and $m \geq p+1$, then each natural linear operator $A : \Omega^p(M;TM) \to \Omega^{p+1}(M;TM)$ is a scalar multiple of $dC(P) \wedge \mathsf{I}$.*

PROOF : From (3.3) it is clear that $P \mapsto dC(P) \wedge \mathsf{I}$ is indeed natural and it is easily seen not to be identically zero if $p \geq 1$ and $m \geq p+1$.

To prove that there are not more linear natural operators we determine all G^2_m–equivariant maps $A_0 : \Lambda^p \mathbf{R}^{m*} \otimes \mathbf{R}^m \otimes \mathbf{R}^{m*} \to \Lambda^{p+1}\mathbf{R}^{m*} \otimes \mathbf{R}^m$. According to (1.9) we start with $GL(m,\mathbf{R})$–equivariant maps, and then check, which linear combinations of them are G^2_m–equivariant, too. By (1.8) these maps can be obtained either by simply alternating, or by performing one contraction, tensorizing with I and then alternating. There are two possibilities to perform one contraction: We may contract the upper index either into a form index (without loss of generality into the first one), or into the derivation index. Thus we must have:

$$(*) \qquad A(P)^i_\gamma = aP^i_{\alpha,k} + bP^m_{m\alpha,k}\delta^i_\ell + cP^m_{\alpha,m}\delta^i_\ell.$$

(Alternation is not explicitly indicated.)

Now we check, which values of a, b and c give rise to $\tilde{G}^2_m$ equivariant maps. Thus we need the transformation laws (1) and (2) of Proposition (2.5). From these we see immediately that the left hand side of $(*)$ is $\tilde{G}^2_m$ invariant, i.e. $\overline{A(P)}^i_\gamma = A(P)^i_\gamma$, and thus equivariancy is equivalent to $A(\bar{P})^i_\gamma = A(P)^i_\gamma$. Now we have:

$$\begin{aligned} A(\bar{P})^i_\gamma =& a(P^i_{\alpha,k} + P^j_\alpha B^i_{jk}) + bP^m_{m\alpha,k}\delta^i_\ell + \\ &+ c(P^m_{\alpha,m}\delta^i_\ell + P^j_\alpha B^m_{jm}\delta^i_\ell - pP^m_{\alpha t}B^t_{m\alpha_p}\delta^i_\ell) \end{aligned}$$

and thus equivariancy is equivalent to:

$$(**) \qquad 0 = aP^j_\alpha B^i_{jk} + cP^j_\alpha B^m_{jm}\delta^i_\ell - cpP^m_{\alpha t}B^t_{m\alpha_p}\delta^i_\ell$$

while the parameter b remains free. The right hand side of this equation represents a bilinear map:

$$\Lambda^p\mathbf{R}^{m*} \otimes \mathbf{R}^m \times S^2\mathbf{R}^{m*} \otimes \mathbf{R}^m \to \Lambda^{p+1}\mathbf{R}^{m*} \otimes \mathbf{R}^m.$$

Now one has to discuss equation $(**)$ for the different values of p and m:

First for $p = 0$ the term corresponding to b in $(*)$ as well as the last term in $(**)$ does not occur. For $m > 1$ equation $(**)$ immediately implies $a = c = 0$, while for $m = 1$ it only gives $c = -a$. But in this case the maps corresponding to a and c are linearly dependent and $c = -a$ gives the zero map. So the first part of the theorem is proved.

Next for $p > 1$, $m \geq p+1$ and for $p = 1$, $m > 2$ equation $(**)$ again leads to $a = c = 0$, while for $p = 1$, $m = 2$ one has to show that the maps in the ansatz $(*)$ become linearly dependent and then deduces that there is only one free parameter.

So for $p \geq 1$, $m \geq p+1$ we have in any case exactly one free parameter b, and thus the dimension of the vector space of natural operators is at most one, and the proof is complete. ■

Theorem 3.7. *There is no nonzero natural linear operator:*

$$A : \Omega^p(M;TM) \to \Omega^{p+2}(M;TM).$$

PROOF : We show, that there is no nonzero G^3_m equivariant linear map

$$A_0 : \Lambda^p\mathbb{R}^{m*} \otimes \mathbb{R}^m \otimes S^2\mathbb{R}^{m*} \to \Lambda^{p+2}\mathbb{R}^{m*} \otimes \mathbb{R}^m.$$

In this case P has two derivation indices, and from (1.10) we know that due to symmetry at most one of them may be free if the resulting map is nonzero. Thus there is only one possibility to get a nonzero $GL(m,\mathbb{R})$ equivariant map, namely to contract the upper index of P into one of the derivation indices, tensorize with $\mathbb{I}$ and then alternate the result. So we must have:

$$(*) \qquad A(P)^i_\gamma = aP^m_{\alpha,km}\delta^i_\ell.$$

Now we check, whether this is $\tilde{G}^3_m$ equivariant. Thus we need the transformation laws (1) and (3) of Proposition (2.6). The left hand side of $(*)$ is $\tilde{G}^3_m$ invariant, and the first term of (2.6)(3) reproduces the original map. So equivariancy is equivalent to $0 = aP^j_\alpha C^m_{jkm}$, and thus clearly to $a = 0$. ■

3.8. Let us now turn to the bilinear case. We are looking for bilinear natural operators

$$A_M : \Omega^p(M;TM) \times \Omega^q(M;TM) \to \Omega^r(M;TM).$$

By (1.10) A_M must be homogeneous of total order between 0 and 4, corresponding to $r = p+q-1,\dots,p+q+3$. Again we start with the case of algebraic operators, i.e. $r = p+q-1$.

Theorem 3.9.

(1) *If $p = q = 0$ or $m := dim(M) < p+q-1$, then there is no nonzero bilinear natural operator $\Omega^p(M;TM) \times \Omega^q(M;TM) \to \Omega^{p+q-1}(M;TM)$*

(2) *For all other values of p, q and m these operators form a vector space linearly generated by the following eight operators:*
$S(P,Q) \wedge \mathbb{I}$, $C(P) \wedge C(Q) \wedge \mathbb{I}$, $i(P)(C(Q)) \wedge \mathbb{I}$, $i(Q)(C(P)) \wedge \mathbb{I}$, $C(P) \wedge Q$, $C(Q) \wedge P$, $i(P)(Q)$ *and* $i(Q)(P)$.
Moreover if $p,q \geq 2$ and $m > p+q$, then these operators form a basis, i.e. are linearly independent.

PROOF : (1) is trivial.
(2): From (3.3) it follows that the operators under consideration are indeed natural, and inserting special forms one shows that they are linearly independent if $p,q \geq 2$ and $m > p+q$.
To show that these operators form a generating system we determine all $GL(m,\mathbb{R})$–equivariant linear maps

$$A_0 : \Lambda^p\mathbb{R}^{m*} \otimes \mathbb{R}^m \otimes \Lambda^q\mathbb{R}^{m*} \otimes \mathbb{R}^m \to \Lambda^{p+q-1}\mathbb{R}^{m*} \otimes \mathbb{R}^m.$$

From (1.8) we know, that these maps are generated by maps obtained by performing one contraction and then alternating the result, or by performing two contractions, tensorizing with $\mathbb{I}$ and then alternating the result. Clearly we have four possibilities to perform one contraction (P into P, P into Q, Q into P and Q into Q), and four possibilities to perform two contractions (both into P, both into Q, each into itself and P into Q and Q into P). So we have eight generators, and they are immediately seen to be (up to signs) the coordinate expressions of the operators listed in the theorem. ∎

3.10. Now we turn to the (probably most interesting) case $r = p + q$ and hence operators homogeneous of total order one. In this case several results were already known: For $p = q = r = 0$ and hence $A : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ unicity of the Lie bracket was proved by D. Krupka and V. Mikolášová ([**Kr-M**]), S. van Strien ([**vS**]) and in a stronger (infinitesimal) sense by M. de Wilde and P. Lecomte ([**dW-L**]). The general result for $p, q \geq 2$ and $dim(M) \geq p + q + 2$ was proved by I. Kolář and P. Michor ([**K-M**]).

