Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 31 (1990), No. 2, 307--313

Persistent URL: http://dml.cz/dmlcz/106860

Property (G) and (K) of Orlicz spaces[1]

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Abstract. For Orlicz spaces, (K)$\Longleftrightarrow$(H) and (G)$\Longleftrightarrow$(HR) are proved.

Keywords: Orlicz space

Classification: 46B20

In 1958, the property (G) of Banach space X was introduced by Fan and Glicksberg [1]. X is said to have (G) if every point on the unit sphere $S(X)$ is denting point of the closed unit ball. Twenty eight years passed, unexpectedly, Lin, Lin and Troyanski [2] discovered that (G) is equivalent to (K) + (R) for any Banach space. X is said to have property (K) if the norm topology and the weak topology coincide on $S(X)$. (R) denotes the rotundity. For Orlicz spaces, the criteria of property (H) were obtained [3], [4]. X is said to have (H) if for any sequence on $S(X)$, weak and norm convergence coincide. In this paper, we proved that (K)$\Longleftrightarrow$(H) and (G)$\Longleftrightarrow$(HR) for either the Orlicz function space $L_M[0,1]$ or the sequence space l_M endowed with either the Orlicz norm $\|\cdot\|_M$ or the Luxemburg norm $\|\cdot\|_{(M)}$.

$M(u)$, $N(v)$ denote a pair of complementary N-functions. For a function $x(t)$, its modulo $\rho_M(x) = \int_0^1 M(x(t))dt$ and for a sequence $(x(j))_1^\infty$, its modulo $\rho_M(x) = \sum_{j=1}^{\infty} M(x(j))$. "$M \in \Delta_2$" denotes that $M(u)$ satisfies the Δ_2 condition for large (small, in the case of the sequence spaces) u, and "$M \in \mathrm{sc}[0,\infty)$" denotes that $M(u)$ is strictly convex on $[0,\infty)$.

Theorem 1.1. *For* $[l_M, \|\cdot\|_{(M)}]$, *(K)*$\Longleftrightarrow M \in \Delta_2$.

PROOF : Necessity. See [**3**].

Sufficiency. Suppose $x \in S(l_M)$ and $\tau > 0$, by $M \in \Delta_2$, there exists $\varepsilon > 0$ such that

$$(1) \qquad \rho_M(x) \le 2\varepsilon \implies \|x\|_{(M)} < \tau/2\,.$$

Again by $M \in \Delta_2$, there exists $\delta > 0$ such that

$$(2) \qquad \rho_M(x) \le 1 \quad \rho_M(x-y) < \delta \implies |\rho_M(x) - \rho_M(y)| < \varepsilon\,,^{[5]}$$

[1]The Project Supported by National Natural Science Foundation of China

Choose j_0 satisfying $\sum\limits_{j=j_0+1}^{\infty} M(x_0(j)) < \varepsilon$. Denote $e_j = (0\ldots0\overset{j\text{-th}}{1}0\ldots)$. Put

$$A_\delta = \{x \in S(l_M) \colon |\langle x - x_0, e_j\rangle| = |x(j) - x_0(j)| < \frac{\delta}{j_0}\,(j = 1, \ldots, j_0)\}\,.$$

For any $x \in A_\delta$, $\sum\limits_{j=1}^{j_0} M(x(j) - x_0(j)) < \sum\limits_{j=1}^{j_0} M\left(\frac{\delta}{j_0}\right) < \sum\limits_{j=1}^{j_0} \frac{\delta}{j_0} = \delta$. From (2)

$$\left|\sum_{j=1}^{j_0} M(x(j)) - \sum_{j=1}^{j_0} M(x_0(j))\right| < \varepsilon\,.$$

Thus

$$\sum_{j=j_0+1}^{\infty} M(x(j)) = 1 - \sum_{j=1}^{j_0} M(x(j)) = \sum_{j=1}^{j_0} M(x_0(j)) - \sum_{j=1}^{j_0} M(x(j)) + \\ + \sum_{j=j_0+1}^{\infty} M(x_0(j)) < 2\varepsilon\,,$$

hence

$$\rho_M\left(\frac{x - x_0}{2}\right) \le \sum_{j=1}^{j_0} M\left(\frac{x(j) - x_0(j)}{2}\right) + \frac{1}{2}\left(\sum_{j=j_0+1}^{\infty} M(x_0(j)) + \right. \\ \left. + \sum_{j=j_0+1}^{\infty} M(x(j))\right) < 2\varepsilon\,,$$

it follows $\|x - x_0\|_{(M)} < \tau$ from (1), i.e. $A_\delta \subset B(x_0, \tau)$. ■

Theorem 1.2. *For* $[l_M, \|\cdot\|_M]$, *(K)*$\Longleftrightarrow M \in \Delta_2$.

