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Abstract. In this paper a new aspect of the Meijer's integral transform is treated, for which its corresponding inversion formula has been duly achieved. It turns out to exist a relation between this transform and Laplace's, which opens the way to define different types of convolutions. Furthermore, some operational rules are obtained.

Keywords: $M_{\alpha,\beta}$-integral transform, Meijer, Laplace, Kratzel, Bessel, Bessel—Clifford, convolution, operational rule

Classification: 44A15

1. Introduction.

In this paper a new version of Meijer's integral transform has been studied, which will be referred to as the $M_{\alpha,\beta}$-integral transform. This variant generalizes those of E. Kratzel's [6], J. Conlan's, E.L. Koh's [3] and J. Rodríguez [9] as well, among others, and it is given as

$$F(s)=\int_0^\infty (st)^{\alpha+\beta-1}L_{\alpha-1}(st)f(t)\,dt \tag{1.1}$$

$$f(t)=\frac{1}{\pi i}\int_{\Gamma_c}(st)^{-\beta}E_{\alpha-1}(st)F(s)\,ds \tag{1.2}$$

with $\Gamma_c=\{s/s\in C, \operatorname{Re}\sqrt{2s}>c>0\}$. The functions $L_{\alpha-1}(t)$ and $E_{\alpha-1}(t)$ appear in their respective kerns, and are solutions of the differential equation [5]

$$ty''+\alpha y'-y=0 \tag{1.3}$$

$E_{\alpha-1}(t)$ admits the following expansion

$$E_{\alpha-1}(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!\,\Gamma(\alpha+n)} \tag{1.4}$$

and it is known as the modified (or hyperbolic) Bessel—Clifford function of first kind and order $(\alpha-1)$. When $(\alpha-1)$ is a non-integer, then $t^{1-\alpha}E_{1-\alpha}(t)$ constitutes in itself another solution of (1.3), which is non-linearly dependent on $E_{\alpha-1}(t)$. Similarly, $L_{\alpha-1}(t)$ will be referred to as the modified Bessel—Clifford function of third kind and order $(\alpha-1)$, and it is given as

$$L_{\alpha-1}(t)=-\frac{\pi}{2\,\mathrm{sen}(\alpha-1)\pi}(E_{\alpha-1}(t)^{1-\alpha}E_{1-\alpha}(t)) \tag{1.5}$$

It is of interest to emphasize the fact that $E_{\alpha-1}(t)$ and $L_{\alpha-1}(t)$ are linked to their corresponding Bessel functions by the following expressions

$$E_{\alpha-1}(t) = t^{-\frac{\alpha-1}{2}} I_{\alpha-1}(2\sqrt{t})$$

$$L_{\alpha-1}(t) = t^{-\frac{\alpha-1}{2}} K_{\alpha-1}(2\sqrt{t}). \tag{1.6}$$

$L_{\alpha-1}(t)$ admits the generalization given in [7] and [8] as

$$\eta(\varrho, \alpha; z) = \int_0^\infty \tau^{-\alpha} e^{-\tau - z\tau^{-\varrho}} dz \quad (\varrho > 0, |\arg z| < \frac{\pi}{2}), \tag{1.7}$$

which for $\varrho = 1$, reduces to

$$\eta(1, \alpha; z) = 2L_{\alpha-1}(z). \tag{1.8}$$

The asymptotic behaviour of $L_{\alpha-1}(t)$ can be interfered from $\eta(1, \alpha; t)$, as follows

$$L_{\alpha-1}(t) \sim \begin{cases} \frac{\Gamma(\alpha-1)}{2} t^{1-\alpha} & \text{if } \operatorname{Re}\alpha - 1 > 0 \\ \frac{\Gamma(\alpha-1)}{2} t^{1-\alpha} + \frac{\Gamma(1-\alpha)}{2} & \text{if } \operatorname{Re}\alpha - 1 = 0, \alpha - 1 \neq 0 \\ -1nt & \text{if } \alpha - 1 = 0 \\ \frac{\Gamma(1-\alpha)}{2} & \text{if } \operatorname{Re}\alpha - 1 < 0 \end{cases} \tag{1.9}$$

for $t \to 0^+$, and

$$L_{\alpha-1}(t) \sim \frac{\sqrt{\pi}}{2} t^{-\frac{2\alpha-1}{4}} e^{-2\sqrt{t}} \tag{1.10}$$

for $t \to +\infty$.

