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Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 2, 357--366

Persistent URL: http://dml.cz/dmlcz/106865

Comment.Math.Univ.Carolinae 31,2(1990)357–366

Invariance principles in $L^2[0,1]$

PAULO EDUARDO OLIVEIRA[1]

Abstract. We study invariance principles in the space $L^2[0,1]$. For this purpose we introduce an isometry with an auto-reproducing Hilbert space which enables us to establish conditions for weak relative compactness. This together with some mixing conditions allows us to establish weak invariance principles.

Keywords: relative compacity, α–mixing, invariance principle

Classification: 60F17

1. Introduction. There has been some interest on studying convergence in distribution on $L^2[0,1]$, see, for example Mason [**8**], where further references are given. In this paper we are concerned with the study of invariance principles on $L^2[0,1]$. That is, instead of studying convergence of the Donsker function on $D[0,1]$ with the Skhorokhod topology as usual (see, for example Herrndorf [**4**], [**5**], Peligrad [**11**]) we are interested on weaker versions of those results. The main problem when proving this kind of results is the proof of the relative compactness of the sequence of $L^2[0,1]$ valued random variables. In order to solve these difficulties we use the method that will be described in section 2. This method is based on the ideas of Berlinet [**2**], ch. 2, where Berlinet is interested on the empirical process for independent and identically distributed random variables. However the method used seems not suitable to the study of the empirical process when we assume only a mixing condition. The study of the relative compactness will be carried on section 3, where it is proved a general condition for weak relative compactness (theorem 3.1), from which we easily derive other conditions already used by other authors when studying similar problems. In section 4, using those characterizations of relative compactness together with some other conditions we derive a general invariance principle in $L^2[0,1]$ (theorem 4.2), from which follow some corollaries concerning particular cases and an interesting result about stochastic integrals (corollary 4.5). Finally, in section 5, we prove some invariance principles with easier verifiable conditions than the ones used on the theorems of section 4.

2. Preliminaries. Consider the kernel $R(s,t) = 1 - \max(s,t)$. This kernel defines an auto-reproducing Hilbert space H_R (see, for example Aronszajan [**1**]). The functions f which are in the space H_R are of the form $f(u) = \int_u^1 g(t)dt$, for some function $g \in L^2[0,1]$. If f', $f'' \in H_R$ are defined by $f'(u) = \int_u^1 g'(t)dt$ and

[1] This work was partially supported by a scholarship of INIC – Instituto Nacional de Investigação Cientifica

$f''(u) = \int_u^1 g''(t)dt$, respectively, then the auto–reproducing inner product is given by $\langle f', f''\rangle_R = \int_0^1 g'(t)g''(t)dt$. This auto–reproducing Hilbert space H_R is isometrically isomorph to the space $L^2[0,1]$ by the isomorphism

$$\Psi\colon L^2[0,1] \longrightarrow H_R$$

$$g \hookrightarrow \int_u^1 g(t)dt\,.$$

So, one can equivalently study convergence problems and relative compactness characterizations of probability distributions on $L^2[0,1]$ or on H_R, which one appears to be the most convenient.

Now take $\mathcal{M}$ to be the space of bounded signed measures on $[0,1]$ endowed with the Borel σ–algebra. According to Suquet [13], the space of measures $\mathcal{M}$ may be embedded in the space H_R by the function $\varphi(\mu)(s) = \int_0^1 R(s,t)d\mu = \int_s^1 \mu[0,u]du$. One easily checks that $\langle f, \varphi(\mu)\rangle_R = \int_0^1 g(u)\mu[0,u]du = \int_0^1 g(u)\mu(du)$, where $f(u) = \int_u^1 g(t)dt$, using the isometry between $L^2[0,1]$ and H_R, which generalizes the formula of integration by parts.

