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Dugundji spaces and topological groups

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Dedicated to the memory of Zdeněk Frolík

Abstract. We give new characterization of Dugundji compact spaces and κ-metrizable compact spaces in terms of special embeddings into topological groups. As a consequence we obtain that a compact retract of a dense subspace of some topological group is Dugundji.

Keywords: Dugundji space, κ-metrizable space, compact space, topological group, embedding, retraction, extension of open sets, set-valued mapping, normal functor, probability measure

Classification: primary 54D30, 54C25, 22A05; secondary 54C15, 54C20, 54C60, 22A10

1. Introduction. Let X be a subspace of Y. A topological space Z is said to be injective with respect to the pair (X,Y) iff every continuous map $f: X \longrightarrow Z$ has a continuous extension $f': Y \longrightarrow Z$. For a compact space X the following are equivalent [**5**]: (i) if Y is a zerodimensional compact space and Z is closed in Y, then X is injective with respect to the pair (Z,Y); (ii) if X is a subspace of a compact space Y, then every compact convex subset of a locally convex topological vector space is injective with respect to the pair (X,Y). A compact space X is called *Dugundji* [**9**] if one of these conditions holds.

Pełczyński [**9**], Ščepin [**10**], Shirokov [**14**] and Dranishnikov [**2**] found different characterizations of Dugundji spaces in terms of special embeddings into the "canonical" Dugundji space, the Tychonoff cube I^τ. The aim of this paper is to demonstrate that in all these characterizations, I^τ can be replaced by a topological group.

Ščepin [**11**], [**12**] introduced the notion of κ-metrizable spaces and showed that Dugundji spaces are κ-metrizable [**12**, Corollary 1]. Shirokov [**14**] gave a characterization of κ-metrizable compact spaces via special embeddings into the Tychonoff cube I^τ. In Section 4 we show that in Shikorov's characterization I^τ can also be replaced by a topological group.

2. Notations, terminology and preliminaries. All topological spaces and groups considered are assumed to be Tychonoff, and all maps are assumed to be continuous if the converse is not stated explicitly. The bar denotes the topological closure. If X is a topological space, then $\mathcal{T}(X)$ stands for denoting the topology of X. Symbol $w(X)$ and $nw(X)$ denote the ***weight*** and the ***network weight*** of a space X respectively [**3**]. If for each $\alpha \in A$ a map $f_\alpha: X \longrightarrow X_\alpha$ is fixed, then the map $f = \Delta\{f_\alpha : \alpha \in A\}: X \longrightarrow \Pi\{X_\alpha : \alpha \in A\}$ defined by $f(x) = \{f_\alpha(x)\}_{\alpha \in A}$ for $x \in X$ is called the ***diagonal product*** of the family $\{f_\alpha : \alpha \in A\}$.

2.1. Ščepin's theorem [10]. *A compact space X is Dugundji if and only if there exist a family $\{X_\alpha : \alpha \in A\}$ of compact metric spaces and a map $f_\alpha : X \longrightarrow X_\alpha$ for each $\alpha \in A$ so that:*

(i) *the diagonal product $f = \Delta\{f_\alpha : \alpha \in A\}: X \longrightarrow \Pi\{X_\alpha : \alpha \in A\}$ is homeomorphic embedding, and*
(ii) *for each subset $B \subset A$, the diagonal subproduct $f_B = \Delta\{f_\alpha : \alpha \in B\}: X \longrightarrow f_B(X) \subset \Pi\{X_\alpha : \alpha \in B\}$ is an open map onto its image.*

2.2. Definition. Let X be a subspace of Y. A map $e: \mathcal{T}(X) \longrightarrow \mathcal{T}(Y)$ is *regular* provided that:

(i) $e(U) \cap X = U$ whenever $U \in \mathcal{T}(X)$,
(ii) if $U, V \in \mathcal{T}(X)$ and $U \cap V = \emptyset$, then $e(U) \cap e(V) = \emptyset$,
(iii) $e(X) = Y$.

This concept was considered first (without being named) by Kuratowski [7] and van Douwen [1]. Lately the name was given by Shirokov [14]. Our next definition is due to Shirokov [14].