As for $E_{\alpha-1}(z)$, it can be referred to from [10] that

$$E_{\alpha-1}(z) \sim \frac{1}{\Gamma(\alpha)} \text{ if } \operatorname{Re}\alpha > 0 \text{ and } z \to 0^+ \tag{1.11}$$

and also that

$$z^{\frac{\alpha}{2}-\frac{1}{4}} E_{\alpha-1}(z) \sim \frac{1}{\sqrt{2\pi}} (e^{2\sqrt{z}} \pm ie^{-2\sqrt{z}+i(\alpha-1)\pi})(1 + 0(|z|^{-1/2})) \tag{1.12}$$

for $z \to +\infty$.

Similarly, the following integral representations for $L_{\alpha-1}(t)$ can be derived from (1.5) through appropriate changes:

$$L_{\alpha-1}(st) = \frac{1}{2} \int_0^\infty \tau^{-\alpha} e^{-\tau - st/\tau} d\tau \tag{1.13}$$

$$L_{\alpha-1}(st) = \frac{1}{2} s^{1-\alpha} \int_0^\infty \tau^{-\alpha} e^{-s\tau - t/\tau} d\tau \tag{1.14}$$

$$L_{\alpha-1}(st) = \frac{1}{2} t^{1-\alpha} \int_0^\infty \tau^{-\alpha-2} e^{-s\tau - t/\tau} d\tau \tag{1.15}$$

which will be used to express the $M_{\alpha,\beta}$-integral transform in terms of the Laplace transform, so as to enable us to obtain convolutions for that transformation.

2. The $M_{\alpha,\beta}$-integral transform.

Its existence is based on the following:

Proposition 1. *Let α, β be complex numbers and $f(t)$ a locally integrable function on $(0,\infty)$, such that*

$$f(t) = \begin{cases} 0(t^{-\beta}) & \text{if } \operatorname{Re}\alpha - 1 \geq 0 \\ 0(t^{1-\alpha-\beta}) & \text{if } \operatorname{Re}\alpha - 1 < 0 \end{cases}$$

for $t \to 0^+$, and

$$f(t) = 0(e^{c\sqrt{2t}})$$

for $t \to +\infty$.

Under these conditions the integral given as

$$F(s) = M_{\alpha,\beta}\{f(t)\} = \int_0^\infty (st)^{\alpha+\beta-1} L_{\alpha-1}(st) f(t)\, dt \tag{2.1}$$

converges for $\operatorname{Re}\sqrt{2s} > c$. Besides, $f(s)$ proves to be analytic on the convergence domain.

PROOF : Set

$$F(s) = \int_0^\varepsilon (st)^{\alpha+\beta-1} L_{\alpha-1}(st) f(t) + \int_\varepsilon^T (st)^{\alpha+\beta-1} L_{\alpha-1}(st) f(t)\, dt + \\ + \int_T^\infty (st)^{\alpha+\beta-1} L_{\alpha-1}(st) f(t)\, dt \quad \text{for } 0 < \varepsilon < T < +\infty.$$

It can be noted that the first integral in the right-hand side exists due to (1.9) together with the hypothesis. The second integral exists because of $f(t)$ being locally integrable and $(st)^{\alpha+\beta-1}L_{\alpha-1}(st)$ a continuous function. Finally, existence for the third integral is guaranteed by (1.10) provided that $\operatorname{Re}\sqrt{2s} > c$.

Analyticity proves obviously. ∎

Now, the following inversion formula can be established.

Proposition 2. *Let α, β be complex numbers with $\operatorname{Re}\alpha > 0$. Assume that $F(s)$ is analytic over the domain $\Omega = \{s/s \in \mathrm{C} \text{ and } \operatorname{Re}\sqrt{2s} > B \geq 0\}$ and also that $|F(s)| \leq M|s|^{-q}$ holds, M and q being real constants non-depending on s and such that $q > -\operatorname{Re}\beta + \frac{5}{4}$. Then, for any fixed real $c > B$, the following expression*

$$F(s) = \int_0^\infty (st)^{\alpha+\beta-1} L_{\alpha-1}(st) f(t)\, dt$$

is valid for $\operatorname{Re}\sqrt{2s} > c$. Here $f(t)$ is given by

$$f(t) = \frac{1}{\pi i} \int_{\Gamma_c} (zt)^{-\beta} E_{\alpha-1}(zt) F(z)\, dz \tag{2.2}$$

with $\Gamma_c = \{z/z \in \mathrm{C}$ *and* $\mathrm{Re}\sqrt{2z} = c\}$.