Let $\{\xi_n\}$ be a sequence of real random variables, and $\sigma > 0$ a real number. For each $n \in \mathbb{N}$ define the random element $\mu_n = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^n \xi_n \delta_{\frac{i}{n}}$. This random element is, for a suitably chosen σ–algebra, a random variable taking values in $\mathcal{M}$. Therefore, taking account of the embedding φ and of the isomorphism ψ, we may interpret this random variable as a random variable taking values in H_R or in $L^2[0,1]$, as is more convenient. As a random variable taking values in the space $L^2[0,1]$, μ_n is interpreted as $\psi^{-1}\varphi(\mu_n)(u) = -\mu_n[0,u]$, and $\mu_n[0,u] = \frac{1}{\sigma\sqrt{n}} S_{[nu]}$. That is, μ_n as a random variable in $L^2[0,1]$ is the function appearing in the invariance principle (see, for example, Billingsley [3]). For this reason we shall call the random measure μ_n the <u>Donsker random measure</u>.

3. Relative compactness of $\{\mu_n[0,u]\}$ in $L^2[0,1]$. Define the functions $g_i(u) = \cos(i+\frac{1}{2})u\pi$, and the real numbers $\lambda_i = ((i+\frac{1}{2})\pi)^{-2}$, $i \in \mathbb{N}$. Then the functions $\sqrt{\lambda_i}g_i$, $i \in \mathbb{N}$, are an orthonormal basis of $L^2[0,1]$ such that $R(s,t) = \sum_{i=0}^{\infty} \lambda_i g_i(s)g_i(t)$, and this series converge uniformly. Moreover, note that the series $\sum_{i=0}^{\infty} \lambda_i$ converge. For simplicity one may choose an orthonormal basis of H_R of the form $G_i = \sqrt{\lambda_i}g_i$. Using the isometry described in section 2 we can prove a general sufficient condition for weak relative compactness.

Theorem 3.1. *Suppose the random variables ξ_n, $n \in \mathbb{N}$, satisfy the condition*

$$\sum_{k=2}^{n} E\xi_k^2 + 2\sum_{i=2}^{n}\sum_{j=1}^{i-1} |E\xi_i\xi_j| = O(n) \tag{1}$$

Then the set $\{\frac{1}{\sigma\sqrt{n}}S_{[nu]},\ n \in \mathbb{N}\}$ is weakly relatively compact in $L^2[0,1]$.

PROOF : To prove this relative compactness we will interpret the Donsker random measure as a random variable taking values in the space H_R and prove

$$\sup_{n\in\mathbb{N}} \int_{H_R} R_N(F)P_n(dF) \longrightarrow 0, \text{ as } N \longrightarrow \infty, \tag{2}$$

where $R_N(F) = \sum_{i=N}^{\infty} \langle F, G_i\rangle_R^2$, and P_n is the probability distribution of μ_n as a random variable in H_R. Then the relative compactness stated will follow (Parthasarathy [10]). Evaluating the integral in (2) we obtain

$$\int_{H_R} R_N(F)P_n(dF) = \int_{H_R} \sum_{i=N}^{\infty}\langle F, G_i\rangle_R^2 P_n(dF) = E\left(\sum_{i=N}^{\infty}\langle \varphi(\mu), G_i\rangle_R^2\right) =$$

$$= E\left(\sum_{i=N}^{\infty}\left(\int G_i(u)\mu_n(du)\right)^2\right) = \sum_{i=N}^{\infty}\frac{\lambda_i}{\sigma^2}\left(\sum_{k,l=1}^{n} g_i\left(\frac{k}{n}\right) g_i\left(\frac{l}{n}\right) E\xi_k\xi_l\right) =$$

$$= \sum_{i=N}^{\infty}\frac{\lambda_i}{\sigma^2 n}\widetilde{g}_{in}^t\Gamma_n\widetilde{g}_{in}, \tag{3}$$

where $\widetilde{g}_{in}^t = (g_i(\frac{1}{n}),\dots,g_i(1))$ and Γ_n is the covariance matrix of $\xi_1,\dots,\xi_n$. Put $O_n = \widetilde{g}_{in}^t\Gamma_n\widetilde{g}_{in}$. Then, noting that $\|\widetilde{g}_{in}\|_\infty \le 1$, we have $|O_n - O_{n-1}| \le \Delta_n = \max_{\|\widetilde{x}\|_\infty<1} \widetilde{x}^t\widehat{\Gamma}_n\widetilde{x}$, where $\widehat{\Gamma}_n$ is obtained from Γ_n by replacing all entries by zero, except the last row and last column. Evidently $O_n \le O_1 + \sum_{k=2}^{n}\Delta_k$. It easily verified that $\Delta_k = E\xi_k^2 + 2\sum_{j=1}^{k-1}|E\xi_j\xi_k|$. Then

$$\sum_{k=2}^{n}\Delta_k = \sum_{k=2}^{n}E\xi_k^2 + 2\sum_{i=2}^{n}\sum_{j=1}^{i-1}|E\xi_i\xi_j|.$$

