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On convexity and smoothness of Banach space

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Abstract. We introduce a function being, in a certain sense, the inverse to the classical modulus of convexity of a Banach space. Several properties of this function and its connections with the moduli of convexity and smoothness are derived.

Keywords: Uniformly convex Banach space, uniformly smooth Banach space, modulus of convexity, modulus of smoothness

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1. Introduction.

The notion of the modulus of convexity introduced by Clarkson [4] plays an important role in many investigations of the geometric theory of Banach spaces and its applications(cf. [1],[3],[6],[7],[8],[12],[13]). This notion allows us to measure the convexity and rotundity of the unit ball of a Banach space. In order to indicate the importance of the modulus of convexity, let us recall that using this modulus one can select a class of Banach space having normal structure and being reflexive [9],[15], among others.

The aim of this paper is to describe and examine a function being, in a certain sense, the inverse to the modulus of convexity. We show that this function has several interesting properties and we point out to its connections with the Clarkson's modulus of convexity. Moreover, we obtain an inequality which corresponds to the well-known Lindenstrauss dual formula.

2. Basic properties of the modulus of convexity.

Let E be a given real Banach space with the norm $\|\cdot\|$ and the zero element θ. Denote by $K(\theta, r)$ the closed ball centered at x and with radius r. For simplicity, we shall denote by B and S the ball $K(\theta, 1)$ and the unit sphere at θ, respectively.

Recall that the modulus of convexity of a space E is the function $\delta_E : [0,2] \to [0,1]$ defined by the formula

$$\delta_E(\varepsilon) = \inf[1 - \frac{\|x+y\|}{2} : x, y \in B, \|x-y\| \geqslant \varepsilon],$$

or equivalently

$$\delta_E(\varepsilon) = \inf[1 - \frac{\|x+y\|}{2} : x, y \in S, \|x-y\| = \varepsilon].$$

The number $\varepsilon_0 = \varepsilon_0(E) = \sup[\varepsilon \geqslant 0 : \delta_E(\varepsilon) = 0]$ is called the characteristic of convexity of a space E. A space E is said to be uniformly convex provided $\varepsilon_0 = 0$.

In what follows we quote basic properties of the modulus of convexity. These properties may be found in [**9**],[**13**],[**15**], for example.

First let us recollect that the function δ_E is nondecreasing on the interval $[0,2]$ and is increasing on the interval $[\varepsilon_0(E),2]$. Moreover, δ_E is continuous on $[0,2)$ and is generally discontinuous at the point $\varepsilon = 2$. The equality $\delta_E(2) = 1$ holds true iff E is strictly convex (i.e. its unit sphere contains no segments).

For any Banach space E its modulus of convexity is bounded from above by a modulus of convexity of a Hilbert space H [**14**]

$$\delta_E(\varepsilon) \leqslant \delta_H(\varepsilon) = 1 - (1 - (\varepsilon/2)^2)^{1/2}.$$

For other facts and details concerning the modulus of convexity we refer to [**6**], [**9**], [**10**].

3. The function β and its properties.

Now we define the function $\beta_E : [0,1] \to [0,2]$ by

$$\beta_E(t) = \sup[\|x - y\| : x, y \in S, \frac{\|x + y\|}{2} = 1 - t]$$

or, in the equivalent form

$$\beta_E(t) = \sup[\|x - y\| : x, y \in S, \frac{\|x + y\|}{2} \geqslant 1 - t].$$

Observe that the function β_E is nondecreasing on the interval $[0,1]$. Thus, there exists the limit $\lim_{t\to 0} \beta_E(t)$. We have

Theorem 3.1. $\lim_{t\to 0} \beta_E(t) = \varepsilon_0(E)$.

PROOF : Fix $\varepsilon < \varepsilon_0(E)$. Then for an arbitrary $\eta > 0$ there exist $x, y \in S$, $\|x - y\| = \varepsilon$ with the property

$$1 - \frac{\|x + y\|}{2} \leqslant \eta$$

what means that

$$\beta_E(\eta) \geqslant \varepsilon$$

and consequently

$$\lim_{t\to 0} \beta_E(t) \geqslant \varepsilon.$$

Hence we get

$$\lim_{t\to 0} \beta_E(t) \geqslant \varepsilon_0(E)$$

Suppose now that

$$\beta_0 = \lim_{t\to 0} \beta_E(t) > \varepsilon_0(E). \tag{3.1}$$

Let us fix $t > 0$ arbitrarily small. Then there exist $x, y \in S$ such that $\frac{\|x+y\|}{2} = 1 - t$ and $\|x - y\| \geqslant \beta_0$. This implies $\delta_E(\beta_0) \leqslant t$ and in view of the arbitrariness of t we conclude that $\delta_E(\beta_0) = 0$. But this statement contradicts to (3.1) what gives the desired equality. ∎