PROOF : Assume s to be fixed and that $1 < R < \infty$. Set:

$$I(s,T) = \int_0^T (st)^{\alpha+\beta-1} L_{\alpha-1}(st) f(t)\, dt =$$

$$= \frac{1}{\pi i} \int_0^T (st)^{\alpha+\beta-1} L_{\alpha-1}(st) \int_{\Gamma_c} (zt)^{-\beta} E_{\alpha-1}(zt) F(z)\, dz \tag{2.3}$$

where

$$\Gamma_c = \left\{ w/w \in \mathrm{C} \text{ and } \mathrm{Re}\sqrt{2w} = c \right\} =$$
$$= \left\{ w = a + bi/a = \frac{1}{2}(c^2 - t^2), b = ct, t \in (-\infty, +\infty) \right\}.$$

Consider, on the other hand, the domain defined as

$$\Lambda = \{(t,z)/t \in [0,T], z \in \Gamma_c\}.$$

To make feasible in (2.3) inversion of the order of integration it suffices to apply Fubini's theorem, previously verifying that

$$((st)^{\alpha+\beta-1} L_{\alpha-1}(st)(zt)^{-\beta} E_{\alpha-1}(zt) F(z))$$

proves an absolutely integrable function on Λ, provided that

$$\mathrm{Re}\,\alpha > 0, \text{ and } q > -\mathrm{Re}\,\beta + \frac{5}{4}.$$

Therefore, the following holds true

$$I(s,T) = \frac{s^{\alpha+\beta-1}}{\pi i} \int_{\Gamma_c} z^{-\beta} F(z) \int_0^T t^{\alpha-1} E_{\alpha-1}(zt) L_{\alpha-1}(st)\, dt dz. \tag{2.4}$$

Now, by invoking equality [11]

$$\int_0^T t^{\alpha-1} E_{\alpha-1}(zt) L_{\alpha-1}(st)\, dt = \frac{T^\alpha}{z-s}(z E_\alpha(zT) L_{\alpha-1}(sT) +$$
$$+ s E_{\alpha-1}(zT) L_\alpha(sT)) - \frac{s^{1-\alpha}}{2(z-s)}$$

and by substituting its right-hand side for the second part of (2.4), we obtain

$$I(s,T) = \frac{s^{\alpha+\beta-1}}{\pi i} \int_{\Gamma_c} z^{-\beta} F(z) \left[\frac{T^\alpha}{z-s}(z E_\alpha(zT) L_{\alpha-1}(sT) + s E_{\alpha-1}(zT) L_\alpha(sT)) - \right.$$
$$\left. - \frac{s^{1-\alpha}}{2(z-s)} \right] dz.$$

Now, by virtue of the asymptotic behaviour of $L_{\alpha-1}(t)$ and $E_{\alpha-1}(t)$, we can have the following inequality:

$$\left|\frac{T^\alpha}{z-s}(zE_\alpha(zT)L_{\alpha-1}(sT)+sE_{\alpha-1}(zT)L_\alpha(sT))\right| \leqslant$$
$$\leqslant N\cdot\frac{|z|^{-\operatorname{Re}\frac{\alpha}{2}}-\frac{1}{4}|s|^{-\operatorname{Re}\frac{\alpha}{2}}+\frac{1}{4}|z|^{1/2}(|z|^{1/2}+|s|^{1/2})}{||z|-|s||}\cdot e^{-\sqrt{2T}(\operatorname{Re}\sqrt{2s}-c)}$$

and, as a consequence,

$$\left|\frac{s^{\alpha+\beta-1}}{\pi i}\int_{\gamma_c} z^{-\beta}F(z)\frac{T^\alpha}{z-s}(zE_\alpha(zT)L_{\alpha-1}(sT)+sE_{\alpha-1}(zT)L_\alpha(sT))\right| <$$
$$< M_1|s|^{\operatorname{Re}\frac{\alpha}{2}+\operatorname{Re}\beta-\frac{3}{4}}e^{-\sqrt{2T}(\operatorname{Re}\sqrt{2s}-c)}\int_{\Gamma_c}|z|^{-q-\operatorname{Re}\frac{\alpha}{2}-\operatorname{Re}\beta-\frac{1}{4}}\,dz$$

is true for $\operatorname{Re}\sqrt{2s}>c$, due to $\frac{|z|^{1/2}(|z|^{1/2}+|s|^{1/2})}{||z|-|s||}$ being a bounded function.