Taking account of condition (1) it follows $O_n = O(n)$, that is, there exists some constant L such that $\frac{1}{n}\widetilde{g}_{in}^t\Gamma_n\widetilde{g}_{in} \le L$, for all $i, n \in \mathbb{N}$. So, an upper bound to (3) is $\frac{L}{\sigma^2}\sum_{i=N}^{\infty}\lambda_i$, which converge to zero as $N \longrightarrow \infty$, thus proving the relative compactness of the sequence $\frac{1}{\sigma\sqrt{n}}S_{[nu]}$ in $L^2[0,1]$. ∎

Corollary 3.2. *Suppose the random variables ξ_n, $n \in \mathbb{N}$, are stationary, and $\sum_{k=0}^{\infty} |E\xi_0\xi_k| < \infty$. Then the set $\{\frac{1}{\sigma\sqrt{n}} S_{[nu]}, n \in \mathbb{N}\}$ is weakly relatively compact in $L^2[0,1]$.*

PROOF : If the ξ_n are stationary an upper bound for the left side of expression (1) is

$$(n-1)E\xi_0^2 + 2\sum_{k=1}^{n-1}(n-k)|E\xi_0\xi_k| \leq (n-1)E\xi_0^2 + 2n\sum_{k=1}^{n-1}|E\xi_0\xi_k| \leq 2n\sum_{k=1}^{\infty}|E\xi_0\xi_k|,$$

so condition (1) is verified. ■

Note that this condition is used by Billingsley [3] to derive weak relative compactness of the sequence $\frac{1}{\sigma\sqrt{n}} S_{[nu]}$ in $D[0,1]$.

When proving invariance principles one often suppose the convergence $\frac{1}{n}ES_n^2 \longrightarrow \sigma^2$. In what regards the proof of relative compacity we shall need only the weaker condition

$$\sup_{n\in\mathbb{N}} \frac{1}{n} ES_n^2 < \infty. \tag{4}$$

Corollary 3.3. *Suppose the random variables ξ_n, $n \in \mathbb{N}$, verify condition (4) and $E\xi_i\xi_j \geq 0$, $i, j \in \mathbb{N}$. Then the set $\{\frac{1}{\sigma\sqrt{n}} S_{[nu]}, n \in \mathbb{N}\}$ is weakly relatively compact in $L^2[0,1]$.*

PROOF : In this case the left side of condition (1) is equal to $ES_n^2 - E\xi_1^2$. So (1) is obviously verified. ■

Corollary 3.4. *Suppose the ξ_n, $n \in \mathbb{N}$, verify condition (4) and are non–correlated. Then the set $\{\frac{1}{\sigma\sqrt{n}} S_{[nu]}, n \in \mathbb{N}\}$ is weakly relatively compact in $L^2[0,1]$.*

PROOF : As the ξ_n are non–correlated $\sum_{i=2}^{n}\sum_{j=1}^{i-1} |E\xi_i\xi_j| = 0$, so (1) follows immediately from (4). ■

In general condition (4) seems not to be sufficient as it involves only the covariances and it seems essential to have some control on the absolute values of the covariances in order to be able to deduce relative compactness.

To finish this section we will present a sufficient condition for weak relative compactness of a different kind. It depends on the eigenvalues of the covariance matrices Γ_n, as n increases. This condition seems not to be comparable with condition (1) as far as the author was able to find.