2.3. Definition. Let X be a subspace of Y. A map $e: \mathcal{T}(X) \longrightarrow \mathcal{T}(Y)$ is *d-regular* provided that:

(i) $e(U) \cap X = U$ for each $U \in \mathcal{T}(X)$,
(ii) $e(U \cap V) = e(U) \cap e(V)$ whenever $U, V \in \mathcal{T}(X)$,
(iii) $e(X) = Y$.

It is clear that each d-regular map is regular.

2.4. Definition. Let X be a subspace of Y. A map $e: \mathcal{T}(X) \longrightarrow \mathcal{T}(Y)$ *preserves inclusions* (or is *inclusion-preserving*) iff $e(U) \subset e(V)$ provided that $U, V \in \mathcal{T}(X)$ and $U \subset V$.

2.5. Lemma. *Suppose that X is a subspace of Y and $e: \mathcal{T}(X) \longrightarrow \mathcal{T}(Y)$ is a regular map. Define $e': \mathcal{T}(X) \longrightarrow \mathcal{T}(Y)$ by $e'(U) = \cup\{e(V): V \in \mathcal{T}(X) \text{ and } V \subset U\}$ for $U \in \mathcal{T}(X)$. Then e' is regular and preserves inclusions. If e is d-regular then so is e'.*

For a topological group G we fix $\mathcal{N}(G)$ for denoting the family of all normal subgroups of G and define $\mathcal{M}(G) = \{N \in \mathcal{N}(G): nw(G/N) \le \omega\}$. For each $N \in \mathcal{N}(G)$ the natural quotient map of G onto G/N will be denoted by π_N. If $N, N' \in \mathcal{N}(G)$ and $N' \subset N$, then there exists the map $\pi_N^{N'} : G/N' \longrightarrow G/N$ such that $\pi_N = \pi_N^{N'} \circ \pi_{N'}$.

2.6. Lemma [15, Lemma 2]. *Suppose that G is a σ-compact group. Then:*

(i) *for every G_δ-subset B of G there exists an $N \in \mathcal{N}(G)$ so that $\overline{B} = \pi_N^{-1}(\pi_N(\overline{B}))$ (in particular, $\overline{B}$ is a G_δ-subset of G),*
(ii) *if $\mathcal{E} \subset \mathcal{M}(G)$ is so that for any $N \in \mathcal{M}(G)$ there exists an $\tilde{N} \in \mathcal{E}$, $\tilde{N} \subset N$, then the diagonal map $\pi_\mathcal{E} = \Delta\{\pi_N: N \in \mathcal{E}\}: G \longrightarrow \prod\{G/N: N \in \mathcal{E}\}$ is a homeomorphic embedding.*
(iii) *if $\{N_j: j \in \omega\} \subset \mathcal{M}(G)$, then $N = \cap\{N_j: j \in \omega\} \in \mathcal{M}(G)$.*

2.7. Lemma. *Let G be a topological group, X its compact subspace which algebraically generates G. If B is a G_δ-subset of G with $X \subset B \subset G$, then there exists a closed G_δ-subset B' of G so that $X \subset B' \subset B$.*

PROOF : Since X algebraically generates G, G is σ-compact and hence normal. If $B = \cap\{U_i : i \in \omega\}$, then by induction we can choose, using normality of G, a sequence $\{V_i : i \in \omega\}$ of open subsets of G so that $X \subset V_i \subset \overline{V_i} \subset U_i \cap V_{i-1}$ for all $i \in \omega$. Then $B' = \cap\{V_i : i \in \omega\}$ is as required. ∎

We fix symbols $F(X)$ and $A(X)$ for denoting the *free topological group* and the *free Abelian topological group* of a space X respectively [8], [4].