The proof of this theorem is similar to that of Theorem 1.

Theorem 1.3. *For* $[L_M[0,1], \|\cdot\|_{(M)}]$, *(K)*$\Longleftrightarrow M \in \Delta_2$ *and* $M \in \mathrm{sc}[0, \infty)$.

PROOF : Necessity. See [4].

Sufficiency. $M \in \Delta_2$ and $M \in \mathrm{sc}[0,\infty)$ implies local uniform rotundity of $L_M[[0,1], \|\cdot\|_{(M)}]$ [6]. LUR implies (G) [7] and (G) implies (K) [2]. ■

Theorem 1.4. *For* $L_M[[0,1], \|\cdot\|_M]$, *(K)*$\Longleftrightarrow M \in \Delta_2$ *and* $M \in \mathrm{sc}[0, \infty)$.

PROOF : Necessity. See [4].

Sufficiency. $\{e_j(t)\}_1^\infty$ denotes the system of Harr functions. Without loss of generality, $\|1\|_N = 1$ and $\|e_j\|_M = 1\,(j = 1, 2\ldots)$ may be assumed.

Suppose $x_0 \in S(L_M)$. There exists $\beta > 0$ such that the measure of $E = \{t \colon |x_0(t)| \ge \beta\}$ is positive. For arbitrary $\tau > 0$, there exists ε, $0 < \varepsilon < \tau$ such that

$$\rho_M(x) < 8\varepsilon \implies \|x\|_M < \tau\,. \tag{3}$$

Take $y_0 \in E_N$, $\|y_0\|_{(N)} = 1$ satisfying

$$(4) \qquad \int_0^1 x_0(t)y_0(t)dt > 1 - \varepsilon .$$

There exists ξ, $0 < \xi < \frac{\operatorname{mes} E}{2}$ such that

$$(5) \qquad \operatorname{mes} F < 3\xi \implies \|\chi_F\|_N < \varepsilon , \quad \|y_0\chi_F\|_N < \varepsilon \text{ and } \|x_0\chi_F\|_M < \varepsilon .$$

Let $z_0(t) = \operatorname{sign} x_0(t)\chi_E(t)$, choose j_0 satisfying

$$(6) \qquad \left\| \sum_{j=j_0+1}^{\infty} y_0(j)e_j \right\|_N < \varepsilon , \quad \left\| \sum_{j=j_0+1}^{\infty} z_0(j)e_j \right\|_N < \varepsilon .$$

For every $x \in L_M[0,1]$, there exists $k_x > 0$ satisfying $\|x\|_M = \frac{1}{k_x}(1 + \rho_M(k_x x))$. Put

$$A_\varepsilon = \{x \in S(L_M) \colon |\langle x - x_0, e_j\rangle| = |x(j) - x_0(j)| < \frac{\varepsilon}{j_0}, (j = 1, \dots, j_0)\} ,$$

then

$$(7) \qquad \inf_{x \in A_\varepsilon} \operatorname{mes}\{t \colon |x(t)| \geq \frac{\beta}{2}\} = d_\varepsilon > 0 ,$$

$$(8) \qquad \sup_{x \in A_\varepsilon} k_x = k_\varepsilon < \infty ,$$

$$(9) \qquad \exists D_\varepsilon > 0 , \ \operatorname{mes}\{t \colon |k_x x(t)| > D_\varepsilon\} < \xi , \ (x \in A_\varepsilon) .$$