Theorem 3.5. *Let Γ_n be the covariance matrix of $\xi_1, \ldots, \xi_n$, and let λ_n be the greatest eigenvalue of Γ_n. Moreover suppose $\inf_{n\in\mathbb{N}} \frac{1}{n}ES_n^2 > 0$ and condition (4). If* *there exist* $\lambda > 0$ *such that* $\sup \lambda_n \le \lambda$ *then the set* $\{\frac{1}{\sigma\sqrt{n}} S_{[nu]}, n \in \mathbb{N}\}$ *is weakly relatively compact in* $L^2[0,1]$.

PROOF : An upper bound of (3) is obtained replacing $\widetilde{g}^t_{in}\Gamma_n\widetilde{g}_{in}$ by $\max_{\|\widetilde{x}\|_\infty \le 1} \widetilde{x}^t\Gamma_n\widetilde{x}$. We shall prove that under the assumptions of the theorem this quantity is comparable with $u^t\Gamma_n u$, where $u^t = (1,\ldots,1)$. For each $n \in \mathbb{N}$, the size of the smallest semi–axe of the hyper–ellipsoid $\widetilde{x}^t\Gamma_n\widetilde{x} = Cu^t\Gamma_n u$, where C is a fixed positive constant, is given by

$$\sqrt{\frac{Cu^t\Gamma_n u}{\lambda_n}} \ge \sqrt{\frac{Cu^t\Gamma_n u}{\lambda}}.$$

If the n–dimensional hyper–cube $\|\widetilde{x}\|_\infty \le 1$ is contained inside the hyper–ellipsoid $\widetilde{x}^t\Gamma_n\widetilde{x} = Cu^t\Gamma_n u$ then it follows $\max_{\|\widetilde{x}\|_\infty \le 1} \widetilde{x}^t\Gamma_n\widetilde{x} \le Cu^t\Gamma_n u = CES_n^2$, $n \in \mathbb{N}$. If we can choose C independent of n then replacing in (3), condition (1) will follow from (4), thus obtaining the weak relative compacity searched. In order to include the hyper–cube we must choose C such that $\frac{1}{c} \le \frac{1}{\lambda} \inf \frac{1}{n} ES_n^2$. ■

Note that when proving invariance principles one often suppose the convergence $\frac{1}{n}ES_n^2 \longrightarrow \sigma^2 > 0$, so trivially $\inf_{n \in \mathbb{N}} \frac{1}{n}ES_n^2 > 0$ and (4). Remark also that the condition $\sup \lambda_n \le \lambda$ avoids complete chaos. In fact, if $\sup \lambda_n = +\infty$ this would mean that, when increasing dimension (that is, the number of random variables considered), it exists some direction (defined by the eigenvector associated with λ_n) where the variance increases without bound. So, as long as the dimension is large enough, the distribution of the random variables will become more and more chaotic. Of course, if one expects to prove some relative compactness or invariance principles it is expected to find some sort of regularity.

4. Some weak invariance principles. In order to prove weak convergence in distribution of a sequence of Hilbert space random variables one must establish weak relative compactness and then prove the convergence of the sequence of characteristic functions. In our case we will not follow this procedure but, instead establish a stronger convergence result. This is due to the fact that the characteristic function of the Donsker random measure as a random variable in $L^2[0,1]$ is not suitable to establish the desired result. To establish the convergence in distribution in $L^2[0,1]$ $\mu_n[0,u] = \frac{1}{\sigma\sqrt{n}} S_{[nu]} \xrightarrow{d} W(u)$, where W is a version of the brownian motion, we will prove the convergence of the marginal distributions

$$(\mu_n[0,u_1],\ldots,\mu_n[0,u_k]) \xrightarrow{d} (W(u_1),\ldots,W(u_k)),$$

for any choice $u_1,\ldots,u_k \in [0,1]$ and $k \in \mathbb{N}$. In what follows we always suppose verified the condition

$$\frac{1}{n}ES_n^2 \longrightarrow \sigma^2 > 0. \tag{5}$$

If we suppose that the real random variables ξ_n, $n \in \mathbb{N}$, satisfy the central limit theorem, then from (5) one easily derive the convergence of the one dimensional marginal distributions

$$\mu_n[0,u] = \frac{1}{\sigma\sqrt{n}} S_{[nu]} \xrightarrow{d} W(u),\ u \in [0,1]. \tag{6}$$

