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On the group of isometries of the Urysohn universal metric space

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Dedicated to the memory of Zdeněk Frolík

Abstract. We show that the group mentioned in the title, equipped with the topology of pointwise convergence, is a universal topological group with a countable base.

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Does there exist a universal topological group with a countable base, i.e. such a group G that every topological group with a countable base is isomorphic (topologically or algebraically) to a subgroup of G ? A positive answer to this question of S. Ulam was given in [**1**]: the group Aut Q of all autohomeomorphisms of the Hilbert cube, equipped with the compact–open topology, is universal. In the present paper we apply Katětov's construction [**2**] of Urysohn universal metric spaces to give another example of a universal topological group with a countable base.

Let us say that a separable metric space M is Urysohn iff for any finite metric space X, any subspace $Y \subseteq X$ and any isometric embedding $f: Y \longrightarrow M$ there exists an isometric embedding $\bar{f}: X \longrightarrow M$ which extends f. There exists a unique (up to an isometry) complete Urysohn space [**2**], [**3**], and there exist non–complete Urysohn spaces [**2**]. For a metric space M, let $\operatorname{Is} M$ be the topological group of isometries of M onto itself, equipped with the topology of pointwise convergence (which coincides with the compact–open topology on $\operatorname{Is} M$). If M is separable, the group $\operatorname{Is} M$ has a countable base.

Theorem. *Let U be the complete Urysohn separable metric space. Then* $\operatorname{Is} U$ *is a universal topological group with a countable base, i.e. every (Hausdorff) topological group with a countable base is isomorphic to a subgroup of* $\operatorname{Is} U$.

The proof is based on Katětov's paper [**2**]. Recall some definitions from [**2**]. For a metric space (X, d) let $E(X)$ be the set of all functions $f: X \longrightarrow R$ such that $|f(p) - f(q)| \le d(p,q) \le f(p) + f(q)$ whenever p, $q \in X$. For f, $g \in E(X)$ put $d^E(f,g) = \sup\{|f(p) - g(p)|: p \in X\}$. Then $(E(X), d^E)$ is a metric space. If $Y \subseteq X$ and $f \in E(Y)$, define $f^* \in E(X)$ as follows: for $p \in X$, let $f^*(p) = \inf\{d(p,q) + f(q): q \in Y\}$. The mapping $f \longmapsto f^*$ is an isometric embedding of $E(Y)$ in $E(X)$, so we can identify $E(Y)$ with a subspace of $E(X)$. Let $E(X, \omega) = \bigcup\{E(Y): Y \subseteq X, Y \text{ is finite}\} \subseteq E(X)$. There is a natural isometric embedding $p \longmapsto f_p$ of X in $E(X,\omega)$, where $f_p(q) = d(p,q)$ for p, $q \in X$, so X can be identified with a subspace of $E(X,\omega)$. Put $X_0 = X$, $X_{n+1} = E(X_n,\omega)$. We regard each X_n as a subspace of X_{n+1}, so we get a chain $X_0 \subset X_1 \subset \dots$ of metric spaces. There is a natural metric on $X_\omega = \bigcup X_n$ which extends the metric on X_n for every n.

Every isometry $\varphi \in \operatorname{Is} X$ extends uniquely to an isometry $E(\varphi) \in \operatorname{Is} E(X)$ [2, Fact 1.6]. Let $\varphi^* \in \operatorname{Is} E(X,\omega)$ be the restriction of $E(\varphi)$ to $E(X,\omega)$. The mapping $\varphi \longmapsto E(\varphi)$ from $\operatorname{Is} X$ to $\operatorname{Is} E(X)$ need not be continuous; however, it follows from [2, Fact 1.7] that the mapping $\varphi \longmapsto \varphi^*$ from $\operatorname{Is} X$ to $\operatorname{Is} E(X,\omega)$ is an isomorphic embedding of topological groups. For $\varphi = \varphi_0 \in \operatorname{Is} X$ let $\varphi_{n+1} = (\varphi_n)^* \in \operatorname{Is} X_{n+1}$. There is a unique isometry φ_ω of X_ω which extends φ_n for every n.

Lemma 1. *The topological group* $\operatorname{Is} X$ *is isomorphic to a subgroup of* $\operatorname{Is} X_\omega$.

PROOF : The mapping $\varphi \longmapsto \varphi_\omega$ from $\operatorname{Is} X$ to $\operatorname{Is} X_\omega$ is an isomorphic embedding of topological groups. ∎

If X is separable, then X_ω is Urysohn [2]. The completion of X_ω is also Urysohn [3], [2, lemma 3.3] and hence isometric to U. This proves

Lemma 2. *If* X *is separable, the topological group* $\operatorname{Is} X_\omega$ *is isomorphic to a subgroup of* $\operatorname{Is} U$.

Lemma 3. *Every topological group with a countable base is isomorphic to a subgroup of* $\operatorname{Is} X$ *for some separable metric space* X.

Actually every topological group is isomorphic to a subgroup of $\operatorname{Is} X$ for some metric space X. For metrizable groups this is obvious: if G is a metrizable group, there exists a left–invariant metric d on G compatible with its topology. For any $g \in G$ the left shift $x \longmapsto gx$ is an isometry of (G,d), and thus we obtain an isomorphic embedding of G in $\operatorname{Is}(G,d)$. This proves lemma 3. ∎

The theorem follows immediately from lemmas 1, 2, 3.

Question 1. *Are the topological groups* $\operatorname{Is} U$ *and* $\operatorname{Aut} Q$ *isomorphic?*

Question 2. *Let* m *be an uncountable cardinal. Does there exist a universal topological group of weight* m*? Is* $\operatorname{Aut} I^m$ *such a group?*

REFERENCES

[1] Uspenskij V. V., *A universal topological group with a countable base*, Funktsional'nyj analiz i ego prilozhenija (Functional analysis and its applications) 20 (1986), 86–87.

[2] Katětov M., *On universal metric spaces*, Proc. Sixth Prague Topological Symp. 1986, Frolík Z. (ed.), Berlin: Heldermann Verlag, 1988, pp. 323–330.

[3] Urysohn P., *Sur un espace métrique universel*, Bull. Sci. Math. 51 (1927), 43–64.

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