Theorem.

(1) *If $m := dim(M) = 0$ or $m < p + q$ then there is no nonzero bilinear natural operator $\Omega^p(M;TM) \times \Omega^q(M;TM) \to \Omega^{p+q}(M;TM)$*

(2) *If $m \geq p + q$ then the these operators form a vector space linearly generated by the following ten operators:*
$[P,Q]$, $dC(P) \wedge Q$, $dC(Q) \wedge P$, $dC(P) \wedge C(Q) \wedge \mathbb{I}$, $C(P) \wedge dC(Q) \wedge \mathbb{I}$, $i(P)(dC(Q)) \wedge \mathbb{I}$, $i(Q)(dC(P)) \wedge \mathbb{I}$, $d(i(P)(C(Q))) \wedge \mathbb{I}$, $d(i(Q)(C(P))) \wedge \mathbb{I}$ *and* $dS(P,Q) \wedge \mathbb{I}$.
These operators form a basis if $p, q \geq 2$ and $m \geq p + q + 1$.

(3) *For all values of p and q, if $m \geq p + q + 1$, then a basis is given by those operators from the list above, that make sense for a p-form P and a q-form Q.*

(4) *If $m = p + q$, then a basis for each value of p and q is given in the list below:*

p	q	*basis*
0	1	$[P,Q]$
0	≥ 2	$[P,Q], dC(Q) \wedge P$
1	1	$[P,Q], dC(P) \wedge Q, dC(Q) \wedge P, dC(P) \wedge C(Q) \wedge \mathbb{I}$, $C(P) \wedge dC(Q) \wedge \mathbb{I}$
1	≥ 2	$[P,Q], dC(P) \wedge Q, dC(Q) \wedge P, dC(P) \wedge C(Q) \wedge \mathbb{I}$, $C(P) \wedge dC(Q) \wedge \mathbb{I}, d(i(P)(C(Q))) \wedge \mathbb{I}$
≥ 2	≥ 2	$[P,Q], dC(P) \wedge Q, dC(Q) \wedge P, dC(P) \wedge C(Q) \wedge \mathbb{I}$, $C(P) \wedge dC(Q) \wedge \mathbb{I}, d(i(P)(C(Q))) \wedge \mathbb{I}, d(i(Q)(C(P))) \wedge \mathbb{I}$.

PROOF : (1) is trivial.

All linear independence results can be proved by inserting special elements.

For the rest we only consider the case $p,q \geq 2$ and indicate how to prove the remaining cases. This case was treated with the method we use in [K–M]. It is shown there that the ansatz is:

$$
\begin{aligned}
A_0(P,Q)^i_\gamma = & aP^m_{m\alpha,k}Q^n_{n\beta}\delta^i_\ell + bP^m_{\alpha,m}Q^n_{n\beta}\delta^i_\ell + cP^m_{\alpha,k}Q^n_{nm\beta}\delta^i_\ell + \\
& + dP^m_{mn\alpha,k}Q^n_\beta\delta^i_\ell + eP^m_{n\alpha,m}Q^n_\beta\delta^i_\ell + fP^m_{n\alpha,k}Q^n_{m\beta}\delta^i_\ell + \\
& + gP^m_{m\alpha,n}Q^n_\beta\delta^i_\ell + hP^m_{\alpha,n}Q^n_{m\beta}\delta^i_\ell + \\
& + iP^m_{m\alpha,k}Q^i_\beta + jP^m_{\alpha,m}Q^i_\beta + kP^m_{\alpha,k}Q^i_{m\beta} + \\
(*) \qquad & + \ell P^i_{\alpha,k}Q^n_{n\beta} + mP^i_{n\alpha,k}Q^n_\beta + nP^i_{\alpha,n}Q^n_\beta + \\
& + AP^m_{m\alpha}Q^n_{n\beta,k}\delta^i_\ell + BP^m_{m\alpha}Q^n_{\beta,n}\delta^i_\ell + CP^m_{mn\alpha}Q^n_{\beta,k}\delta^i_\ell + \\
& + DP^m_\alpha Q^n_{nm\beta,k}\delta^i_\ell + EP^m_\alpha Q^n_{m\beta,n}\delta^i_\ell + FP^m_{n\alpha}Q^n_{m\beta,k}\delta^i_\ell + \\
& + GP^m_\alpha Q^n_{n\beta,m}\delta^i_\ell + HP^m_{n\alpha}Q^n_{\beta,m}\delta^i_\ell + \\
& + IP^i_\alpha Q^n_{n\beta,k} + JP^i_\alpha Q^n_{\beta,n} + KP^i_{n\alpha}Q^n_{\beta,k} + \\
& + LP^m_{m\alpha}Q^i_{\beta,k} + MP^m_\alpha Q^i_{m\beta,k} + NP^m_\alpha Q^i_{\beta,m}.
\end{aligned}
$$

(Alternation is not explicitly indicated, δ^i_ℓ is the coordinate expression of I and lower case greek letters represent different groups of free form indices, whose actual values in each term are clear from the context.)

It is then shown there, that for $m > p+q+1$ G^2_m–equivariancy of A_0 is equivalent to: $b = e = h = j = \ell = m = B = E = H = J = L = M = 0$.
$a, c, f, g, i, n, A, C, G, I$ are ten free parameters, which determine all other parameters uniquely by the following equations:

$$
\begin{gathered}
d = (-1)^{p-1}(p-1)g + (-1)^q C, \quad k = -qn \\
D = (-1)^q c + (-1)^{q-1}(q-1)G, \quad F = (-1)^{q-1}f, \quad K = (-1)^{p+q-1}pn, \quad N = -n.
\end{gathered}
$$

A detailed analysis of the equations derived in [K–M] shows that this continues to hold for $m = p+q+1$.

Thus the dimension of the vector space of bilinear natural operators is at most ten, and the theorem holds in this case.

Moreover the free parameters listed above correspond (up to signs) directly to the operators listed in the theorem as follows:

$$
\begin{array}{lll}
a \leftrightarrow dC(P) \wedge C(Q) \wedge \mathsf{I}, & c \leftrightarrow d(i(P)(C(Q))) \wedge \mathsf{I}, & f \leftrightarrow dS(P,Q) \wedge \mathsf{I}, \\
g \leftrightarrow i(Q)(dC(P)) \wedge \mathsf{I}, & i \leftrightarrow dC(P) \wedge Q, & n \leftrightarrow [P,Q], \\
A \leftrightarrow C(P) \wedge dC(Q) \wedge \mathsf{I}, & C \leftrightarrow d(i(Q)(C(P))) \wedge \mathsf{I}, & G \leftrightarrow i(P)(dC(Q)) \wedge \mathsf{I}, \\
I \leftrightarrow dC(Q) \wedge P. & &
\end{array}
$$

This is easily verified by calculating the coordinate expressions of the operators. ∎

Remark. In [K–M] the generator $dS(P,Q)\wedge \mathbb{I}$ was replaced by $dC(i(P)(Q))\wedge \mathbb{I}$. A short computation shows that these two operators are related as:

$$dC(i(P)(Q))\wedge \mathbb{I} = dS(P,Q)\wedge \mathbb{I} + (-1)^{p+1} d(i(P)(C(Q)))\wedge \mathbb{I}.$$

I chose $dS(P,Q)\wedge \mathbb{I}$ here since it corresponds directly to the parameter f.