On the other hand, the last integral converges because

$$q>-\operatorname{Re}\beta+1>-\operatorname{Re}\frac{\alpha}{2}-\operatorname{Re}\beta+\frac{3}{4}.$$

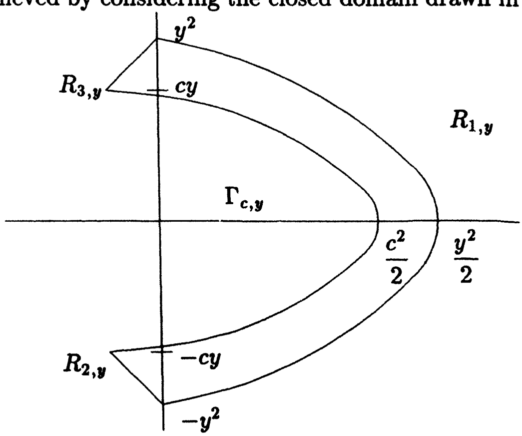
Thus, for every fixed s, with $\operatorname{Re}\sqrt{2s}>c>0$, this integral proves uniformly convergent on $1<T<\infty$ and then it is valid to take up the limit for $T\to\infty$:

$$\lim_{T->\infty} I(s,T)=\int_0^\infty(st)^{\alpha+\beta-1}L_{\alpha-1}(t)f(t)\,dt=\frac{s^\beta}{2\pi i}\int_{\Gamma_c}\frac{z^{-\beta}F(z)}{s-z}\,dz$$

To finish the proof, it only remains to perform the evaluation of the integral

$$\int_{\Gamma_c}\frac{z^{-\beta}F(z)}{s-z}\,dz$$

which can be achieved by considering the closed domain drawn in this figure:



$$R_y = \Gamma_{c,y} + \sum_{i=1}^{3} R_{i,y},$$

whose contour (considered by J. Betancor [1]) admits the following parametric representation

$$R_{1,y} = \begin{cases} a(t) = \frac{1}{2}(y^2 - t^2) \\ b(t) = yt \end{cases} \quad t \in [-y, y]$$
$$R_{2,y} = \begin{cases} a(t) = \frac{1}{2}(t^2 - y^2) \\ b(t) = -yt \end{cases} \quad t \in [c, y]$$
$$R_{3,y} = \begin{cases} a(t) = \frac{1}{2}(t^2 - y^2) \\ b(t) = ty \end{cases} \quad t \in [c, y]$$
$$\Gamma_{c,y} = \begin{cases} a(t) = \frac{1}{2}(c^2 - t^2) \\ b(t) = ct \end{cases} \quad t \in [-y, y].$$

If $F(z)$ is holomorphic on $\Omega = \{z/z \in \mathrm{C} \text{ and } \operatorname{Re}\sqrt{2z} > B > 0\}$, then it follows from Cauchy' theorem that

$$\int_{R_y} \frac{z^{-\beta} F(z)}{s - z}\, dz = 2\pi i s^{-\beta} F(s).$$

But according to the previously established bounds we can write

$$\left|\int_{R_{1,y}} \frac{z^{-\beta} F(z)}{s - z}\, dz\right| \leqslant \frac{M}{d(s)} \cdot y^{-2(q + \operatorname{Re}\beta - 1)}$$

which tends to zero for $y \to +\infty$ in view that $q > -\operatorname{Re}\beta + 1$. Here $d(s)$ denotes the distance from s to $R_{1,y}$.

The same procedure and conditions lead to

$$\left|\int_{R_{2,y}} \frac{z^{-\beta} F(z)}{s - z}\, dz\right| \to 0, \quad \left|\int_{R_{3,y}} \frac{z^{-\beta} F(z)}{s - z}\, dz\right| \to 0,$$

for $y \to \infty$.

Hence

$$\int_{R,y} \frac{z^{-\beta} F(z)}{s - z}\, dz = \int_{-\Gamma_{c,y}} \frac{z^{-\beta} F(z)}{s - z}\, dz$$

and, as a consequence,

$$\lim_{T -> \infty} I(s, T) = F(s)$$

can be easily inferred. ■

In the following, several propositions will be given in order to express the $M_{\alpha,\beta}$-integral transform in terms of Laplace's. We always take the assumption that every integral is absolutely convergent.

Proposition 3. *The integral transform*

$$F(s) = M_{\alpha,\beta}\{f(t)\} = \int_0^\infty (st)^{\alpha+\beta-1} L_{\alpha-1}(st) f(t)\,dt$$

can be re-written for $\operatorname{Re}\alpha > \frac{1}{2}$ *as:*

$$F(s) = M_{\alpha,\beta}\{f(t)\} = \tag{2.5}$$

$$= \frac{\sqrt{\pi}}{\Gamma(\alpha-1/2)} s^{\alpha+\beta-1} \mathfrak{L}\left\{\xi^{2\alpha+2\beta-1} \int_0^1 (1-\tau)^{\alpha-\frac{3}{2}} \tau^\beta f(\xi^2\tau)\,d\tau; 2\sqrt{s}\right\}$$