In fact, if (7) is false, then there exists $x \in A_\varepsilon$, $F = \{t \colon |x(t)| \geq \frac{\beta}{2}\}$, $\operatorname{mes} F < \xi$. Hence

$$\int_{E\setminus F} (x_0(t) - x(t)) \operatorname{sign} x_0(t)dt \geq \int_{E\setminus F} |x_0(t)|dt - \int_{E\setminus F} |x(t)|dt \geq$$
$$\geq \frac{\beta}{2}(\operatorname{mes} E - \xi) \geq \frac{\beta}{4} \operatorname{mes} E .$$

However, from (5) and (6),

$$\int\limits_{E\setminus F} (x_0(t)-x(t))\operatorname{sign} x_0(t)dt \le \left|\int\limits_{E} (x_0(t)-x(t))\operatorname{sign} x_0(t)dt\right| +$$

$$\left|\int\limits_{F} (x_0(t)-x(t))\operatorname{sign} x_0(t)dt\right| \le$$

$$\le \left|\int_0^1 (x_0(t)-x(t))z_0(t)dt\right| + \|x_0-x\|_M \|\chi_F\|_N \le$$

$$\le \left|\int_0^1 \left(\sum_{j=1}^{\infty}(x_0(j)-x(j))e_j(t)\right)\left(\sum_{j=1}^{\infty} z_0(j)e_j(t)\right)dt\right| + 2\varepsilon =$$

$$= \left|\int_0^1 \left(\sum_{j=1}^{j_0}(x_0(j)-x(j))e_j(t)\right)\left(\sum_{j=1}^{j_0} z_0(j)e_j(t)\right)dt\right| +$$

$$+ \left|\int_0^1 \left(\sum_{j=j_0+1}^{\infty}(x_0(j)-x(j))e_j(t)\right)\left(\sum_{j=j_0+1}^{\infty} z_0(j)e_j(t)\right)dt\right| + 2\varepsilon \le$$

$$\le \left\|\sum_{j=1}^{j_0}(x_0(j)-x(j))e_j\right\|_M \|z_0\|_N + \|x_0-x\|_M \left\|\sum_{j=j_0+1}^{\infty} z_0(j)e_j\right\|_N + 2\varepsilon < 5\varepsilon .$$

This is a contradiction when ε is small enough. Therefore (7) is true.

For any $x \in A_\varepsilon$,

$$1 = \frac{1}{k_x}(1+\rho_M(k_x x)) \ge \frac{1}{k_x} \int\limits_{\{t:\ |x(t)|\ge\beta/2\}} M\left(k_x\frac{\beta}{2}\right)dt \ge \frac{1}{k_x} M\left(k_x\frac{\beta}{2}\right) d_\varepsilon ,$$

combined with $\lim\limits_{u\to\infty}\frac{M(u)}{u} = \infty$, it is easy to see that the set $\{k_x\}$ has an upper bound k_ε depending on d_ε. i.e. (8) is true.

For any $x \in A_\varepsilon$, by (8), $1 \ge \frac{1}{k_x} \int\limits_{\{t:\ |k_x x(t)|>D\}} M(D)dt \ge \frac{1}{k_\varepsilon} M(d)\operatorname{mes}\{t\colon |k_x x(t)| > D\}$. Let $D = M^{-1}\left(\frac{k_\varepsilon}{\xi}\right)$, then $\operatorname{mes}\{t\colon |k_x x(t)| \ge D\} < \xi$. i.e. (9) is true.

By the strict convexity of $M(u)$ on $[0,\infty)$, there exists δ_ε, $0<\delta_\varepsilon<1$, such that $|u|$, $|v| \le D_\varepsilon$, $|u-v|\ge\varepsilon$, $0 < \frac{1}{1+k_\varepsilon} \le \alpha \le \frac{1}{1+1/k_\varepsilon} < 1$ implies

$$M(\alpha u + (1-\alpha)v) \le (1-\delta_\varepsilon)(\alpha M(u) + (1-\alpha)M(v)) . \tag{10}$$