3.11. Let us next discuss the case $p,q\geq 2$, $m=p+q$.

Lemma. *In this case the following equations hold:*
(1) $P^m_{\alpha,m}Q^n_{n\beta}\delta^i_\ell = (-1)^{p-1}P^m_{m\alpha,k}Q^n_{n\beta}\delta^i_\ell + P^m_{\alpha,k}Q^n_{nm\beta}\delta^i_\ell + (-1)^{q-1}P^i_{\alpha,k}Q^n_{n\beta}$
(2) $P^m_{\alpha,m}Q^i_\beta = (-1)^{p-1}P^m_{m\alpha,k}Q^i_\beta + P^m_{\alpha,k}Q^i_{m\beta}$
(3) $P^m_{n\alpha,m}Q^n_\beta\delta^i_\ell = (-1)^{p-1}P^m_{\alpha,m}Q^n_{n\beta}\delta^i_\ell + (-1)^{p+q-1}P^m_{\alpha,m}Q^i_\beta$
(4) $P^m_{m\alpha,n}Q^n_\beta\delta^i_\ell = P^m_{m\alpha,k}Q^n_{n\beta}\delta^i_\ell + (-1)^p P^m_{mn\alpha,k}Q^n_\beta\delta^i_\ell + (-1)^q P^m_{m\alpha,k}Q^i_\beta$
(5) $P^m_{\alpha,n}Q^n_{m\beta}\delta^i_\ell = (-1)^{p-1}P^m_{m\alpha,n}Q^n_\beta\delta^i_\ell + (-1)^{q-1}P^i_{\alpha,n}Q^n_\beta$
(6) $P^m_{n\alpha,k}Q^n_{m\beta}\delta^i_\ell = (-1)^{p-1}P^m_{\alpha,n}Q^n_{m\beta}\delta^i_\ell + (-1)^{p-1}P^m_{\alpha,k}Q^n_{nm\beta}\delta^i_\ell + (-1)^{p+q-1}P^m_{\alpha,k}Q^i_{m\beta}$
(7) $P^i_{n\alpha,k}Q^n_\beta = (-1)^p P^i_{\alpha,k}Q^n_{n\beta} + (-1)^{p-1}P^i_{\alpha,n}Q^n_\beta$.

PROOF : All these equations can be proved by inserting basis elements for P. For $\varphi := dx^1\wedge\cdots\wedge dx^p \in \Omega^p(\mathbb{R}^m)$ it suffices to consider the basis elements $\varphi\otimes\frac{\partial}{\partial x^1}\otimes dx^1$, $\varphi\otimes\frac{\partial}{\partial x^2}\otimes dx^1$, $\varphi\otimes\frac{\partial}{\partial x^{p+1}}\otimes dx^1$, $\varphi\otimes\frac{\partial}{\partial x^1}\otimes dx^{p+1}$, $\varphi\otimes\frac{\partial}{\partial x^{p+1}}\otimes dx^{p+1}$ and $\varphi\otimes\frac{\partial}{\partial x^{p+2}}\otimes dx^{p+1}$ for P, since any other basis element can be brought to one of these forms by simply renumbering the coordinates. Inserting each of these elements into the equations one then computes directly that they hold for arbitrary basis elements inserted for Q. ∎

3.12. Looking at the ansatz $(*)$ in (3.10) we see that Lemma (3.11) implies that all maps in it, which correspond to lower case letters are linearly generated by those which correspond to the parameters a, c, d, i, k, l and n. Clearly similar equations hold for the maps in which Q is differentiated, and thus we may reduce the ansatz to:

$$\begin{aligned} A_0(P,Q)^i_\gamma =& aP^m_{m\alpha,k}Q^n_{n\beta}\delta^i_\ell + cP^m_{\alpha,k}Q^n_{nm\beta}\delta^i_\ell + dP^m_{mn\alpha,k}Q^n_\beta\delta^i_\ell + \\ &+ iP^m_{m\alpha,k}Q^i_\beta + kP^m_{\alpha,k}Q^i_{m\beta} + \ell P^i_{\alpha,k}Q^n_{n\beta} + nP^i_{\alpha,n}Q^n_\beta + \\ &+ AP^m_{m\alpha}Q^n_{n\beta,k}\delta^i_\ell + CP^m_{mn\alpha}Q^n_{\beta,k}\delta^i_\ell + DP^m_\alpha Q^n_{nm\beta,k}\delta^i_\ell + \\ &+ IP^i_\alpha Q^n_{n\beta,k} + KP^i_{n\alpha}Q^n_{\beta,k} + LP^m_{m\alpha}Q^i_{\beta,k} + NP^m_\alpha Q^i_{\beta,m}. \end{aligned}$$

But now repeating the computations of [K–M] it is quite easy to show that for $m=p+q$ G^2_m–equivariancy of A_0 is equivalent to: $\ell = L = 0$,
a, c, i, n, A, C and I are seven free parameters, which determine all other parameters uniquely by the equations:

$$d=(-1)^q C,\quad k=-qn,\quad D=(-1)^q c,\quad K=(-1)^{p+q-1}pn,\quad N=-n.$$

Thus Theorem (3.10) continues to hold in this case.

3.13. To finish the discussion of Theorem (3.10) we indicate how to prove the remaining cases:

First one takes those maps from (3.10)($*$), that make sense for p– and q–forms. Then if $m \geq p+q+2$ the proof is like for $p, q \geq 2$, $m \geq p+q+1$. If $m = p+q+1$, then one shows, that two maps in A_0 can be expressed as a linear combination of the other ones. Eliminating these maps the proof works as before. Finally for $m = p+q$ one shows, that one may eliminate several maps from (3.10)($*$), and then again the proof works.

3.14. Next we consider second order bilinear operators, i.e. the case $r = p+q+1$. From now on all results seem to be new.

Theorem.

(1) *If $p = q = 0$ or $m := dim(M) < p+q+1$, then there is no nonzero bilinear natural operator $A : \Omega^p(M; TM) \times \Omega^q(M; TM) \to \Omega^{p+q+1}(M; TM)$.*

(2) *For all other values of p, q and m these operators form a vector space, linearly generated by the operators*

$$dC(P) \wedge dC(Q) \wedge \mathsf{I}, \quad d(i(P)(dC(Q))) \wedge \mathsf{I} \quad \textit{and} \quad d(i(Q)(dC(P))) \wedge \mathsf{I}$$

and these three operators form a basis if $p, q \geq 1$ (and $m \geq p+q+1$).

(3) *If $p = 0$, $q \geq 1$ and $m \geq q+1$, then each bilinear natural operator*

$$A : \mathcal{X}(M) \times \Omega^q(M; TM) \to \Omega^{q+1}(M; TM)$$

is a scalar multiple of $d(i(P)(dC(Q))) \wedge \mathsf{I}$.