To justify this we will invoke the well-known connection existing between the K-integral transform and Laplace's [4], given as

$$\int_0^\infty (xy)^{1/2} K_{\alpha-1}(xy) g(x)\,dx =$$

$$= \frac{\sqrt{\pi}\cdot 2^{1-\alpha}}{\Gamma(\alpha-\frac{1}{2})} y^{\alpha-\frac{1}{2}} \int_0^\infty e^{-yx} \int_0^x (x^2-r^2)^{\alpha-\frac{3}{2}} r^{\frac{3}{2}-\alpha} g(r)\,dr dx$$

Now, by performing the changes of variable $x = \sqrt{t}$ $y = 2\sqrt{s}$ and $r = \sqrt{t\tau}$ and also by using the relation

$$L_{\alpha-1}(x) = x^{-\frac{\alpha-1}{2}} K_{\alpha-1}(2\sqrt{x})$$

we obtain

$$\int_0^\infty (st)^{\alpha+\beta-1} L_{\alpha-1}(st) f(t)\,dt =$$

$$= \frac{\sqrt{\pi} s^{\alpha+\beta-1}}{2\Gamma(\alpha-\frac{1}{2})} \int_0^\infty e^{-2\sqrt{st}} t^{\alpha+\beta-1} \int_0^1 (1-\tau)^{\alpha-\frac{3}{2}} \tau^\beta f(t\tau)\,d\tau dt \tag{2.6}$$

where $f(t) = t^{-\frac{\alpha}{2}-\beta+\frac{1}{4}} g(\sqrt{t})$.

Finally, the new change $t = \xi^2$ in the right-hand side of (2.6) leads to the result stated in (2.5).

Proposition 4. *The* $M_{\alpha,\beta}$*-integral transform can be expressed as*

$$F(s) = M_{\alpha,\beta}\{f(t)\} = \frac{s^\beta}{2} \mathfrak{L}\{f_{-\alpha,\alpha+\beta-1}(t)\} \tag{2.7}$$

provided that $\operatorname{Re} ts > 0$, *which proves equivalent to stating that*

$$F(s) = M_{\alpha,\beta}\{f(t)\} = \frac{s^{\alpha+\beta-1}}{2} \mathfrak{L}\{f_{\alpha-2,\beta}(t)\} \tag{2.8}$$

with

$$f_{\lambda,\gamma}(t) = \int_0^\infty t^\lambda \tau^\gamma e^{-\frac{\tau}{t}} f(\tau)\, d\tau. \tag{2.9}$$

In fact, by substituting the integral representation of (1.14) for the last part of (2.1), we have

$$F(s) = M_{\alpha,\beta}\{f(t)\} = \frac{1}{2}\int_0^\infty (st)^{\alpha+\beta-1} f(t) s^{1-\alpha} \int_0^\infty x^{-\alpha} e^{-xs-\frac{t}{x}}\, dx dt =$$
$$= \frac{s^\beta}{2}\int_0^\infty e^{-sx}\, dx \int_0^\infty x^{-a} t^{\alpha+\beta-1} f(t) e^{-\frac{t}{x}}\, dt = \frac{s^\beta}{2}\mathfrak{L}\{f_{-\alpha,\alpha+\beta-1}(x); s\}$$

once the integration order has been inverted.

Now, to obtain (2.8) substitute (1.15) for (2.1) and invert the order of integration.

Proposition 5. *The $M_{\alpha,\beta}$-integral transform can be given for* $\operatorname{Re} ts > 0$ *as:*

$$F(s) = M_{\alpha,\beta}\{f(t)\} = \frac{s^\beta}{2}\mathfrak{L}\left\{x^{-\alpha}\mathfrak{L}\left\{t^{\alpha+\beta-1}f(t); x^{-1}\right\}; s\right\} \tag{2.10}$$

or else

$$F(s) = M_{\alpha,\beta}\{f(t)\} = \frac{s^{\alpha+\beta-1}}{2}\mathfrak{L}\left\{x^{\alpha-2}\mathfrak{L}\left\{t^\beta f(t) : x^{-1}\right\}; s\right\} \tag{2.11}$$

3. Convolutions for the $M_{\alpha,\beta}$-integral transform.

In this section several convolutions for the $M_{\alpha,\beta}$-integral transform are given.

a). Define convolution $*$ of two functions $f(t)$ and $g(t)$, as:

$$f(t) * g(t) = \frac{1}{2}t^{-\beta} I^{\alpha-1}\int_0^t (t-\xi)^\beta\, d\xi \int_0^1 \eta^{\alpha+\beta-1}(1-\eta)^{\alpha+\beta-1}\cdot$$
$$\cdot f(\xi\eta) g[(1-\eta)(t-\xi)]\, d\eta, \tag{3.1}$$

where $I^{\alpha-1}$ stands for the Riemann—Liouville fractional integral [10].