Take η satisfying

$$0 < \eta < \delta_\varepsilon M\left(\frac{\varepsilon}{2}\right)\xi/3k_\varepsilon^2 , \tag{11}$$

and $j_0' \geq j_0$ satisfying

$$\left\| \sum_{j=j_0'+1}^{\infty} x_0(j)e_j \right\|_M < \eta . \tag{12}$$

Put

$$A_\eta = \{x \in S(L_M) \colon |x(j) - x_0(j)| < \frac{\eta}{j_0'} \ (j = 1, \ldots, j_0')\};$$

obviously $A_\eta \subset A_\varepsilon$. For any $x \in A_\eta$, by (12) we have

$$\|x + x_0\|_M \geq \left\| \sum_{j+1}^{j_0'} (x(j) + x_0(j))e_j \right\|_M \geq 2 \left\| \sum_{j+1}^{j_0'} x_0(j)e_j \right\|_M -$$

$$- \left\| \sum_{j+1}^{j_0'} (x_0(j) - x(j))e_j \right\|_M \geq 2(1-\eta) - \eta = 2 - 3\eta . \tag{13}$$

Put

$$G_x = \{t \colon |k_x x(t)| \leq D_\varepsilon, \ |k_0 x_0(t)| \leq D_\varepsilon, \ |k_x x(t) - k_0 x_0(t)| \geq \varepsilon\},$$

By (13),

$$2 = \|x\|_M + \|x_0\|_M = \frac{k_x + k_0}{k_x k_0} \left(1 + \frac{k_x}{k_x + k_0} \rho_M(k_0 x_0) + \frac{k_0}{k_x + k_0} \rho_M(k_x x)\right) \geq$$

$$\geq \frac{k_x + k_0}{k_x k_0} \left(1 + \rho_M\left(\frac{k_x k_0}{k_x + k_0}(x_0 + x)\right)\right) \geq \|x_0 + x\|_M \geq 2 - 3\eta .$$

Combine with (10) and notice that $k_x > 1$ ($x \in S(L_M)$),

$$3\eta \geq$$

$$\geq \frac{k_x+k_0}{k_x k_0} \int_0^1 \left\{ \frac{k_x}{k_x+k_0} M(k_0 x_0(t)) + \frac{k_0}{k_x+k_0} M(k_x x(t)) - M\left(\frac{k_x k_0}{k_x+k_0}(x(t)+x_0(t))\right) \right\} dt$$

$$\geq \frac{k_x+k_0}{k_x k_0} \int_{G_x} \left\{ \frac{k_x}{k_x+k_0} M(k_0 x_0(t)) + \frac{k_0}{k_x+k_0} M(k_x x(t)) - M\left(\frac{k_x k_0}{k_x+k_0}(x(t)+x_0(t))\right) \right\} dt$$

$$\geq \frac{k_x+k_0}{k_x k_0} \delta_\varepsilon \int_{G_x} \left\{ \frac{k_x}{k_x+k_0} M(k_0 x_0(t)) + \frac{k_0}{k_x+k_0} M(k_x x(t)) \right\} dt$$

$$\geq \frac{k_x+k_0}{k_\varepsilon^2} \delta_\varepsilon \frac{1}{k_x+k_0} M\left(\frac{\varepsilon}{2}\right) \operatorname{mes} G_x = \frac{\delta_\varepsilon}{k_\varepsilon^2} M\left(\frac{\varepsilon}{2}\right) \operatorname{mes} G_x .$$

Combine with (11), $\operatorname{mes} G_x < \xi$ is obtained. Put

$$G_x' = \{t \colon |k_x x(t) - k_0 x_0(t)| \geq \varepsilon\} .$$

It follows from (9) that

$$\operatorname{mes} G'_x < 3\xi . \tag{14}$$

By (6), it is easy to deduce that $|\int_0^1 (x_0(t) - x(t))y_0(t)dt| < 3\varepsilon$ $(x \in A_\eta)$, hence from (4), $\int_0^1 x(t)y_0(t)dt > 1 - 4\varepsilon$. Combine with the definition of G'_x, (14) and (5):