PROOF : Naturality of the operators occurring in the theorem follows from (3.3), and they are easily seen to be linearly independent for $p, q \geq 1$ and $m \geq p+q+1$ by inserting special elements for P and Q.

To show that there are not more bilinear natural operators we determine all G^3_m equivariant maps $A = A_{20} + A_{11} + A_{02}$, where

$$\begin{aligned} A_{20} &: \Lambda^p\mathbb{R}^{m*} \otimes \mathbb{R}^m \otimes S^2\mathbb{R}^{m*} \otimes \Lambda^q\mathbb{R}^{m*} \otimes \mathbb{R}^m \to \Lambda^{p+q+1}\mathbb{R}^{m*} \otimes \mathbb{R}^m \\ A_{11} &: \Lambda^p\mathbb{R}^{m*} \otimes \mathbb{R}^m \otimes \mathbb{R}^{m*} \otimes \Lambda^q\mathbb{R}^{m*} \otimes \mathbb{R}^m \otimes \mathbb{R}^{m*} \to \Lambda^{p+q+1}\mathbb{R}^{m*} \otimes \mathbb{R}^m \\ A_{02} &: \Lambda^p\mathbb{R}^{m*} \otimes \mathbb{R}^m \otimes \Lambda^q\mathbb{R}^{m*} \otimes \mathbb{R}^m \otimes S^2\mathbb{R}^{m*} \to \Lambda^{p+q+1}\mathbb{R}^{m*} \otimes \mathbb{R}^m \end{aligned}$$

are linear. We do this only for the case $p, q \geq 2$ and indicate how to treat the remaining cases. As before we take all $GL(m, \mathbb{R})$ equivariant maps as an ansatz, and these are generated by all maps obtained by performing one contraction and alternating the result, or by performing two contractions, tensorizing with I and then alternating.

Let us first consider A_{20}: Here the first form has two (symmetric) derivation indices, and to get a nonzero map at least one of them must be contracted (c.f. (1.10)). Thus we have only two possibilities to perform one contraction and five possibilities to perform two contractions. This gives a seven parameter family for

A_{20} (lower case letters in the list below). Similarly we get a seven parameter family for A_{02} (upper case letters in the list below).

Now consider A_{11}: Since there are four groups of lower indices, we clearly have eight possibilities to perform one contraction (upper case letters with tildes in the list below). If we want to perform two contractions, the situation is as follows: We have four possibilities to contract the first upper index. If it is contracted into a form index, we have four possibilities to contract the second upper index (since $p, q \geq 2$), but if it is contracted into a derivation index, there remain only three possibilities for the second upper index. Thus we have 14 possibilities to perform two contractions (lower case letters with tildes in the list below). So we get the ansatz:

$$\begin{aligned}
A(P,Q)^i_\gamma =& aP^m_{n\alpha,km}Q^n_\beta\delta^i_\ell + bP^m_{\alpha,mn}Q^n_\beta\delta^i_\ell + cP^m_{\alpha,km}Q^n_{n\beta}\delta^i_\ell + \\
&+dP^m_{m\alpha,kn}Q^n_\beta\delta^i_\ell + eP^m_{\alpha,kn}Q^n_{m\beta}\delta^i_\ell + fP^m_{\alpha,km}Q^i_\beta + gP^i_{\alpha;kn}Q^n_\beta + \\
&+\tilde{a}P^m_{mn\alpha,k}Q^n_{\beta,r}\delta^i_\ell + \tilde{b}P^m_{m\alpha,n}Q^n_{\beta,r}\delta^i_\ell + \tilde{c}P^m_{m\alpha,k}Q^n_{n\beta,r}\delta^i_\ell + \tilde{d}P^m_{m\alpha,k}Q^n_{\beta,n}\delta^i_\ell + \\
&+\tilde{e}P^m_{n\alpha,m}Q^n_{\beta,r}\delta^i_\ell + \tilde{f}P^m_{\alpha,m}Q^n_{n\beta,r}\delta^i_\ell + \tilde{g}P^m_{\alpha,m}Q^n_{\beta,n}\delta^i_\ell + \\
&+\tilde{h}P^m_{n\alpha,k}Q^n_{m\beta,r}\delta^i_\ell + \tilde{i}P^m_{\alpha,n}Q^n_{m\beta,r}\delta^i_\ell + \tilde{j}P^m_{\alpha,k}Q^n_{nm\beta,r}\delta^i_\ell + \tilde{k}P^m_{\alpha,k}Q^n_{m\beta,n}\delta^i_\ell + \\
(*) \qquad &+\tilde{\ell}P^m_{n\alpha,k}Q^n_{\beta,m}\delta^i_\ell + \tilde{m}P^m_{\alpha,n}Q^n_{\beta,m}\delta^i_\ell + \tilde{n}P^m_{\alpha,k}Q^n_{n\beta,m}\delta^i_\ell + \\
&+\tilde{A}P^m_{m\alpha,k}Q^i_{\beta,r} + \tilde{B}P^m_{\alpha,m}Q^i_{\beta,r} + \tilde{C}P^m_{\alpha,k}Q^i_{m\beta,r} + \tilde{D}P^m_{\alpha,k}Q^i_{\beta,m} + \\
&+\tilde{E}P^i_{n\alpha,k}Q^n_{\beta,r} + \tilde{F}P^i_{\alpha,n}Q^n_{\beta,r} + \tilde{G}P^i_{\alpha,k}Q^n_{n\beta,r} + \tilde{H}P^i_{\alpha,k}Q^n_{\beta,n} + \\
&+AP^m_\alpha Q^n_{m\beta,nr}\delta^i_\ell + BP^m_\alpha Q^n_{\beta,mn}\delta^i_\ell + CP^m_{m\alpha}Q^n_{\beta,nr}\delta^i_\ell + \\
&+DP^m_\alpha Q^n_{n\beta,mr}\delta^i_\ell + EP^m_{n\alpha}Q^n_{\beta,mr}\delta^i_\ell + FP^i_\alpha Q^n_{\beta,nr} + GP^m_\alpha Q^i_{\beta,mr}.
\end{aligned}$$

As before alternation is not explicitly indicated. ■

3.15. We start checking equivariancy with $\tilde{G}^3_m$. Thus we need the transformation laws (1), (2) and (3) of Proposition (2.6).

From these one immediately sees that the right hand side of (3.14)(*) as well as the whole map A_{11} is $\tilde{G}^3_m$–invariant. Then one computes that equivariancy is equivalent to:

$$\begin{aligned}
0 =& a(P^j_{n\alpha}C^m_{jkm} - P^m_{t\alpha}C^t_{kmn})Q^n_\beta\delta^i_\ell + \\
&+b(P^j_\alpha C^m_{jmn} - pP^m_{\alpha t}C^t_{mn\alpha_p})Q^n_\beta\delta^i_\ell + \\
&+cP^j_\alpha C^m_{jkm}Q^n_{n\beta}\delta^i_\ell + eP^j_\alpha C^m_{jkn}Q^n_{m\beta}\delta^i_\ell + \\
&+fP^j_\alpha C^m_{jkm}Q^i_\beta + gP^j_\alpha C^i_{jkn}Q^n_\beta + \\
&+AP^m_\alpha(Q^j_{m\beta}C^n_{jnr} - Q^n_{t\beta}C^t_{mnr})\delta^i_\ell + \\
&+BP^m_\alpha(Q^j_\beta C^n_{jmn} - qQ^n_{\beta t}C^t_{mn\beta_q})\delta^i_\ell + \\
&+CP^m_{m\alpha}Q^j_\beta C^n_{jnr}\delta^i_\ell + EP^m_{n\alpha}Q^j_\beta C^n_{jmr}\delta^i_\ell + \\
&+FP^i_\alpha Q^j_\beta C^n_{jnr} + GP^m_\alpha Q^j_\beta C^i_{jmr}.
\end{aligned}$$