Now let us observe that $\beta_E(1) = 2$. In order to characterize a set of these t for which $\beta(t) = 2$, let us denote

$$\delta_1 = \delta_1(E) = \lim_{\varepsilon \to 2} \delta_E(\varepsilon).$$

Then we have

Theorem 3.2. *If $t \in [\delta_1, 1]$, then $\beta_E(t) = 2$.*

PROOF : Assume that $\delta_1 < 1$. Next, fix $\eta > 0$ arbitrarily small. So there exists $\zeta > 0$ such that for $\varepsilon \in [2 - \zeta, 2)$ we have

$$\delta_1 - \eta \leqslant \delta_E(\varepsilon) \leqslant \delta_1.$$

The above inequalities allow us to infer that for an arbitrary $\chi > 0$ there exist $x, y \in S, \|x - y\| = \varepsilon$ such that

$$1 - \frac{\|x+y\|}{2} \leqslant \delta_E(\varepsilon) + \chi \leqslant \delta_1 + \chi.$$

Hence

$$1 - (\delta_1 + \chi) \leqslant \frac{\|x+y\|}{2}$$

and consequently

$$\beta_E(\delta_1 + \chi) \geqslant \varepsilon \geqslant 2 - \zeta.$$

By virtue of the fact that ζ and χ were chosen arbitrarily we obtain the assertion of our theorem. ∎

By the similar reasoning we can prove that $\beta_E(t) = 2$ implies that $t \in [\delta_1, 1]$. Thus we have

Corollary 3.1. $\beta_E(t) = 2$ *iff* $t \in [\delta_1, 1]$.

Corollary 3.2. $\delta_1 = \inf[t \leqslant 1 : \beta_E(t) = 2]$.

Now we show some relationship between the functions $\beta = \beta_E$ and $\delta = \delta_E$.

Theorem 3.3. $\beta(t) = 2(1 - \delta(2 - 2t))$ *for any* $t \in [0, 1]$.

PROOF : Keeping in mind the definition of the function β_E we can write the following sequence of equalities

$$\begin{aligned}
\beta(t) &= \sup[\|x + z\| : x, z \in S, \frac{\|x - z\|}{2} = 1 - t] \\
&= 2(\sup[1 - (1 - \frac{\|x+z\|}{2} : x, z \in S, \|x - z\| = 2 - 2t]) \\
&= 2(1 - \inf[1 - \frac{\|x+z\|}{2} : x, z \in S, \|x - z\| = 2 - 2t]) \\
&= 2(1 - \delta(2 - 2t)).
\end{aligned}$$

This ends the proof. ∎

As an immediate consequence of the above theorem we obtain the following corollaries.

Corollary 3.3. $\beta(0) = 2(1-\delta(2))$ *and E is strictly convex if and only if* $\beta(0) = 0$.

Corollary 3.4. $\lim_{t\to 0}\beta(t) = \varepsilon_0 = 2(1-\delta_1) = 2(1-\lim_{\varepsilon\to 2}\delta(\varepsilon))$ *and E is uniformly convex iff* $\delta_1 = 1$.

Corollary 3.5. *The function β is continuous on the interval* $(0,2]$ *and is increasing on the interval* $[0,\delta_1]$.

Apart from that we conclude that the function $\beta_E(t)$ may be eventually discontinuous at the point $t = 0$ only.

The below given theorem shows that the functions $\beta(t)$ and $\delta(\varepsilon)$ are, in a certain sense, inverse of one another.

Theorem 3.4. $\delta(\beta(t)) = t$ *for any* $t \in (0,\delta_1)$.