Proposition 6. *If we define convolution $f(t) * g(t)$ as in* (3.1)*; $f(t), g(t), f(t) * g(t)$ being $M_{\alpha,\beta}$-transformable functions for* $\operatorname{Re}\sqrt{2s} > c > 0$*, then*

$$M_{\alpha,\beta}\{f(t) * g(t)\} = s^{1-\alpha-\beta} M_{\alpha,\beta}\{f(t)\}\cdot M_{\alpha,\beta}\{g(t)\}$$

is true.

Proof : In fact, from (2.11) it follows that

$$F(s) = M_{\alpha,\beta}\{f(t)\} = \frac{s^{\alpha+\beta-1}}{2}\int_0^\infty e^{-s\tau}\tau^{\alpha-2}\int_0^\infty e^{-\tau^{-1}t} t^\beta f(t)\, dt d\tau =$$
$$= \frac{s^{\alpha+\beta-1}}{2}\mathfrak{L}\left\{\tau^{\alpha-2} f_0(\tau); s\right\}$$

where $f_0(\tau) = \int_0^\infty e^{-\tau^{-1}t} t^\beta f(t)\,dt$.

Similarly,

$$G(s) = M_{\alpha,\beta}\{g(t)\} = \frac{s^{\alpha+\beta-1}}{2}\mathfrak{L}\left\{\tau^{\alpha-2} g_0(\tau); s\right\}.$$

Hence,

$$M_{\alpha,\beta}\{f(t)\}\cdot M_{\alpha,\beta}\{g(t)\} = \frac{s^{2\alpha+2\beta-2}}{4}\mathfrak{L}\left\{\tau^{\alpha-2} f_0(\tau)\right\}\cdot \mathfrak{L}\left\{\tau^{\alpha-2} g_0(\tau)\right\} =$$
$$\frac{s^{2\alpha+2\beta-2}}{4}\mathfrak{L}\left\{\int_0^t \xi^{\alpha-2} f_0(\xi)(t-\xi)^{\alpha-2} g_0(t-\xi)\,d\xi\right\}.$$

Now, the change $\xi = tu$ leads to

$$\frac{s^{2\alpha+2\beta-2}}{4}\mathfrak{L}\left\{t^{2\alpha-3}\int_0^1 u^{\alpha-2} f_0(tu) g_0[(1-u)t]\,du\right\} = \frac{s^{2\alpha+2\beta-2}}{4}\cdot$$
$$\mathfrak{L}\left\{t^{2\alpha-3}\int_0^\infty\int_0^\infty\int_0^1 u^{\alpha-2}(1-u)^{\alpha-2} e^{-t^{-1}[u^{-1}\tau+(1-u)^{-1}y]}\tau^\beta y^\beta f(\tau) g(y)\,du d\tau dy\right\}$$

which combined with

$$x = u^{-1}\tau + (1-u)^{-1}y, \quad \xi = u^{-1}\tau$$

yields

$$\frac{s^{2\alpha+2\beta-2}}{4}\mathfrak{L}\left\{t^{2\alpha-3}\int_0^\infty e^{-t^{-1}x}\,dx\int_0^x (x-\xi)^\beta \xi^\beta\,d\xi\cdot\right.$$
$$\left.\cdot\int_0^1 u^{\alpha+\beta-1}(1-u)^{\alpha+\beta-1} f(u\xi) g[(x-\xi)(1-u)]\,du\right\} = \tag{3.2}$$
$$= \frac{s^{2\alpha+2\beta-2}}{4}\mathfrak{L}\left\{t^{\alpha-2} t^{\alpha-1}\int_0^\infty e^{-t^{-1}x} H(f,g;x)\,dx\right\}$$

where

$$H(f,g;x) = \int_0^x (x-\xi)^\beta \xi^\beta\,d\xi\int_0^1 u^{\alpha+\beta-1}(1-u)^{\alpha+\beta-1} f(u\xi) g[(x-\xi)(1-u)]\,du$$

and by taking into account that

$$t^{\alpha-1}\int_0^\infty e^{-t-1x} H(f,g;x)\,dx = \int_0^\infty e^{-t-1x} I^{\alpha-1} H(f,g;x)\,dx$$

holds, then it can be easily inferred that (3.2) can be re-written as:

$$s^{\alpha+\beta-1}\left[\frac{s^{\alpha+\beta-1}}{2}\mathfrak{L}\left\{t^{\alpha-2}\mathfrak{L}\left\{x^\beta\frac{x^{-\beta}}{2} I^{\alpha-1} H(f,g;x); t^{-1}\right\}; s\right\} =\right.$$
$$s^{\alpha+\beta-1} M_{\alpha,\beta}\{f(t) * g(t)\}.$$