The right hand side of this equation represents a trilinear map:

$$\Lambda^p\mathbf{R}^{m*}\otimes\mathbf{R}^m\times S^3\mathbf{R}^{m*}\otimes\mathbf{R}^m\times\Lambda^q\mathbf{R}^{m*}\otimes\mathbf{R}^m\to\Lambda^{p+q+1}\mathbf{R}^{m*}\otimes\mathbf{R}^m.$$

Now one uses antisymmetry in the free lower indices to bring all terms into a standard form and gets the following maps and coefficients:

$$(*)\qquad\begin{aligned}
&P^j_{n\alpha}C^m_{jkm}Q^n_\beta\delta^i_\ell : a\\
&P^m_{t\alpha}C^t_{kmn}Q^n_\beta\delta^i_\ell :\ -a+(-1)^p pb+(-1)^q E\\
&P^j_\alpha C^m_{jmn}Q^n_\beta\delta^i_\ell :\ b+B\\
&P^j_\alpha C^m_{jkm}Q^n_{n\beta}\delta^i_\ell : c\\
&P^j_\alpha C^m_{jkn}Q^n_{m\beta}\delta^i_\ell : e+(-1)^q A-qB\\
&P^m_\alpha C^n_{jnr}Q^j_{m\beta}\delta^i_\ell :(-1)^{q-1}A\\
&P^m_{m\alpha}C^n_{jnr}Q^j_\beta\delta^i_\ell : (-1)^q C\\
&P^j_\alpha C^m_{jkm}Q^i_\beta :\quad f\\
&P^j_\alpha C^i_{jkn}Q^n_\beta :\quad g+(-1)^q G\\
&P^i_\alpha C^n_{jnr}Q^j_\beta :\quad (-1)^q F.
\end{aligned}$$

Next inserting special elements for P, Q and C one proves that for $p,q\geq 1$ and $m\geq p+q+2$ the maps listed above are linearly independent as trilinear maps, and thus $\tilde G^3_m$–equivariancy is equivalent to all coefficients listed in $(*)$ being zero and thus to:
$a=c=f=A=C=F=0$, $e=-qb$, $B=-b$, $E=(-1)^{p+q+1}pb$ and $G=(-1)^{q-1}g$, while the parameters b, g, d and D and all parameters corresponding to maps from A_{11} remain free.

3.16. Now we check, which $\tilde G^3_m$–equivariant maps are $\tilde G^2_m$–equivariant, too. Thus we need the transformation laws (1), (2) and (3) of Proposition (2.5). Using these we get:

$$\begin{aligned}
&A(\bar P,\bar Q)^i_\gamma=\\
&=b(P^m_{\alpha,mn}+P^j_{\alpha,n}B^m_{jm}-P^j_\alpha B^m_{jt}B^t_{mn}-pP^m_{\alpha t,m}B^t_{n\alpha_p}-pP^m_{\alpha t,n}B^t_{m\alpha_p}-\\
&\qquad -pP^j_{\alpha t}B^m_{jm}B^t_{n\alpha_p}-pP^j_{\alpha t}B^m_{jn}B^t_{m\alpha_p}+pP^m_{\alpha t}B^t_{ms}B^s_{n\alpha_p}+pP^m_{\alpha t}B^t_{ns}B^s_{m\alpha_p}+\\
&\qquad +pP^m_{\alpha t}B^t_{s\alpha_p}B^s_{mn}+p(p-1)P^m_{\alpha ts}B^t_{m\alpha_{p-1}}B^s_{n\alpha_p})Q^n_\beta\delta^i_\ell+\\
&+d(P^m_{m\alpha,kn}-P^m_{m\alpha,t}B^t_{kn}-(p-1)P^m_{m\alpha t,k}B^t_{n\alpha p})Q^n_\beta\delta^i_\ell+\\
&+e(P^m_{\alpha,kn}-P^m_{\alpha,t}B^t_{kn}+P^j_{\alpha,k}B^m_{jn}+P^j_{\alpha,n}B^m_{jk}-P^j_\alpha B^m_{jt}B^t_{kn}-pP^m_{\alpha t,k}B^t_{n\alpha_p}-\\
&\qquad -pP^j_{\alpha t}B^m_{jk}B^t_{n\alpha_p}+pP^m_{\alpha t}B^t_{ks}B^s_{n\alpha_p}+pP^m_{\alpha t}B^t_{s\alpha_p}B^s_{kn})Q^n_{m\beta}\delta^i_\ell+\\
&+g(P^i_{\alpha,kn}-P^i_{\alpha,t}B^t_{kn}+P^j_{\alpha,k}B^i_{jn}+P^j_{\alpha,n}B^i_{jk}-P^j_\alpha B^i_{jt}B^t_{kn}-pP^i_{\alpha t,k}B^t_{n\alpha_p}-\\
&\qquad -pP^j_{\alpha t}B^i_{jk}B^t_{n\alpha_p}+pP^i_{\alpha t}B^t_{ks}B^s_{n\alpha_p}+pP^i_{\alpha t}B^t_{s\alpha_p}B^s_{kn})Q^n_\beta+
\end{aligned}$$

