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Dedicated to the memory of Zdeněk Frolík

Abstract. Every nowhere dense set in $\beta\omega \setminus \omega$ is a 2^ω–set if and only if every nowhere dense set in $\beta\omega \setminus \omega$ is a nowhere dense subset of another nowhere dense set.

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The aim of this short note is to show that the famous Hechler's conjecture is equivalent to a statement concerning the natural order of the family of all nowhere dense subsets of $\beta\omega \setminus \omega$.

Let X be a topological space, τ a cardinal number. A set $Z \subset X$ is called to be a τ–set provided there is a pairwise disjoint family $\{U_\alpha : \alpha \in \tau\}$ consisting of open subsets of X such that $Z \subset \overline{U_\alpha}$ for every $\alpha \in \tau$. In [He], S. H. Hechler studied the nowhere dense sets in the space $\beta\omega \setminus \omega$, the Čech–Stone remainder of a countable discrete space ω. His conjecture, *every nowhere dense set in $\beta\omega \setminus \omega$ is a 2^ω–set*, presents an open problem up to now. Hechler proved it assuming MA. Since then, a lot of set–theoretical assumptions implying Hechler's conjecture is known (see [BS]).

During the last Winter School on Abstract Analysis, V. I. Malykhin turned the author's attention to the following problem. (Malykhin attributes it to A. I. Veksler.) Denote by **NWD** the family of all nowhere dense subsets of $\beta\omega \setminus \omega$. For K, L in **NWD**, let us write $K < L$, if $K \subset L$ and if moreover the set K is nowhere dense in L. Malykhin's (or Veksler's) problem reads simply as follows: *Is there a maximal element in the partial order* $(\mathbf{NWD}, <)$*?* Here we shall prove the following.

Theorem. *Every nowhere dense set in $\beta\omega \setminus \omega$ is a 2^ω–set if and only if* $(\mathbf{NWD}, <)$ *has no maximal element.*

Before giving the proof, let us fix some notation and recall the necessary auxiliary facts. The symbol ω stands for a countable discrete space, $\beta\omega$ is its Čech–Stone compactification, $\omega^* = \beta\omega \setminus \omega$ is the space of all uniform ultrafilters on ω. For $A \subset \omega$, $A^* = \overline{A} \setminus A$. Notice that for A, $B \subset \omega$, $A^* \subset B^*$ iff $A \setminus B$ is finite, $A^* \cap B^* = \emptyset$ if $A \cap B$ is finite. A family $\mathcal{A}$ of subsets of ω is called almost disjoint, if all its members are infinite and the intersection of any two distinct members of $\mathcal{A}$ is finite, a MAD family is a maximal almost disjoint family. If $\mathcal{A}$ and $\mathcal{B}$ are two families of sets, then $\mathcal{B}$ *refines* $\mathcal{A}$ ($\mathcal{B} \prec \mathcal{A}$), if for each $B \in \mathcal{B}$ there is some $A \in \mathcal{A}$ with $B \subset A$.

A family $\mathcal{M}$ of subsets of ω has an *almost disjoint refinement*, if there is an almost disjoint family $\mathcal{A}$ such that for each $M \in \mathcal{M}$ there is some $A \in \mathcal{A}$ with $A \subset M$. If $\mathcal{A}$ is an almost disjoint family, then $\mathcal{J}^+(\mathcal{A})$ will stand for the collection

$$\mathcal{J}^+(\mathcal{A}) = \{M \subset \omega \colon |\{A \in \mathcal{A} \colon |A \cap M| = \omega\}| \geq \omega\}.$$

The forthcoming three lemmas will provide us with combinatorial facts useful for the proof of the Theorem.