3. Technical lemmas.

3.1. Notations and conventions. Throughout this section we fix a topological group G and its compact subspace X that algebraically generates G. It is clear that G is σ-compact. Furthermore, we fix a closed set B with $X \subset B \subset G$, a regular map $e: \mathcal{T}(X) \longrightarrow \mathcal{T}(B)$ preserving inclusions and an $N^* \in \mathcal{M}(G)$ such that $B = \pi_{N^*}^{-1}(\pi_{N^*}(B))$. Let $\mathcal{N}^*(G) = \{N \in \mathcal{N}(G) : N \subset N^*\}$ and $\mathcal{M}^*(G) = \mathcal{N}^*(G) \cap \mathcal{M}(G)$. For $N \in \mathcal{N}^*(G)$ define $X_N = \pi_N(X)$, $B_N = \pi_N(B)$, $\varphi_N = \pi_N |_X : X \longrightarrow X_N$ and $\psi_N = \pi_N \mid B : B \longrightarrow B_N$. If $N, N' \in \mathcal{N}^*(G)$ and $N' \subset N$, then we set $\varphi_N^{N'} = \pi_N^{N'} |_{X_{N'}} : X_{N'} \longrightarrow X_N$ and $\psi_N^{N'} = \pi_N^{N'} |_{B_{N'}} : B_{N'} \longrightarrow B_N$.

3.2. Lemma. *For each $N \in \mathcal{N}^*(G)$ the map ψ_n is open. If $N, N' \in \mathcal{N}^*(G)$ and $N' \subset N$, then the map $\psi_N^{N'}$ is open too.*

PROOF : Since $N \in \mathcal{N}^*(G)$, $B = \pi_N^{-1}(\pi_N(B))$, and so ψ_N is open as restriction of the open map π_N to the full preimage. Furthermore, for $N, N' \in \mathcal{N}^*(G)$ with $N' \subset N$ we have $\psi_N = \psi_N^{N'} \circ \psi_{N'}$. Since ψ_N is open, we conclude that $\psi_N^{N'}$ is also open. ∎

3.3. Lemma. *The set B_N is closed in G/N for each $N \in \mathcal{N}^*(G)$.*

PROOF : Since $N \subset N^*$, $B = \pi_N^{-1}(B_N)$. Since π_N is quotient and B is closed, B_N is closed. ∎

3.4. Definition. For $N \in \mathcal{N}^*(G)$ we say that a set $P \in B$ is *N-cylindrical* provided that $\overline{P} = \psi_N^{-1}(\overline{\psi_N(P)})$ (note that $\overline{P} \subset B$ and $\overline{\psi_N(P)} \subset B_N$ by the lemma above).

3.5. Lemma. *If $N, N^* \in \mathcal{N}^*(G)$, $N' \subset N$ and P is N-cylindrical, then P is also N'-cylindrical.*

PROOF : By Lemma 3.2 the map $\psi_N^{N'}$ is open, so the result follows from the equality $\psi_N = \psi_N^{N'} \circ \psi_{N'}$. ∎

3.6. Lemma. *If both sets $P, Q \subset B$ are N-cylindrical for some $N \in \mathcal{N}^*(G)$, then so is $P \cap Q$.*

3.7. Lemma. *If $N \in \mathcal{N}^*(G)$ and sets $U, V \in \mathcal{T}(B)$ are N-cylindrical, then $\psi_N(U) \cap \psi_N(V) = \emptyset$ is equivalent to $U \cap V = \emptyset$.*

PROOF : Suppose that $\psi_N(U) \cap \psi_N(V) = W \neq \emptyset$. Since ψ_N is open, W is open, and hence so is $\psi_N^{-1}(W)$. Since U and V are N-cylindrical, $\psi_N^{-1}(W) \subset \psi_N^{-1}(\overline{W}) \subset \psi_N^{-1}(\overline{\psi_N(U)} \cap \overline{\psi_N(V)}) = \overline{U} \cap \overline{V}$. Since $\psi_N^{-1}(W)$ is open, we have $U \cap V \neq \emptyset$. ■

3.8. Definition. We will say that a subgroup $N \in \mathcal{N}^*(G)$ is *e-admissible* if there exists a base $\mathcal{B}_N$ of the space X_N such that:

(i) for all $U \in \mathcal{B}_N$ the set $e(\varphi_N^{-1}(U))$ is N-cylindrical,
(ii) $X_N \cap \psi_N(e(\varphi_N^{-1}(U))) \subset \overline{U}$ for each $U \in \mathcal{B}_N$.