$$
\begin{aligned}
&+\tilde{a}(P^m_{mn\alpha,k} - P^m_{mt\alpha}B^t_{kn})(Q^n_{\beta,r} + Q^j_\beta B^n_{jr})\delta^i_\ell +\\
&+\tilde{b}(P^m_{m\alpha,n} - (p-1)P^m_{m\alpha t}B^t_{n\alpha_p})(Q^n_{\beta,r} + Q^j_\beta B^n_{jr})\delta^i_\ell +\\
&+\tilde{c}P^m_{m\alpha,k}Q^n_{n\beta,r}\delta^i_\ell +\\
&+\tilde{d}P^m_{m\alpha,k}(Q^n_{\beta,n} + Q^j_\beta B^n_{jn} - qQ^n_{\beta t}B^t_{n\beta_q})\delta^i_\ell +\\
&+\tilde{e}(P^m_{n\alpha,m} + P^j_{n\alpha}B^m_{jm} - P^m_{t\alpha}B^t_{mn} - (p-1)P^m_{n\alpha t}B^t_{m\alpha_p})(Q^n_{\beta,r} + Q^j_\beta B^n_{jr})\delta^i_\ell +\\
&+\tilde{f}(P^m_{\alpha,m} + P^j_\alpha B^m_{jm} - pP^m_{\alpha t}B^t_{m\alpha_p})Q^n_{n\beta,r}\delta^i_\ell +\\
&+\tilde{g}(P^m_{\alpha,m} + P^j_\alpha B^m_{jm} - pP^m_{\alpha t}B^t_{m\alpha_p})(Q^n_{\beta,n} + Q^j_\beta B^n_{jn} - qQ^n_{\beta t}B^t_{n\beta_q})\delta^i_\ell +\\
&+\tilde{h}(P^m_{n\alpha,k} + P^j_{n\alpha}B^m_{jk} - P^m_{t\alpha}B^t_{kn})(Q^n_{m\beta,r} + Q^j_{m\beta}B^n_{jr} - Q^n_{t\beta}B^t_{mr})\delta^i_\ell +\\
&+\tilde{i}(P^m_{\alpha,n} + P^j_\alpha B^m_{jn} - pP^m_{\alpha t}B^t_{n\alpha_p})(Q^n_{m\beta,r} + Q^j_{m\beta}B^n_{jr} - Q^n_{t\beta}B^t_{mr})\delta^i_\ell +\\
&+\tilde{j}(P^m_{\alpha,k} + P^j_\alpha B^m_{jk})(Q^n_{nm\beta,r} - Q^n_{nt\beta}B^t_{mr})\delta^i_\ell +\\
&+\tilde{k}(P^m_{\alpha,k} + P^j_\alpha B^m_{jk})(Q^n_{m\beta,n} + Q^j_{m\beta}B^n_{jn} - Q^n_{t\beta}B^t_{mn} - (q-1)Q^n_{m\beta t}B^t_{n\beta_q})\delta^i_\ell +\\
&+\tilde{\ell}(P^m_{n\alpha,k} + P^j_{n\alpha}B^m_{jk} - P^m_{t\alpha}B^t_{kn})(Q^n_{\beta,m} + Q^j_\beta B^n_{jm} - qQ^n_{\beta t}B^t_{m\beta_q})\delta^i_\ell +\\
&+\tilde{m}(P^m_{\alpha,n} + P^j_\alpha B^m_{jn} - pP^m_{\alpha t}B^t_{n\alpha_p})(Q^n_{\beta,m} + Q^j_\beta B^n_{jm} - qQ^n_{\beta t}B^t_{m\beta_q})\delta^i_\ell +\\
&+\tilde{n}(P^m_{\alpha,k} + P^j_\alpha B^m_{jk})(Q^n_{n\beta,m} - (q-1)Q^n_{n\beta t}B^t_{m\beta_q})\delta^i_\ell +\\
&+\tilde{A}P^m_{m\alpha,k}(Q^i_{\beta,r} + Q^j_\beta B^i_{jr}) +\\
&+\tilde{B}(P^m_{\alpha,m} + P^j_\alpha B^m_{jm} - pP^m_{\alpha t}B^t_{m\alpha_p})(Q^i_{\beta,r} + Q^j_\beta B^i_{jr}) +\\
&+\tilde{C}(P^m_{\alpha,k} + P^j_\alpha B^m_{jk})(Q^i_{m\beta,r} + Q^j_{m\beta}B^i_{jr} - Q^i_{t\beta}B^t_{mr}) +\\
&+\tilde{D}(P^m_{\alpha,k} + P^j_\alpha B^m_{jk})(Q^i_{\beta,m} + Q^j_\beta B^i_{jm} - qQ^i_{\beta t}B^t_{m\beta_q}) +\\
&+\tilde{E}(P^i_{n\alpha,k} + P^j_{n\alpha}B^i_{jk} - P^i_{t\alpha}B^t_{kn})(Q^n_{\beta,r} + Q^j_\beta B^n_{jr}) +\\
&+\tilde{F}(P^i_{\alpha,n} + P^j_\alpha B^i_{jn} - P^i_{\alpha t}B^t_{n\alpha_p})(Q^n_{\beta,r} + Q^j_\beta B^n_{jr}) +\\
&+\tilde{G}(P^i_{\alpha,k} + P^j_\alpha B^i_{jk})Q^n_{n\beta,r} +\\
&+\tilde{H}(P^i_{\alpha,k} + P^j_\alpha B^i_{jk})(Q^n_{\beta,n} + Q^j_\beta B^n_{jn} - qQ^n_{\beta t}B^t_{n\beta_q}) +\\
&+BP^m_\alpha(Q^n_{\beta,mn} + Q^j_{\beta,m}B^n_{jn} - Q^j_\beta B^n_{jt}B^t_{mn} - qQ^n_{\beta t,m}B^t_{n\beta_q} - qQ^n_{\beta t,n}B^t_{m\beta_q} -\\
&\qquad - qQ^j_{\beta t}B^n_{jm}B^t_{n\beta_q} - qQ^j_{\beta t}B^n_{jn}B^t_{m\beta_q} + qQ^n_{\beta t}B^t_{ms}B^s_{n\beta_q} +\\
&\qquad + qQ^n_{\beta t}B^t_{ns}B^s_{m\beta_q} + qQ^n_{\beta t}B^t_{s\beta_q}B^s_{mn} + q(q-1)Q^n_{\beta ts}B^t_{m\beta_{q-1}}B^s_{n\beta_q})\delta^i_\ell +\\
&+DP^m_\alpha(Q^n_{n\beta,mr} - Q^n_{n\beta,t}B^t_{mr} - (q-1)Q^n_{n\beta t,r}B^t_{m\beta_q})\delta^i_\ell +\\
&+EP^m_{n\alpha}(Q^n_{\beta,mr} - Q^n_{\beta,t}B^t_{mr} + Q^j_{\beta,m}B^n_{jr} + Q^j_{\beta,r}B^n_{jm} - Q^j_\beta B^n_{jt}B^t_{mr} -\\
&\qquad - qQ^n_{\beta t,r}B^t_{m\beta_q} - qQ^j_{\beta t}B^n_{jr}B^t_{m\beta_q} + qQ^n_{\beta t}B^t_{rs}B^s_{m\beta_q} +\\
&\qquad + qQ^n_{\beta t}B^t_{s\beta_q}B^s_{mr})\delta^i_\ell +\\
&+GP^m_\alpha(Q^i_{\beta,mr} - Q^i_{\beta,t}B^t_{mr} + Q^j_{\beta,m}B^i_{jr} + Q^j_{\beta,r}B^i_{jm} - Q^j_\beta B^i_{jt}B^t_{mr} -\\
&\qquad - qQ^i_{\beta t,r}B^t_{m\beta_q} - qQ^j_{\beta t}B^i_{jr}B^t_{m\beta_q} + qQ^i_{\beta t}B^t_{rs}B^s_{m\beta_q} + qQ^i_{\beta t}B^t_{s\beta_q}B^s_{mr}).
\end{aligned}
$$

Since the left hand side of the ansatz $(*)$ of (3.14) is $\tilde{G}^2_m$ invariant, $\tilde{G}^2_m$ equivariancy is equivalent to $A(\bar{P},\bar{Q})^i_\gamma = A(P,Q)^i_\gamma$. Moreover the first terms (respectively the products of the first terms of each bracket) reproduce the original map (3.14)$(*)$, and thus equivariancy is equivalent to the rest being zero. This is again a linear combination of maps, but this time there are three essentially different types of maps: First there are trilinear maps:

$$\Lambda^p\mathbf{R}^{m*} \otimes \mathbf{R}^m \otimes \mathbf{R}^{m*} \times S^2\mathbf{R}^{m*} \otimes \mathbf{R}^m \times \Lambda^q\mathbf{R}^{m*} \otimes \mathbf{R}^m \to \Lambda^{p+q+1}\mathbf{R}^{m*} \otimes \mathbf{R}^m$$

(those in which P has a derivation index). Second there are trilinear maps:

$$\Lambda^p\mathbf{R}^{m*} \otimes \mathbf{R}^m \times S^2\mathbf{R}^{m*} \otimes \mathbf{R}^m \times \Lambda^q\mathbf{R}^{m*} \otimes \mathbf{R}^m \otimes \mathbf{R}^{m*} \to \Lambda^{p+q+1}\mathbf{R}^{m*} \otimes \mathbf{R}^m$$

(those in which Q has a derivation index), and finally there are maps:

$$\Lambda^p\mathbf{R}^{m*} \otimes \mathbf{R}^m \times S^2\mathbf{R}^{m*} \otimes \mathbf{R}^m \times \Lambda^q\mathbf{R}^{m*} \otimes \mathbf{R}^m \to \Lambda^{p+q+1}\mathbf{R}^{m*} \otimes \mathbf{R}^m$$

which are linear in the first and third factor and quadratic in the second factor. Since the value of a vector valued differential form at zero and the values of its partial derivatives at zero can be chosen independently, we may split the equation into three parts, corresponding to these three types of maps.