In order to prove the convergence of the k–dimensional marginal distributions we will need to impose some mixing and integrability conditions. For that purpose define the mixing coefficients

$$\alpha_n(k) = \begin{cases} \sup\{|P(A\cap B) - P(A)P(B)|,\ A \in \sigma(\xi_i,\ 1 \le i \le m) & \\ B \in \sigma(\xi_i, m+k \le i \le n), i \le m \le n-k\} & k = 1,\ldots,n-1 \\ 0 & k \ge n, \end{cases}$$

and

$$\alpha(k) = \sup_{n\in\mathbb{N}} \alpha_n(k),\ k \in \mathbb{N}$$

The sequence $\{\xi_n\}$ is called α–mixing if $\alpha(k) \longrightarrow 0$, as $k \longrightarrow \infty$.

Lemma 4.1. *Suppose the sequence of random variables $\{\xi_n\}$ satisfy the central limit theorem, condition (5), is α–mixing and verifies*

$$\sup\{\frac{1}{n} E(S_{m+n} - S_m)^2,\ m\,, n \in \mathbb{N}\} < \infty \tag{7}$$

Then, for every choice $k \in \mathbb{N}$ and $u_1, \ldots, u_k \in [0,1]$

$$(\mu_n[0,u_1], \ldots, \mu_n[u_k]) \xrightarrow{d} (W(u_1), \ldots, W(u_k)).$$

PROOF : We will proceed as in Herrndorf [5] to prove this convergence. By the classical Prokhorov characterization of weak relative compactness and the fact that one dimensional marginals are weakly relatively compact one easily derives that $\{(\mu_n[0,u_1], \ldots, \mu_n[0,u_k]), n \in \mathbb{N}\}$ is weakly relatively compact. So this sequence has a subsequence which is convergent in distribution to some probability distribution Q on $\mathbb{R}^k$. Let π_{u_i} be the projection associated with the component u_i, and choose $\{r_n\}$ some sequence of non–negative reals such that $r_n \longrightarrow 0$ and $nr_n \longrightarrow +\infty$. As the convergence (6) holds we deduce that every $Q\pi_{u_i}^{-1}$ is gaussian with variance u_i. From (7) one easily derive $E(\mu_n[0,u_i+r_n] - \mu_n[0,u_i])^2 \longrightarrow 0$, taking account of the choice of the sequence $\{r_n\}$. If follows, from this and the asymptotic independence of the increments, that the probability distribution $Q(\pi_{u_1}, \pi_{u_2} - \pi_{u_1}, \ldots, \pi_{u_k} - \pi_{u_{k-1}})^{-1}$ is the weak limit of some subsequence of $\{(\mu_n[0,u_1], \mu_n[0,u_2] - \mu_n[0,u_1+r_n], \ldots, \mu_n[0,u_k] - \mu_n[0,u_{k-1}+r_n])\}$. Now, from the α–mixing condition and $nr_n \longrightarrow +\infty$, we derive that the functions π_{u_1}, $\pi_{u_2} - \pi_{u_1}$, $\ldots$, $\pi_{u_k} - \pi_{u_{k-1}}$ are independent under Q, so Q is the distribution of $(W(u_1), \ldots,$

$W(u_k))$. We will describe this proof for $k=2$, the general case follows by a recursive application of this process. Let A_1, A_2 be Borel sets of $[0,1]$. Then

$$\begin{aligned}
&|Q(\pi_1,\pi_2-\pi_1)^{-1}(A_1\times A_2)-Q\pi_1^{-1}(A_1)Q(\pi_2-\pi_1)^{-1}(A_2)| = \\
&\quad = \lim |P\{\mu_n[0,u_1]\in A_1,\ \mu_n[0,u_2]-\mu_n[0,u_1+r_n]\in A_2\}- \\
&\quad - P\{\mu_n[0,u_1]\in A_1\}P\{\mu_n[0,u_2]-\mu_n[0,u_1+r_n]\in A_2\}| \leq \\
&\qquad\qquad \leq \alpha_n([nr_n]) \leq \alpha([nr_n]) \longrightarrow 0,
\end{aligned}$$

remarking that $\{\mu_n[0,u_1]\in A_1\}\in\sigma(\xi_i,\ 1\leq i\leq [nu_1])$ and $\{\mu_n[0,u_2]-\mu_n[0,u_1+r_n]\in A_2\}\in\sigma(\xi_i,\ [nu_1+nr_n]\leq i\leq [nu_2])$. ∎

Now combining this lemma with some relative compactness condition we derive a weak invariance principle.