Lemma 1 ([**BV**], Theorem 1.5). *Let $\mathcal{R}$ be a countably infinite almost disjoint family on ω. Then $\mathcal{J}^+(\mathcal{R})$ has an almost disjoint refinement $\mathcal{B}$ such that $B \cap R$ is finite for all $R \in \mathcal{R}$, $B \in \mathcal{B}$.*

Lemma 2 ([**BDS**], Proposition 1.9). *The following are equivalent:*

(a) *Every nowhere dense subset of ω^* is a 2^ω-set;*
(b) *for every infinite* MAD *family $\mathcal{A}$ on ω, $\mathcal{J}^+(\mathcal{A})$ has an almost disjoint refinement.*

Lemma 3. *The following are equivalent:*

(a) *There is no maximal element in* (**NWD**, $<$);
(b) *for every infinite* MAD *family $\mathcal{A}$ on ω there is some* MAD *family $\mathcal{B}$ on ω such that $\mathcal{B}$ refines $\mathcal{A}$ and for every $M \in \mathcal{J}^+(\mathcal{A})$ there is some $A \in \mathcal{A}$ with $A \cap M \in \mathcal{J}^+(\mathcal{B})$.*

PROOF : (a)$\longrightarrow$(b): Let $\mathcal{A}$ be a maximal almost disjoint family. By the maximality of $\mathcal{A}$, the set $K = \omega^* \setminus \bigcup\{A^* \colon A \in \mathcal{A}\}$ is nowhere dense in ω^*. Since K is not maximal, there is some nowhere dense set L with $K < L$. Find some almost disjoint family $\mathcal{B}$, which refines $\mathcal{A}$ and such that $B^* \cap L = \emptyset$ for all $B \in \mathcal{B}$, and which is a maximal one having these two properties. $\mathcal{B}$ is a MAD family, because L is nowhere dense. If $M \in \mathcal{J}^+(\mathcal{A})$, then M^* meets K. Since K is nowhere dense in L, there is a clopen set $N^* \subset M^*$ satisfying $N^* \cap L \neq \emptyset$, $N^* \cap K = \emptyset$. Thus $N \in \mathcal{J}^+(\mathcal{B})$, but $N \notin J^+(\mathcal{A})$. So there are only finitely many members from $\mathcal{A}$, which meet N in an infinite set, but infinitely many such from $\mathcal{B}$. Consequently, there is some $A \in \mathcal{A}$ such that $A \cap N \in \mathcal{J}^+(\mathcal{B})$. Since $N^* \subset M^*$, we have $A \cap M \in \mathcal{J}^+(\mathcal{B})$, too.

(b)$\longrightarrow$(a): Let K be a nowhere dense subset of ω. Choose a MAD family $\mathcal{A}$ such that for each $A \in \mathcal{A}$, $A^* \cap K = \emptyset$. Let $\mathcal{B}$ be a MAD family as in (b). The set $L = \omega^* \setminus \bigcup\{B^* \colon B \in \mathcal{B}\}$ is nowhere dense by the maximality of $\mathcal{B}$; we need to show that $K < L$. Clearly $K \subset L$, because $\mathcal{B} \prec \mathcal{A}$. Let M^* be an arbitrary clopen subset of $\beta\omega \setminus \omega$ which meets L. There is nothing to prove if $M^* \cap K = \emptyset$. Otherwise $M \in \mathcal{J}^+(\mathcal{A})$ and by (b) there is some $A \in \mathcal{A}$ such that $A \cap M \in \mathcal{J}^+(\mathcal{B})$. Now, A^* obviously does not meet K, the same must hold for its subset $(A \cap M)^*$. However, $(A \cap M)^* \cap L$ is non-void, because $A \cap M \in \mathcal{J}^+(\mathcal{B})$. This shows that K is nowhere dense in L. ∎