3.17. Let us first consider the maps, in which P has a derivation index. Using antisymmetry in the free lower indices we bring all terms into a standard form and use the fact that the free lower indices must be the same in each term. Then we get the equation

$$0 = bP^j_{\alpha,n}B^m_{jm}Q^n_\beta\delta^i_\ell + \cdots + (-1)^q q\tilde{H}P^i_{\alpha,k}B^t_{n\beta_q}Q^n_{\beta t}$$

with the following maps and coefficients:

$$\begin{aligned}
&P^j_{\alpha,n}B^m_{jm}Q^n_\beta\delta^i_\ell: && b\\
&P^m_{\alpha t,m}B^t_{n\alpha_p}Q^n_\beta\delta^i_\ell: && -pb+(-1)^{p+q-1}\tilde{e}\\
&P^m_{\alpha t,n}B^t_{m\alpha_p}Q^n_\beta\delta^i_\ell: && -pb\\
&P^m_{m\alpha,t}B^t_{kn}Q^n_\beta\delta^i_\ell: && -d+(-1)^q\tilde{b}\\
&P^m_{m\alpha t,k}B^t_{n\alpha_p}Q^n_\beta\delta^i_\ell: && (p-1)d+(-1)^{p+q}\tilde{a}\ (*)\\
&P^m_{\alpha,t}B^t_{kn}Q^n_{m\beta}\delta^i_\ell: && -e+(-1)^{q-1}\tilde{i}\\
&P^j_{\alpha,k}B^m_{jn}Q^n_{m\beta}\delta^i_\ell: && e-\tilde{k}\\
&P^j_{\alpha,n}B^m_{jk}Q^n_{m\beta}\delta^i_\ell: && e+(-1)^q\tilde{i}-q\tilde{m}\\
&P^m_{\alpha t,k}B^t_{n\alpha_p}Q^n_{m\beta}\delta^i_\ell: && pe+(-1)^{p+q}\tilde{h}\\
&P^m_{m\alpha,k}B^n_{jn}Q^j_\beta\delta^i_\ell: && \tilde{d}\\
&P^m_{m\alpha,k}B^t_{n\beta_q}Q^n_{\beta t}\delta^i_\ell: && (-1)^q q\tilde{d}
\end{aligned}$$

$$\begin{array}{ll}
P^m_{\alpha,m}B^n_{jn}Q^j_\beta\delta^i_\ell : & \tilde g \\
P^m_{\alpha,m}B^t_{n\beta_q}Q^n_{\beta t}\delta^i_\ell : & (-1)^q q\tilde g \\
P^m_{n\alpha,k}B^t_{mr}Q^n_{t\beta}\delta^i_\ell : & (-1)^q\tilde h - q\tilde\ell \\
P^m_{\alpha,k}B^t_{mr}Q^n_{nt\beta}\delta^i_\ell : & (-1)^{q-1}\tilde j - (q-1)\tilde n \\
P^m_{\alpha,k}B^n_{jn}Q^j_{m\beta}\delta^i_\ell : & \tilde k \\
P^m_{\alpha,k}B^t_{n\beta_q}Q^n_{m\beta t}\delta^i_\ell : & (-1)^{q-1}(q-1)\tilde k \\
P^m_{n\alpha,k}B^n_{jm}Q^j_\beta\delta^i_\ell : & \tilde\ell \\
P^m_{\alpha,n}B^n_{jm}Q^j_\beta\delta^i_\ell : & \tilde m \\
P^i_{\alpha,t}B^t_{kn}Q^n_\beta : & -g + (-1)^q\tilde F \\
P^j_{\alpha,k}B^i_{jn}Q^n_\beta : & g + \tilde D \\
P^j_{\alpha,n}B^i_{jk}Q^n_\beta : & g \\
P^i_{\alpha t,k}B^t_{n\alpha_p}Q^n_\beta : & pg + (-1)^{p+q-1}\tilde E \\
P^m_{m\alpha,k}B^i_{jr}Q^j_\beta : & (-1)^q\tilde A \\
P^m_{\alpha,m}B^i_{jr}Q^j_\beta : & (-1)^q\tilde B \\
P^m_{\alpha,k}B^i_{jr}Q^j_{m\beta} : & (-1)^{q-1}\tilde C \\
P^m_{\alpha,k}B^t_{mr}Q^i_{t\beta} : & (-1)^q\tilde C - q\tilde D \\
P^i_{\alpha,k}B^n_{jn}Q^j_\beta : & \tilde H \\
P^i_{\alpha,k}B^t_{n\beta_q}Q^n_{\beta t} : & (-1)^q q\tilde H.
\end{array}$$

The signs in the term marked with $(*)$ are due to the fact, that in d the expression $P^m_{m\alpha t,k}B^t_{n\alpha_p}$ corresponds to the form $dx^\alpha\wedge dx^{\alpha_p}\wedge dx^k$ and not to $dx^\alpha\wedge dx^k\wedge dx^{\alpha_p}$. This argument applies to several other terms containing factors from A_{20} or A_{02}, too.

Inserting special forms for P, Q and B one then shows that for $m\geq p+q+2$ this implies that all coefficients in the above list have to be zero.

Similar computations show that considering the terms in which Q has a derivation index one gets the equations:

$$\tilde f = \tilde G = \tilde a + (-1)^p(p-1)\tilde b = \tilde j - (q-1)D = \tilde n + (-1)^q D = 0$$

and from the last kind of maps one gets:

$$\tilde a + (-1)^p(p-1)\tilde b = \tilde j + (-1)^q(q-1)\tilde n = 0.$$

3.18. Thus for $p,q\geq 2$, $m\geq p+q+1$ equivariancy under $\tilde G^2_m$ and $\tilde G^3_m$ is equivalent to: $a=b=c=e=f=g=\tilde d=\tilde e=\tilde f=\tilde g=\tilde h=\tilde i=\tilde k=\tilde\ell=\tilde m=\tilde A=\tilde B=\tilde C=\tilde D=\tilde E=\tilde F=\tilde G=\tilde H=A=B=C=E=F=G=0$,
d, D and $\tilde c$ are three independent free parameters, which determine all other parameters uniquely by the equations:
$\tilde a=(-1)^{p+q-1}(p-1)d$, $\tilde b=(-1)^q d$, $\tilde j=(q-1)D$ and $\tilde n=(-1)^{q-1}D$.

Thus there can be at most three linearly independent bilinear natural operators in this case and Theorem (3.14) holds.

3.19. To finish the discussion of Theorem (3.14) we indicate how to prove the theorem in the remaining cases:

For $p, q \geq 2$ and $m = p + q + 1$ one shows that the maps in the ansatz $(*)$ of (3.14) become linearly dependent and that one may eliminate ten of them and still has a generating system. Then it is easy to see that the proof works as before.