3.9. Lemma. *If $N \in \mathcal{N}^*(G)$ is e-admissible, then $X_N \cap \psi_N(e(U)) \subset \varphi_N(\overline{U})$ for all $U \in \mathcal{T}(X)$.*

PROOF : Suppose the contrary, and let $U \in \mathcal{T}(X)$, $x \in e(U)$, $\psi_N(x) \in X_N \setminus \varphi_N(\overline{U})$. Since X is compact and $\overline{U} \subset X$, $\overline{U}$ is compact. Thus $\varphi_N(\overline{U})$ is closed in X_N. Since $\mathcal{B}_N$ is a base of X_N, there exists a $W \in \mathcal{B}_N$ with $\psi_N(x) \in W$ and $W \cap \varphi_N(U) = \emptyset$. In particular,

$$U \cap \varphi_N^{-1}(W) = \emptyset. \tag{1}$$

Observe that $\psi_N(x) \in W \subset \psi_N(e(\varphi_N^{-1}(W)))$. Since $W \in \mathcal{B}_N$, it follows that $e(\varphi_N^{-1}(W))$ is N-cylindrical by Definition 3.8(i). So $x \in \overline{e(\varphi_N^{-1}(W))}$. Now $x \in U(e)$ implies that $e(U) \cap e(\varphi_N^{-1}(W)) \neq \emptyset$, a contradiction with (1) and the regularity of the map e. ■

3.10. Lemma. *If $N \in \mathcal{N}^*(G)$ is e-admissible, then φ_N is an open map.*

PROOF : Assume that $V \in \mathcal{T}(X)$ and $y \in \varphi_N(V)$. Fix an $x \in V$ with $\varphi_N(x) = y$. Choose an $U \in \mathcal{T}(X)$ so that $x \in \overline{U} \subset V$. The set $W = \psi_N(e(U))$ is open in B_N (Lemma 3.2), and the previous lemma yields $y \in X_N \cap W \subset \varphi_N(\overline{U}) \subset \varphi_N(V)$. Therefore $\varphi_N(V) \in \mathcal{T}(X_N)$. ■

3.11. Lemma. *Assume that $N, N' \in \mathcal{N}^*(G)$, $N' \subset N$ and N is e-admissible. Then the map $\varphi_N^{N'}: X_{N'} \longrightarrow X_N$ is open.*

PROOF : Since $\varphi_N = \varphi_N^{N'} \circ \varphi_{N'}$ and φ_N is open (Lemma 3.10), we conclude that $\varphi_N^{N'}$ is open. ■

3.12. Lemma. *If $N \in \mathcal{N}^*(G)$ is e-admissible, then there exists a regular map $e_N: \mathcal{T}(X_N) \longrightarrow \mathcal{T}(B_N)$*

PROOF : For every $U \in \mathcal{B}_N$ define $W_U = \psi_N(e(\varphi_N^{-1}(U))) \in \mathcal{T}(B_N)$. Since $\mathcal{B}_N$ is a base of X_N, from Definition 3.8(ii) it follows that $\{X_N \cap W_U : U \in \mathcal{B}_N\}$ is also a base of X_N. Therefore, for every $O \in \mathcal{T}(X_N)$ we would have $e_N(O) = \bigcup\{W_U : U \in \mathcal{B}_N, X_N \cap W_U \subset O\} \in \mathcal{T}(B_N)$ and $X_N \cap e_N(O) = O$. From Definition 3.8(i) and Lemma 3.7 it follows that $W_U \cap W_V = \emptyset$ whenever $U, V \in \mathcal{B}_N$ and $U \cap V = \emptyset$. So $O_1, O_2 \in \mathcal{T}(X_N)$ and $O_1 \cap O_2 = \emptyset$ imply $e_N(O_1) \cap e_N(O_2) = \emptyset$, i.e. $e_N: \mathcal{T}(X_N) \longrightarrow \mathcal{T}(B_N)$ is regular. ■

3.13. Lemma. *For every $N \in \mathcal{N}^*(G)$ there exists a e-admissible $\widetilde{N} \in \mathcal{N}^*(G)$ so that $\widetilde{N} \subset N$ and $w(X_{\widetilde{N}}) \leq w(X_N)$.*