PROOF of the Theorem: Assume that every nowhere dense subset of ω^* is a 2^*-set. Let $\mathcal{A}$ be an arbitrary infinite MAD family on ω. By (b) from Lemma 2, there is some almost disjoint refinement $\mathcal{C}$ of $\mathcal{J}^+(\mathcal{A})$. We may assume that $\mathcal{C}$ is a MAD family. For each $C \in \mathcal{C}$ choose some infinite MAD family $\mathcal{B}(C)$ on C and define $\mathcal{B} = \bigcup\{\mathcal{B}(C): C \in \mathcal{C}\}$. Since $\mathcal{C}$ as well as all $\mathcal{B}(C)$'s are MAD families, $\mathcal{B}$ is a MAD family. Obviously $\mathcal{B}$ refines $\mathcal{A}$. Let $M \in \mathcal{J}^+(\mathcal{A})$. Since $\mathcal{C}$ is an almost disjoint refinement of $\mathcal{J}^+(\mathcal{A})$, there is some $C \in \mathcal{C}$, $C \subset M$. Therefore all members from the infinite family $\mathcal{B}(C)$ are subsets of M, so $C \cap M \in \mathcal{J}^+(\mathcal{B})$. Clearly, the same holds for the set $A \cap M$, where A is the member of $\mathcal{A}$, which contains C. We have verified (b) from Lemma 3.

For the converse implication assume Lemma 3.(b), and choose an arbitrary infinite MAD family $\mathcal{A} = \mathcal{A}_0$ on ω. Our aim is to find an almost disjoint refinement of $\mathcal{J}^+(\mathcal{A})$.

Applying Lemma 3.(b) inductively, we shall find a collection $\{\mathcal{A}_n : n \in \omega\}$ of MAD families such that for all $n \in \omega$, $\mathcal{A}_{n+1} \prec \mathcal{A}_n$ and for every $M \in \mathcal{J}^+(\mathcal{A}_n)$ there is some $A \in \mathcal{A}_n$ with $A \cap M \in \mathcal{J}^+(\mathcal{A}_{n+1})$. For every decreasing chain $\mathcal{C} = \{A_0 \supset A_1 \supset A_2 \supset \dots\}$ with $A_n \in \mathcal{A}_n$ select an almost disjoint family $\mathcal{B}(\mathcal{C})$ using Lemma 1 as follows: Let $R_0 = \omega \setminus A_0$, $R_{n+1} = A_n \setminus A_{n+1}$, $\mathcal{R} = \{R_n : n \in \omega\}$. Let $\mathcal{B}(\mathcal{C})$ be the result of an application of Lemma 1 to this particular almost disjoint family $\mathcal{R}$.

Notice that for distinct chains $\mathcal{C}$, $\mathcal{C}'$, if $B \in \mathcal{B}(\mathcal{C})$ and $B' \in \mathcal{B}(\mathcal{C}')$, then $B \cap B'$ is finite. Indeed, there is some $n \in \omega$ such that $A_n \in \mathcal{C}$ and $A'_n \in \mathcal{C}'$ are distinct. By our definition and by Lemma 1 we have that both sets $B \setminus A_n$ and $B' \setminus A'_n$ are finite, and the set $A_n \cap A'_n$ is finite too. Hence $|B \cap B'| < \omega$.

It remains to show that the family

$$\mathcal{B} = \bigcup\{\mathcal{B}(\mathcal{C}) : \mathcal{C} \text{ is a decreasing chain meeting every } \mathcal{A}_n\}$$

is the desired almost disjoint refinement of $\mathcal{J}^+(\mathcal{A})$. As already observed, $\mathcal{B}$ is almost disjoint.

Let $M \in \mathcal{J}^+(\mathcal{A})$ be arbitrary. For $n \in \omega$, find inductively $A_n \in \mathcal{A}_n$ such that

$$M \cap A_0 \cap A_1 \cap \dots \cap A_n \in \mathcal{J}^+(\mathcal{A}_{n+1}).$$

Now it is clear that $|M \cap (A_n \setminus A_{n+1})| = \omega$ for all n, hence $M \in \mathcal{J}^+(\mathcal{R})$, where $\mathcal{R}$ is determined by the chain $\mathcal{C} = \{A_n : n \in \omega\}$. By Lemma 1, there is some $B \in \mathcal{B}(\mathcal{C})$ with $B \subset M$. This completes the proof. ∎

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