The whole proof works as before if $p \geq 2$ or $q \geq 2$ or $m \geq p + q + 3$ simply taking as an ansatz those maps from (3.14)$(*)$ that make sense for a p–form P and a q–form Q, while for $p, q < 2$ and $m \leq p+q+2$ one has to prove that several maps in the ansatz may be omitted and then the proof works as before.

3.20. Our final task is to show that there are no other bilinear natural operators.

Theorem. *There is no nonzero bilinear natural operator:*

$$\Omega^p(M;TM) \times \Omega^q(M;TM) \to \Omega^{p+q+2}(M;TM).$$

Proof : One shows that there is no nonzero G^4_m equivariant map $A = A_{30} + A_{21} + A_{12} + A_{03}$, where

$$A_{30} : \Lambda^p\mathbf{R}^{m*} \otimes \mathbf{R}^m \otimes S^3\mathbf{R}^{m*} \otimes \Lambda^q\mathbf{R}^{m*} \otimes \mathbf{R}^m \to \Lambda^{p+q+2}\mathbf{R}^{m*} \otimes \mathbf{R}^m$$
$$A_{21} : \Lambda^p\mathbf{R}^{m*} \otimes \mathbf{R}^m \otimes S^2\mathbf{R}^{m*} \otimes \Lambda^q\mathbf{R}^{m*} \otimes \mathbf{R}^m \otimes \mathbf{R}^{m*} \to \Lambda^{p+q+2}\mathbf{R}^{m*} \otimes \mathbf{R}^m$$
$$A_{12} : \Lambda^p\mathbf{R}^{m*} \otimes \mathbf{R}^m \otimes \mathbf{R}^{m*} \otimes \Lambda^q\mathbf{R}^{m*} \otimes \mathbf{R}^m \otimes S^2\mathbf{R}^{m*} \to \Lambda^{p+q+2}\mathbf{R}^{m*} \otimes \mathbf{R}^m$$
$$A_{03} : \Lambda^p\mathbf{R}^{m*} \otimes \mathbf{R}^m \otimes \Lambda^q\mathbf{R}^{m*} \otimes \mathbf{R}^m \otimes S^3\mathbf{R}^{m*} \to \Lambda^{p+q+2}\mathbf{R}^{m*} \otimes \mathbf{R}^m$$

are linear.

First one shows that in this case the ansatz is:

$$\begin{aligned} A(P,Q)^i_\gamma =& \tilde{a}P^m_{\alpha,kmn}Q^n_\beta\delta^i_\ell + \tilde{b}P^m_\alpha Q^n_{\beta,mnr}\delta^i_\ell + \\ &+aP^m_{n\alpha,km}Q^n_{\beta,r}\delta^i_\ell + bP^m_{\alpha,mn}Q^n_{\beta,r}\delta^i_\ell + cP^m_{\alpha,km}Q^n_{n\beta,r}\delta^i_\ell + \\ &+dP^m_{\alpha,km}Q^n_{\beta,n}\delta^i_\ell + eP^m_{m\alpha,kn}Q^n_{\beta,r}\delta^i_\ell + fP^m_{\alpha,kn}Q^n_{m\beta,r}\delta^i_\ell + \\ &+gP^m_{\alpha,kn}Q^n_{\beta,m}\delta^i_\ell + hP^m_{\alpha,km}Q^i_{\beta,r} + iP^i_{\alpha,kn}Q^n_{\beta,r} + \\ &+AP^m_{\alpha,k}Q^n_{m\beta,nr}\delta^i_\ell + BP^m_{\alpha,k}Q^n_{\beta,mn}\delta^i_\ell + CP^m_{m\alpha,k}Q^n_{\beta,nr}\delta^i_\ell + \\ &+DP^m_{\alpha,m}Q^n_{\beta,nr}\delta^i_\ell + EP^m_{\alpha,k}Q^n_{n\beta,mr}\delta^i_\ell + FP^m_{n\alpha,k}Q^n_{\beta,mr}\delta^i_\ell + \\ &+GP^m_{\alpha,n}Q^n_{\beta,mr}\delta^i_\ell + HP^i_{\alpha,k}Q^n_{\beta,nr} + IP^m_{\alpha,k}Q^i_{\beta,mr}. \end{aligned}$$

Then one shows that equivariancy under $\tilde{G}^4_m$, $\tilde{G}^3_m$ and $\tilde{G}^2_m$ implies that all coefficients have to be zero. ■

Theorem 3.21. *There is no nonzero bilinear natural operator:*

$$\Omega^p(M;TM) \times \Omega^q(M;TM) \to \Omega^{p+q+3}(M;TM).$$

Proof : This is similar to the proof of (3.20) but much easier since in this case the ansatz reduces to:

$$\begin{aligned} A(P,Q)^i_\gamma =& aP^m_{\alpha,kmn}Q^n_{\beta,r}\delta^i_\ell + \\ &+bP^m_{\alpha,km}Q^n_{\beta,nr}\delta^i_\ell + cP^m_{\alpha,kn}Q^n_{\beta,mr}\delta^i_\ell + \\ &+dP^m_{\alpha,k}Q^n_{\beta,mnr}\delta^i_\ell. \end{aligned}$$

■

Remark 3.22. It is shown in [Ca] that each of the bilinear natural concomitants of vector valued differential forms except the Frölicher–Nijenhuis bracket can be written as a linear combination of compositions of linear concomitants and insertion operators. There it is also shown how the results obtained here can be used to determine all linear and bilinear operators between ordinary differential forms which are natural under local diffeomorphisms and do not involve top degree forms.

References

[C–dW–G] Cahen M., de Wilde M., Gutt S., *Local cohomology of the algebra of C^∞-functions on a connected manifold*, Lett. Math. Phys. **4** (1980), 157–167.

[Ca] Cap A., *Natural operators between vector valued differential forms*, Proc. Winter School on Geometry and Physics, Srní 1990, to appear.

[D-C] Dieudonné J.A., Carrell J.B., *Invariant Theory, Old and New*, Academic Press, New York—London, 1971.

[dW-L] de Wilde M., Lecomte P., *Algebraic characterizations of the algebra of functions and of the Lie algebra of vector fields on a manifold*, Composito Math. **45** (1982), 199–205.

[K-M] Kolář I., Michor P., *All natural concomitants of vector valued differential forms*, Proc. Winter School on Geometry and Physics, Srní 1987, Supp. ai Rend. Circolo Matematico di Palermo **II–16** (1987), 101–108.

[K-M-S] Kolář I., Michor P., Slovák J., *Natural Operators in Differential Geometry*, to appear in Springer Ergebnisse.

[Ko] Kolář I., *Some natural operators in differential geometry*, Proceedings of the Conference on Differential Geometry and its Applications, Brno 1986, D. Reidl.

[Kr-M] Krupka D., Mikolášová V., *On the uniqueness of some differential invariants: d, $[\ ,\ ]$, ∇*, Czechoslovak Math. J. **34** (1984), 588–597.

[Mi] Michor P., *Remarks on the Frölicher–Nijenhuis bracket*, Proceedings of the Conference on Differential Geometry and its Applications, Brno 1986, D. Reidl.

[Sl] Slovák J., *Peetre Theorem for Nonlinear Operators*, Ann. Global Anal. Geom. **6/3** (1988), 273–283.

[vS] van Strien S., *Unicity of the Lie Product*, Compositio Math. **40** (1980), 79–85.

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