■

b). If we define convolution $\bar{*}$ of two functions $f(t), g(t)$ as

$$f(t)\bar{*}g(t) = \frac{t^{1-\alpha-\beta}}{2} I^{1-\alpha} \int_0^t (t-\xi)^{\alpha+\beta-1} \xi^{\alpha+\beta-1}\, d\xi \cdot \cdot \int_0^1 \eta^{\beta}(1-\eta)^{\beta} f(\xi\eta) g[(1-\eta)(t-\xi)]\, d\eta, \tag{3.3}$$

the following holds true:

Proposition 7. *If convolution $f(t)\bar{*}g(t)$ is defined as in (3.3) and if $f(t), g(t)$ and $f(t)\bar{*}g(t)$ are $M_{\alpha,\beta}$-transformable functions for* $\operatorname{Re}\sqrt{2s} > c > 0$, *then*

$$M_{\alpha,\beta}\{f(t)\bar{*}g(t)\} = s^{-\beta} M_{\alpha,\beta}\{f(t)\} \cdot M_{\alpha,\beta}\{g(t)\}$$

holds.

By using of (2.10), proof follows a similar procedure as in the previous proposition.

c). Let α, β be real numbers with $\alpha > 1$. It is feasible to define a convolution for the $M_{\alpha,\beta}$-integral transform in the space $C(\beta)$, which is made up of all complex functions of the form $f(t) = t^{\gamma-\beta} f_1(t)$, where $\gamma > -1$ and $f_1(t)$ being a continuous function on $[0, \infty)$.

Define in $C(\beta)$ the following operation:

$$f(t) \circ g(t) = \frac{t^{\alpha+\beta}}{2\Gamma(\alpha-1)} \int_0^1 \int_0^1 \int_0^1 t_3^{1+2\beta}(1-t_3)^{\alpha-2}(t_2(1-t_2))^{\beta} \cdot \cdot (t_1(1-t_1))^{\alpha+\beta-1} f(tt_1t_2t_3) g(tt_3(1-t_1)(1-t_2))\, dt_1 dt_2 dt_3 \tag{3.4}$$

By virtue of Weierstrass' approximation theorem the operation $(\circ)$ is completely defined by invoking

$$t^{\gamma-\beta+p} \circ t^{\gamma-\beta+q} = \frac{\Gamma(\gamma+\alpha+p)\Gamma(\gamma+\alpha+q)\Gamma\gamma+p+1)\Gamma\gamma+q+1)}{\Gamma(2\gamma+2\alpha+p+q)\Gamma(2\gamma+\alpha+p+q+1)} t^{\alpha+2\gamma-\beta+p+q}$$

for each $p, q \in \mathbb{N}$ with $\gamma > -1$.

Let us now consider the integral transform

$$T_{\alpha,\beta}\{f(t)\} = \int_0^{\infty} t^{\alpha+\beta-1} L_{\alpha-1}(st) f(t)\, dt$$

which is closely related to $M_{\alpha,\beta}$.

It is proved in [2] that the $(\circ)$-operation proves a convolution for the transformation $T_{\alpha,\beta}$ in the subset of $C(\beta)$ denoted as $C(\beta, c)$, with $c > 0$, and defined as follows:

$$C(\beta, c) = \left\{ f(t)/f(t) \in C(\beta) \quad \text{and } f(t) = 0(e^{c\sqrt{2t}}) \quad \text{for } t \to \infty \right\}.$$

Note that $M_{\alpha,\beta}\{f(t)\} = s^{\alpha+\beta-1}T_{\alpha,\beta}\{f(t)\}$ and also that

$$\frac{2}{\Gamma(-\alpha-\beta+1)\Gamma(-2\alpha-\beta+2)}T_{\alpha,\beta}\left\{t^{-2\alpha-2\beta+1}\right\} = s^{\alpha+\beta-1}$$

hold provided that $-\alpha-\beta+1>0$ and $-2\alpha-\beta+2>0$.