$$\begin{aligned}
1 - 4\varepsilon &< \int_0^1 x(t)y_0(t)dt \\
&\le \Big|\int_{[0,1]\setminus G'_x} \left(\frac{k_0}{k_x}x_0(t) - x(t)\right) y_0(t)dt\Big| + \Big|\int_{[0,1]\setminus G'_x} \frac{k_0}{k_x}x_0(t)y_0(t)dt\Big| + \\
&\quad \Big|\int_{G'_x} x_0(t)y_0(t)dt\Big| \le \\
&\le \left\|\left(\frac{k_0}{k_x}x_0 - x\right)\chi_{[0,1]\setminus G'_x}\right\|_M \|y_0\|_{(N)} + \frac{k_0}{k_x}\|x_0\|_M + \|x\|_M \|y_0\chi_{G'_x}\|_N \\
&< \frac{k_0}{k_x} + 2\varepsilon .
\end{aligned}$$

i.e. $(k_0/k_x) - 1 > -6\varepsilon$. In addition,

$$\begin{aligned}
1 &\ge \int_0^1 |x(t)y_0(t)|dt \ge \int_{[0,1]\setminus G'_x} |x(t)y_0(t)|dt \ge \\
&\ge \int_{[0,1]\setminus G'_x} \frac{k_0}{k_x}x_0(t)y_0(t)dt - \int_{[0,1]\setminus G'_x} \Big|\frac{k_0}{k_x}x_0(t) - x(t)\Big||y_0(t)|dt \\
&\ge \frac{k_0}{k_x}\Big(\int_0^1 x_0(t)y_0(t)dt - \int_{G'_x} |x_0(t)y_0(t)|dt\Big) - \varepsilon \ge \frac{k_0}{k_x}(1 - 2\varepsilon) - \varepsilon .
\end{aligned}$$

i.e. $(k_0/k_x) - 1 < 6\varepsilon$ if $\varepsilon < 1/2$. Therefore

$$\left|\frac{k_0}{k_x} - 1\right| < 6\varepsilon \quad (x \in A_\eta) . \tag{15}$$

Thus, for $x \in A_\eta$,

$$\begin{aligned}
\|(x - x_0)\chi_{[0,1]\setminus G'_x}\|_M &\le \frac{1}{k_x}\|(k_x x - k_0 x_0)\chi_{[0,1]\setminus G'_x}\|_M + \left|\frac{k_0}{k_x} - 1\right| \|x_0\chi_{[0,1]\setminus G'_x}\|_M \\
&< \varepsilon + 6\varepsilon = 7\varepsilon . \qquad (16)
\end{aligned}$$

Combine with (5), $\|x\chi_{[0,1]\setminus G'_x}\|_M \geq \|x_0\|_M - \|x_0\chi_{G'_x}\|_M > 1-\varepsilon$, hence $\|x\chi_{[0,1]\setminus G'_x}\|_M > 1-8\varepsilon$.

Because of

$$\begin{aligned} 1 = \|x\|_M &= \frac{1}{k_x}(1 + \int\limits_{[0,1]\setminus G'_x} M(k_x x(t))dt + \int\limits_{G'_x} M(k_x x(t))dt) \\ &\geq \|x\chi_{[0,1]\setminus G'_x}\|_M + \int\limits_{G'_x} M(x(t))dt > 1 - 8\varepsilon + \rho_M(x\chi_{G'_x}), \end{aligned}$$

$\rho_M(x\chi_{G'_x}) < 8\varepsilon$. It follows from (3), $\|x\chi_{G'_x}\| < \tau$. Thus by (16)

$$\begin{aligned} \|x - x_0\|_M &\leq \|(x-x_0)\chi_{[0,1]\setminus G'_x}\|_M + \|x_0\chi_{G'_x}\|_M + \|x\chi_{G'_x}\|_M \\ &\leq 7\varepsilon + \varepsilon + \tau < 9\tau . \end{aligned}$$

This means $A_\eta \subset B(x_0, 9\tau)$. ∎

It is easy to deduce from the theorem in [2]

Theorem 2. *For* $[L_M[0,1], \|\cdot\|_M]$, $[L_M[0,1], \|\cdot\|_{(M)}]$, $[l_M[0,1], \|\cdot\|_M]$ *or* $[l_M[0,1], \|\cdot\|_{(M)}]$

$$(G) \Longleftrightarrow (H) + (R).$$

The author expresses his gratitute to Prof. Shi Zhongrui for his discussions.

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(*Received* February 27, 1989)