PROOF : Let $w(X_N) = \tau$. By induction we will define a decreasing sequence $\{N_k : k \in \omega\} \subset \mathcal{N}^*(G)$ and a sequence $\{\mathcal{B}_k : k \in \omega\}$ such that for every $k \in \omega$ the following properties will be satisfied:

(1_k) $\mathcal{B}_k$ is a base of X_{N_k} with $|\mathcal{B}_k| \leq \tau$,

(2_k) if $U \in \mathcal{B}_k$, then both sets $e(\varphi_{N_k}^{-1}(U))$ and $e(X \setminus \overline{\varphi_{N_k}^{-1}(U)})$ are N-cylindrical.

Define $N_0 = N$ and choose any base $\mathcal{B}_0$ of X_N of size τ (it is possible because $w(X_N) = \tau$). Now we assume that N_i and $\mathcal{B}_i$ have already been defined for $i \leq k$ so that (1_1), ..., (1_k) and (2_1), ..., (2_{k-1}) hold. Let us define N_{k+1} and $\mathcal{B}_{k+1}$ satisfying (1_{k+1}) and (2_k). Set

$$\gamma_k = \{e(\varphi_{N_k}^{-1}(U)) : U \in \mathcal{B}_k\} \cup \{e(X \setminus \overline{\varphi_{N_k}^{-1}(U)}) : U \in \mathcal{B}_k\}.$$

By (1_k), $|\gamma_k| \leq |\mathcal{B}_k| \leq \tau$. Each $W \in \gamma_k$ is open in the G_δ-set B, so W is a G_δ-subset of G. Therefore there exists an $N_W \in \mathcal{M}^*(G)$ so that W is N_W-cylindrical (Lemma 2.6(i)). Define $\mathcal{E}_k = \{N_W : W \in \gamma_k\} \cup \{N_k\}$ and $N_{k+1} = \bigcap \mathcal{E}_k$. Clearly, $N_{k+1} \in \mathcal{N}^*(G)$ and $N_{k+1} \subset N_k$. Let $j_k = \Delta\{\psi_N : N \in \mathcal{E}_k\} : B \longrightarrow \prod\{B_N : N \in \mathcal{E}_k\}$ be the diagonal product. There exists the natural one–to–one map $i_k : B_{N_{k+1}} \longrightarrow j_k(B)$ so that $j_k = i_k \circ \psi_{N_{k+1}}$. Since $X_{N_{k+1}}$ is compact, the restriction of i_k to $X_{N_{k+1}}$ is a homeomorphism, and so $X_{N_{k+1}}$ is homeomorphic to $j_k(X)$. For each $W \in \gamma_k$ the set X_{N_W} is a compact subspace of B_{N_W} and $N_W \in \mathcal{M}^*(G)$, so $w(X_{N_W}) = nw(X_{N_W}) \leq nw(B_{N_W}) \leq nw(G/N_W) \leq \omega$. Since $j_k(X) \subset \prod\{X_{N_W} : W \in \gamma_k\} \times X_{N_k}$, $|\gamma_k| \leq \tau$ and $w(X_{N_k}) \leq \tau$, we conclude that $w(X_{N_{k+1}}) = w(j_k(X)) \leq \tau$. Now it suffices to choose any base $\mathcal{B}_{k+1}$ of $X_{N_{k+1}}$ with $|\mathcal{B}_{k+1}| \leq \tau$. Then ($1_{k+1}$) is trivial, and ($2_k$) follows from Lemma 3.5 and the inclusion $N_{k+1} \subset N_W$ which holds for each $W \in \gamma_k$.

Define $\widetilde{N} = \bigcap\{N_k : k \in \omega\}$. Then $\widetilde{N} \subset N_0 = N$ and $\widetilde{N} \in \mathcal{N}^*(G)$. For each $k \in \omega$ let $\mu_k = \{(\varphi_{N_k}^{\widetilde{N}})^{-1}(U) : U \in \mathcal{B}_k\}$. Let $\mathcal{B}_{\widetilde{N}} = \bigcup\{\mu_k : k \in \omega\}$.