Theorem 4.2. *Suppose the random variables ξ_n, $n\in\mathbb{N}$, satisfy the central limit theorem, conditions (1), (5) and (7), and are α-mixing. Then $\frac{1}{\sigma\sqrt{n}}S_{[nu]}$ converge in distribution to W in $L^2[0,1]$.*

PROOF : It follows directly from the lemma 4.1 and theorem 3.1. ∎

Note that in this theorem we do not require the existence of moments of order higher that 2, as we require directly the central limit theorem, and impose no condition on the mixing coefficients as, for example Herrndorf [5]. As corollaries of this theorem we state the corresponding weak invariance principles of some particular cases.

Corollary 4.3. *Suppose the random variables ξ_n, $n\in\mathbb{N}$, satisfy the central limit theorem, conditions (5) and (7), are α-mixing, and non-correlated. Then $\frac{1}{\sigma\sqrt{n}}S_{[nu]}$ converge in distribution to W in $L^2[0,1]$.*

PROOF : Use lemma 4.1 and corollary 3.4. ∎

Note that this corollary includes the independent and identically distributed case. Condition (7) is trivially verified in this case, so the invariance principle follows from condition (5) alone.

Corollary 4.4. *Suppose the random variables ξ_n, $n\in\mathbb{N}$, satisfy the central limit theorem, condition (5), are α-mixing, stationary and verify $\sum\limits_{k=0}^{\infty}|E\xi_0\xi_k|<\infty$. Then $\frac{1}{\sigma\sqrt{n}}S_{[nu]}$ converge in distribution to W in $L^2[0,1]$.*

PROOF : Use lemma 4.1 and corollary 3.2. ∎

From theorem 4.2 we can also derive some convergence results concerning stochastic integrals.

Corollary 4.5. *Suppose verified the conditions of theorem 4.2. Let $f\in L^2[0,1]$ and define $F(u)=\psi f(u)=\int\limits_u^1 f(t)dt$. Then $\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^n \xi_n F\left(\frac{i}{n}\right)\xrightarrow{d}\int\limits_0^1 F(u)dW_u$.*

PROOF : As the conditions of theorem 4.2 are satisfied it follows the convergence $\frac{1}{\sigma\sqrt{n}}S_{[nu]}\xrightarrow{d}W$ in $L^2[0,1]$. So the corresponding characteristic functions converge.

That is, for $f \in L^2[0,1]$

$$Ee^{i\langle f,\mu_n\rangle} \longrightarrow Ee^{i\langle f,W\rangle},$$

where $\langle .,. \rangle$ stands for the $L^2[0,1]$ inner product. Using the isometry ψ between $L^2[0,1]$ and H_R, and the embedding φ of $\mathcal{M}$ in H_R, we obtain

$$\langle f,\mu_n\rangle = \langle F, \int_u^1 \mu_n[0,t]dt\rangle_R = \langle F, \varphi(\mu_n)\rangle_R = \int_0^1 F(u)\mu_n(du)$$

as seen before. Recalling the expression of the Donsker random measure, it follows

$$\int_0^1 F(u)\mu_n(du) = \frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n} \xi_n F\left(\frac{i}{n}\right).$$

On the other hand $\langle f, W\rangle = \int_0^1 F(u)W(u)du = \int_0^1 F(u)dW_u$, according to the definition of F and elementary properties of the stochastic integrals (see, for example, Hida [6]). Then the convergence of the characteristic functions proves the result stated. ∎

5. Further weak invariance principles. Until now we have proved weak invariance principles assuming both α-mixing and the central limit theorem. The α-mixing condition may be used, together with some other suitably chosen conditions, to derive the central limit theorem needed to use lemma 4.1. For this purpose introduce the following condition

$$\sup\{\frac{1}{\sqrt{n}}E|S_{m+n}-S_m|^{2+\epsilon},\, m,\, n \in \mathbb{N}\} < \infty. \tag{8}$$

Using Hölder's inequality one easily derives that (8) implies (7), so imposing (8) we need no extra condition to use lemma 4.1.