Under these conditions we define the operation

$$f(t)\bar{\circ}g(t) = \frac{2}{\Gamma(-\alpha-\beta+1)\Gamma(-2\alpha-\beta+2)}t^{-2\alpha-2\beta+1}\circ(f(t)\circ g(t))$$

and then the following can be established:

Proposition 8. *If $\alpha>1, -\alpha-\beta+1>0$ and $-2\alpha-\beta+2>0$ for each $f(t), g(t)\in C(\beta,c)$ in such a way that the expressions $t^{-2\alpha-2\beta+1}\circ(f(t)\circ g(t)$ and $f(t)\circ g(t)\in C(\beta,c)$ belong to $C(\beta,c)$, then the following holds:*

$$M_{\alpha,\beta}\{f(t)\bar{\circ}g(t)\} = M_{\alpha,\beta}\{f(t)\}\cdot M_{\alpha,\beta}\{g(t)\}$$

for $\mathrm{Re}\sqrt{2s}>c$.

PROOF : It suffices to note that

$$\begin{gathered}M_{\alpha,\beta}\{f(t)\bar{\circ}g(t)\} = s^{\alpha+\beta-1}T_{\alpha,\beta}\{f(t)\bar{\circ}g(t)\} = \\ \frac{2s^{\alpha+\beta-1}}{\Gamma(-\alpha-\beta+1)\Gamma(-2\alpha-\beta+2)}T_{\alpha,\beta}\left\{t^{-2\alpha-2\beta+1}\circ(f(t)\circ g(t))\right\} = \\ \frac{2s^{\alpha+\beta-1}}{\Gamma(-\alpha-\beta+1)\Gamma(-2\alpha-\beta+2)}T_{\alpha,\beta}\left\{t^{-2\alpha-2\beta+1}\right\}\cdot T_{\alpha,\beta}\{(f(t)\circ g(t))\} = \\ s^{2\alpha+2\beta-2}T_{\alpha,\beta}\{f(t)\}\cdot T_{\alpha,\beta}\{g(t)\} = M_{\alpha,\beta}\{f(t)\}\cdot M_{\alpha,\beta}\{g(t)\}.\end{gathered}$$

■

4. Operational rules.

The following operational rule, which relates the operator $A_{\alpha,\beta} = t^{1-\alpha-\beta}Dt^{\alpha}Dt^{\beta}$ to the $M_{\alpha,\beta}$-integral transform, comes in very useful in numerous applications.

Proposition 9. *Let $f(t)\in C^2((0,\infty))$, with*

$$\begin{gathered}f(t) = 0(t^m) \quad \text{if } m>\max(-\mathrm{Re}\,\beta, -\mathrm{Re}(\alpha+\beta)) \\ Dt^{\beta}f(t) = 0(t^n) \quad \text{if } n>\max(-1,-\mathrm{Re}\,\alpha)\end{gathered}$$

for $t\to 0^+$ and

$$f(t) = 0(e^{c\sqrt{2t}})$$

for $t\to+\infty$.

Then

$$M_{\alpha,\beta}\{A_{\alpha,\beta}f(t)\} = sM_{\alpha,\beta}\{f(t)\}$$

holds.

In fact

$$M_{\alpha,\beta}\left\{A_{\alpha,\beta}f(t)\right\} = \int_0^\infty (st)^{\alpha+\beta-1}L_{\alpha-1}(st)t^{1-\alpha-\beta}Dt^\alpha Dt^\beta f(t)\,dt =$$
$$= s^{\alpha+\beta-1}(a_1 - A_2 + \int_0^t t^\beta Dt^\alpha DL_{\alpha-1}(st)f(t)\,dt = sM_{\alpha,\beta}\left\{f(t)\right\},$$

can be stated after performing two integrations by parts and verifying that

$$A_1 = t^\alpha Dt^\beta f(t)L_{\alpha-1}(st)]_0^\infty = 0$$
$$A_2 = t^\beta f(t)DL_{\alpha-1}(st)]_0^\infty = 0$$

in view of the behaviour of $f(t)$ and $L_{\alpha-1}(st)$.

This result can be extended by induction as it is shown in the following:

Proposition 10. *Let k be a positive integer and $f(t)\in C^{2k}((0,\infty))$, with*

$$A_{\alpha,\beta}^{k-1}f(t) = 0(t^p), \qquad \text{if } p > \max(-\operatorname{Re}\beta, -\operatorname{Re}(\alpha+\beta))$$
$$Dt^\beta A_{\alpha,\beta}^{k-1}f(t) = 0(t^q) \qquad \text{if } q > \max(-1, -\operatorname{Re}\alpha)$$

for $t\to 0^+$, and

$$f(t) = 0(e^{c\sqrt{2t}})$$

for $t\to +\infty$.

Then, the following holds

$$M_{\alpha,\beta}\left\{A_{\alpha,\beta}^k f(t)\right\} = s^k M_{\alpha,\beta}\left\{f(t)\right\}.$$

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