3.14. Claim. *$\mathcal{B}_{\widetilde{N}}$ is a base of $X_{\widetilde{N}}$.*

PROOF : Using compactness of X, one can easily see that $X_{\widetilde{N}}$ is homeomorphic to the limit of the inverse spectrum $\mathbf{S} = \{X_{N_k}, \varphi_{N_k}^{N_m}, k, m \in \omega, m > k\}$ whose limit projections coincide with $\varphi_{N_k}^{\widetilde{N}} : X_{\widetilde{N}} \longrightarrow X_{N_k}$, and the result follows. ∎

Claim 3.14 and (1_k) for all $k \in \omega$ yield $w(X_{\widetilde{N}}) \leq |\mathcal{B}_{\widetilde{N}}| \leq \tau = w(X_N)$. Thus it remains only to show that $\widetilde{N}$ is e-admissible.

3.15. Claim. *For all $W \in \mathcal{B}_{\widetilde{N}}$ both sets $e(\varphi_{\widetilde{N}}^{-1}(W))$ and $e(X \setminus \overline{\varphi_{\widetilde{N}}^{-1}(W)})$ are $\widetilde{N}$-cylindrical.*

PROOF : Each $W \in \mathcal{B}_{\widetilde{N}}$ is of the form $W = (\varphi_{N_k}^{\widetilde{N}})^{-1}(U)$ for some $k \in \omega$ and $U \in \mathcal{B}_k$. By (2_k) sets

$$e(\varphi_{N_k}^{-1}(U)) = e(\varphi_{\widetilde{N}}^{-1}((\varphi_{N_k}^{\widetilde{N}})^{-1}(U))) = e(\varphi_{\widetilde{N}}^{-1}(W))$$

and

$$e(X \setminus \overline{\varphi_{N_k}^{-1}(U)} = e(X \setminus \overline{\varphi_{\widetilde{N}}^{-1}(W)})$$

are N_{k+1}-cylindrical. Since $\widetilde{N} \subset N_{k+1}$, the same sets are also $\widetilde{N}$-cylindrical by Lemma 3.5. ■

Let us show that $\widetilde{N}$ is e-admissible and that $\mathcal{B}_{\widetilde{N}}$ witnesses this. Item (i) of Definition 3.8 follows from Claim 3.15. To check (ii) fix an $U \in \mathcal{B}_{\widetilde{N}}$. Since e is regular, Claim 3.15 and Lemma 3.7 yield that

$$(2) \qquad \psi_{\widetilde{N}}(e(\varphi_{\widetilde{N}}^{-1}(U))) \cap \psi_{\widetilde{N}}(e(X \setminus \overline{\varphi_{\widetilde{N}}^{-1}(U)})) = \emptyset .$$

Since e preserves inclusions, $X_{\widetilde{N}} \setminus \overline{U} \subset \psi_{\widetilde{N}}(e(X \setminus \overline{\varphi_{\widetilde{N}}^{-1}(U)}))$, and from (2) we obtain that $X_{\widetilde{N}} \cap \psi_{\widetilde{N}}(e(\varphi_{\widetilde{N}}^{-1}(U))) \subset \overline{U}$. ■

3.16. Lemma. *Let $\{N_\alpha : \alpha < \tau\} \subset \mathcal{N}^*(G)$ be decreasing chain of e-admissible subgroups of G. Then $N = \bigcap\{N_\alpha : \alpha < \tau\} \in \mathcal{N}^*(G)$ is also e-admissible.*

PROOF : For all $\alpha < \tau$ fix a base $\mathcal{B}_{N_\alpha}$ of X_{N_α} witnessing that N_α is e-admissible. Since X_N is compact, it is homeomorphic to the limit of the natural inverse spectrum $\mathbf{S} = \{X_{N_\alpha}, \varphi_{N_\alpha}^{N_\beta}, \alpha < \beta < \tau\}$, so $\mathcal{B}_N = \{(\varphi_{N_\alpha}^{N})^{-1}(U) : \alpha < \tau, U \in \mathcal{B}_{N_\alpha}\}$ is a base of X_N. We will show that $\mathcal{B}_N$ witnesses that N is e-admissible. Pick a $W \in \mathcal{B}_N$. Then $W = (\varphi_{N_\alpha}^{N})^{-1}(U)$ for some $\alpha < \tau$ and $U \in \mathcal{B}_{N_\alpha}$. Since $\mathcal{B}_{N_\alpha}$ witnesses that X_{N_α} is e-admissible,the set