Theorem 5.1. *Suppose the random variables ξ_n, $n \in \mathbb{N}$, satisfy conditions (1), (5), (8) and are α-mixing. Moreover suppose they satisfy at least one of the conditions*

(A) $\sum_{k=0}^{\infty} \sup_{|l-m|>k} E\xi_l\xi_m < \infty,$

or

(B) $\sup_{l,m\leq n} E\xi_l\xi_m = O\left(\frac{1}{n}ES_n^2\right)$, $\sum_{j=q_n}^{n-1} \sup_{|l-m|\geq j,\ l,m\leq n} E\xi_l\xi_m \longrightarrow 0$ *as* $q_n \longrightarrow +\infty$.

Then $\frac{1}{\sigma\sqrt{n}}S_{[nu]}$ *converge in distribution to* W *in* $L^2[0,1]$.

PROOF : From conditions (5), (8), the α-mixing and (A) or (B) we easily derive that conditions of corollary 2.1 of Withers [15] are satisfied. Therefore the random variables ξ_n satisfy the central limit theorem. As remarked before, (8) implies (7), so the conditions of lemma 4.1 are satisfied. Finally (1) ensures the weak compacity, so the result follows. ∎

In some interesting particular cases these conditions become much more simplified. The case which is the most simplified is when the random variables are non-correlated.

Corollary 5.2. *Suppose the random variables ξ_n, $n \in \mathbb{N}$, satisfy conditions (5), (8), are α-mixing and non-correlated. Then $\frac{1}{\sigma\sqrt{n}} S_{[nu]} \xrightarrow{d} W(u)$ in $L^2[0,1]$.*

PROOF : As the ξ_n, $n \in \mathbb{N}$, are non-correlated (5) is sufficient to the relative weak compacity of $\frac{1}{\sigma\sqrt{n}} S_{[nu]}$ in $L^2[0,1]$. Further, condition (A) of theorem 5.1 becomes $\sup E\xi_k^2 < \infty$, and this follows from (8). So ξ_n, $n \in \mathbb{N}$, satisfy the central limit theorem. Then using lemma 4.1 and corollary 3.4 the result follows. ∎

Notice that until now we have imposed no conditions on the mixing coefficients, besides the α-mixing condition itself. In the stationary case we will need to impose some condition on mixing coefficients in order to derive a central limit theorem.

Theorem 5.3. *Suppose the ξ_n, $n \in \mathbb{N}$, satisfy condition (8), are α-mixing, strictly stationary and the mixing coefficients satisfy*

$$\sum_{k=1}^{\infty} \alpha(k)^{\frac{\varepsilon}{2+\varepsilon}} < \infty$$

for the same ε of condition (8). Then $\frac{1}{\sigma\sqrt{n}} S_{[nu]} \xrightarrow{d} W(u)$ in $L^2[0,1]$ with $\sigma = E\xi_1^2 + 2\sum_{k=1}^{\infty} E\xi_1\xi_{k+1}$.

PROOF : According to the theorem 1.7 of Ibragimov [7] from the assumptions it follows that the series defining σ is convergent and the ξ_n satisfy the central limit theorem. Then use lemma 4.1 and corollary 3.2 to prove the result stated. ∎

It is possible to state a theorem supposing only weak stationarity using theorem 9 of Philipp [12]. This requires another mixing condition (φ-mixing) and condition on the fourth moments of the random variables. We could also use a theorem of Mori, Yoshihara [9], which gives a necessary and sufficient condition for the central limit theorem for strictly stationary case using uniform integrability, and an extension of this theorem to the non-stationary case, Volný [14]. For sake of brevity we will not state these results here.

Finally remark that, although not mentioned, every theorem involving condition (1) has an evident duplicate replacing this condition by the condition on the eigenvalues of the covariance matrices used in theorem 3.5. Also remark that every set of conditions used evidently imply the conclusion of corollary 4.5.

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(*Received* November 14, 1989)