Proof : Let us fix $t \in (0,\delta_1)$ and $\eta > 0$. Then there exist $x, y \in S$, $\frac{\|x+y\|}{2} \geqslant 1-t$ such that

$$\beta(t) - \eta \leqslant \|x-y\| \leqslant \beta(t)$$

what implies

$$\delta(\beta(t)-\eta) \leqslant 1 - \frac{\|x+y\|}{2} \leqslant t.$$

Thus, taking into account the continuity of the function δ we get $\delta(\beta(t)) \leqslant t$. Now let us suppose that there is $t_0 \in (0,\delta_1)$ such that

$$\delta(\beta(t_0)) < t_0.$$

This permits us to deduce that for any $\gamma > 0$ there exist $x, y, \in S$, $\|x-y\| = \beta(t_0)$ such that

$$1 - \frac{\|x+y\|}{2} \leqslant t_0 - \gamma.$$

From the above inequality we have

$$1-(t_0-\gamma) \leqslant \frac{\|x+y\|}{2}$$

and consequently

$$\beta(t_0-\gamma) \geqslant \beta(t_0).$$

But this yields a contradiction to Corollary 3.5 what finishes the proof. ∎

Corollary 3.6. *β_E is the inverse function to the function δ_E on the interval* $(0,\delta_1)$, *i.e.* $\beta_E(t) = \delta_E^{-1}(t)$ *for* $t \in (0,\delta_1)$.

Apart from that the earlier obtained results allow us to deduce

Corollary 3.7. *$\beta(1-\frac{1}{2}\beta(t)) = 2-2t$ for $t \in (0, \delta_1)$.*

In case when E is uniformly convex, from the above proved facts we can conclude that the function β_E is continuous and increasing on the whole interval $[0,1]$ and $\beta_E(0) = 0, \beta_E(1) = 2$. Obviously, in such a case $\beta_E(t) = \delta_E^{-1}(t)$ for any $t \in [0,1]$. Thus, if $E = H$ (Hilbert space) we get

$$\beta_H(t) = 2\sqrt{1-(1-t)^2}.$$

The theorems proved here and the result due to Nordlander **[14]** imply that every function $\beta_E(t)$ is bounded from below by the function $\beta_H(t)$ associated with a Hilbert space H

$$\beta_E(t) \geqslant \beta_H(t) = 2\sqrt{1-(1-t)^2}, t \in [0,1].$$

Finally let us notice the following useful implication. Assume that $d \geqslant a > 0$ and let $x, y \in E$. Then

$$\|x\| \leqslant d, \|y\| \leqslant d \text{ and } \frac{\|x+y\|}{2} \geqslant a \Rightarrow \|x-y\| \leqslant d\beta_E(\frac{d-a}{a}).$$

This implication is very often used in the case of uniformly convex space E (with β_E replaced by δ_E^{-1}; cf. **[9]**,**[13]**, **[15]**).

4. An inequality associated with modulus of smoothness.

The notion of the modulus of smoothness was introduced in implicit form by Day **[5]** (cf. also **[10]**, **[11]**). The connection between the modulus of convexity of a Banach space E and the modulus of smoothness of its dual E^* has been established by Lindenstrauss in the paper **[11]**.

In this section we shall use the modulus of smoothness which seems to be defined in more natural way than the modulus due to Day **[2]**. Namely, for $\varepsilon \in [0,2]$ we define

$$\varrho_E(\varepsilon) = \sup[1 - \frac{\|x+y\|}{2} : x, y \in S, \|x-y\| \leqslant \varepsilon].$$

Some properties of the function ϱ_E may be found in **[2]**.

In the sequel we derive an equality being a counterpart of the above mentioned Lindenstrauss's formula. This inequality allows us to estimate the modulus of smoothness ϱ_E by the function β_{E^*}.

We start with the following two simple lemmas.

Lemma 4.1. *If $a, b \in [0,2]$ and $a \neq 0$, then $(2a+2b-4)/a \leqslant b$.*

PROOF : Indeed, the inequality $a(2-b) \leqslant 2(2-b)$ is equivalent to the inequality $2a - ab \leqslant 4 - 2b$ what implies

$$2a + 2b - 4 \leqslant ab.$$

This yields the desired result. ∎

Lemma 4.2. *Let $x, y \in E$ and $f, g \in E^*$ be such that $f(x) = g(y) = 1$. Then*

$$\|f+g\|\|x+y\| + \|f-g\|\|x-y\| \geqslant 4.$$

PROOF : Taking into account our assumptions we have

$$\|f+g\|\|x+y\|+\|f-g\|\|x-y\| \geqslant (f+g)(x+y)+(f-g)(x-y) = 2f(x)+2g(y) = 4,$$

and the proof is complete. ∎

Now we show the theorem announced before.

Theorem 4.1. *For an arbitrary $\varepsilon \in [0, 1/2]$ the following inequality holds true*

$$\varrho_E(4\varepsilon) \leqslant \varepsilon\beta_{E^*}(2\varepsilon).$$

PROOF : In the case when $\varepsilon = 1/2$ we have $\varrho_E(2) = 1$ and $\beta_{E^*}(1) = 2$ so our inequality is satisfied. Now, assume that $\varepsilon < 1/2$ and take $x, y \in S$ such that $\|x - y\| \leqslant 4\varepsilon$. Next, choose $f, g \in S^*$ such that $f(x) = g(y) = 1$. Then applying Lemma 4.2, we get

$$4 \leqslant \|f+g\|\|x+y\| + 4\varepsilon\|f-g\|. \tag{4.1}$$

Hence, using the fact that $x + y \neq \theta$ (because $\|x - y\| \leqslant 4\varepsilon < 2$) we obtain

$$\|f+g\|\|x+y\| \geqslant \frac{4 - 4\varepsilon\|f-g\|}{\|x+y\|}$$

and consequently

$$\begin{aligned} 1 - \frac{\|f+g\|}{2} &\leqslant \frac{\|x+y\| + 2\varepsilon\|f-g\| - 2}{\|x+y\|} \\ &= (1/2)\frac{2\|x+y\| + 4\varepsilon\|f-g\| - 4}{\|x+y\|}. \end{aligned}$$

Combining the last inequality with Lemma 4.1, we deduce

$$1 - \frac{\|f+g\|}{2} \leqslant (1/2)2\varepsilon\|f-g\| \leqslant 2\varepsilon,$$

what implies

$$\|f-g\| \leqslant \sup[\|k-h\| : h, k \in S, \frac{\|h+k\|}{2} \geqslant 1 - 2\varepsilon] = \beta_{E^*}(2\varepsilon). \tag{4.2}$$

Further, taking into account the fact that $\varepsilon < 1/2$ we conclude that $f + g \neq \theta$, so using the inequality (4.1) and Lemma 4.1, we get

$$\|x+y\| \geqslant \frac{4 - 4\varepsilon\|f-g\|}{\|f+g\|}.$$

Consequently, in view of (4.2) we have

$$
\begin{aligned}
1-\frac{\|x+y\|}{2} \leqslant \frac{\|f+g\|+2\varepsilon\|f-g\|-2}{\|f+g\|} = \\
=(1/2)\frac{2\|f+g\|+4\varepsilon\|f-g\|-4}{\|f+g\|} \leqslant \\
\leqslant (1/2)\frac{2\|f+g\|+4\varepsilon\beta_{E^*}(2\varepsilon)-4}{\|f+g\|} \leqslant (1/2)2\varepsilon\beta_{E^*}(2\varepsilon)=\varepsilon\beta_{E^*}(2\varepsilon).
\end{aligned}
$$

The last inequality, in virtue of the arbitrariness of $x, y \in S$ such that $\|x-y\| \leqslant 4\varepsilon$, yields the inequality from thesis. Thus the proof is complete. ∎

Finally, let us pay attention to the fact that writing the inequality from Theorem 4.1 in the form

$$\varrho_E(\varepsilon)/\varepsilon \leqslant \frac{1}{4}\beta_{E^*}(\varepsilon/2)$$

and taking into account Theorem 3.1 we have

Corollary 4.1. *If the dual space E^* is uniformly convex, then the space E is uniformly smooth.*

Moreover, we get

Corollary 4.2. *If E is uniformly convex, then E^* is uniformly smooth.*

The last Corollary follows from the fact that in view of Theorem 4.1 we obtain the following inequality

$$\varrho_{E^*}(\varepsilon)/\varepsilon \leqslant \frac{1}{4}\beta_{E^{**}}(\varepsilon/2)$$

for any $\varepsilon \in [0,2]$. This inequality together with the fact saying that the image of the unit ball B under the canonical embedding is weakly star dense in B^{**}, implies

$$\varrho_{E^*}(\varepsilon)/\varepsilon \leqslant \frac{1}{4}\beta_E(\varepsilon/2).$$

Hence we obtain our Corollary.

Remark. Note that Corollaries 4.1 and 4.2 are well known(cf.[**10**]).

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