$$e(\varphi_N^{-1}(W)) = e(\varphi_N^{-1}((\varphi_{N_\alpha}^{N})^{-1}(U))) = e(\varphi_{N_\alpha}^{-1}(U))$$

is N_α-cylindrical, and so it is also N-cylindrical because $N \subset N_\alpha$ (Lemma 3.5). Furthermore, applying item (ii) of Definition 3.8 to X_{N_α} and $\mathcal{B}_{N_\alpha}$ we obtain

$$\begin{gathered} X_N \cap \psi_N(e(\varphi_N^{-1}(W))) = X_N \cap \psi_N(e(\varphi_{N_\alpha}^{-1}(U))) \subset \\ X_N \cap (\psi_{N_\alpha}^{N})^{-1}(\psi_{N_\alpha}(e(\varphi_{N_\alpha}^{-1}(U)))) \subset X_N \cap (\psi_{N_\alpha}^{N})^{-1}(\overline{U}) = (\varphi_{N_\alpha}^{N})^{-1}(\overline{U}) = \overline{W} . \end{gathered}$$

Observe that the last equality holds because $\varphi_{N_\alpha}^{N}$ is open by Lemma 3.11. ■

3.17. Lemma. *Let $\{N_\alpha : \alpha \in A\} \subset \mathcal{N}^*(G)$ be a family consisting of e-admissible subgroups of G. If we additionally assume that the map $e : \mathcal{T}(X) \longrightarrow \mathcal{T}(B)$ is d-regular, then $N = \bigcap\{N_\alpha : \alpha \in A\} \in \mathcal{N}^*(G)$ would be e-admissible too.*

PROOF : For any $\alpha \in A$ fix base $\mathcal{B}_{N_\alpha}$ of X_{N_α} in accordance with Definition 3.8, and define $\gamma_\alpha = \{(\varphi_{N_\alpha}^{N})^{-1}(U) : U \in \mathcal{B}_{N_\alpha}\}$. Let

$$\mathcal{B}_N = \{V_{\alpha_1} \cap \cdots \cap V_{\alpha_k} : \alpha_1, \ldots, \alpha_k \in A, V_{\alpha_i} \in \gamma_{\alpha_i}, i = 1, \ldots, k, k \in \omega\} .$$

Since X_N is compact, $\mathcal{B}_N$ is a base of X_N. We will show that $\mathcal{B}_N$ witnesses that N is e-admissible. Pick an $U \in \mathcal{B}_N$. Choose $k \in \omega$, $\alpha_1, \ldots, \alpha_k \in A$ and $V_{\alpha_i} \in \gamma_{\alpha i}$,

$i = 1, \dots, k$ such that $W = V_{\alpha_1} \cap \dots \cap V_{\alpha_k}$. For every $i \le k$ fix an $U_{\alpha_i} \in \mathcal{B}_{N_{\alpha_i}}$ with $W_{\alpha_i} = (\varphi^N_{N_{\alpha_i}})^{-1}(U_{\alpha_i})$.

(i) By Definition 3.8(i) each set $e(\varphi^{-1}_{N_{\alpha_i}}(U_{\alpha_i})) = e(\varphi^{-1}_N(V_{\alpha_i}))$ is N_{α_i}-cylindrical, and so it is also N-cylindrical because $N \subset N_{\alpha_i}$ (Lemma 3.5). Now Lemma 3.6 yields that the set

$$e(\varphi^{-1}_N(V_{\alpha_1})) \cap \dots \cap e(\varphi^{-1}_N(V_{\alpha_k})) = e(\varphi^{-1}_N(V_{\alpha_1}) \cap \dots \cap \varphi^{-1}_N(V_{\alpha_k})) =$$
$$e(\varphi^{-1}_N(V_{\alpha_1} \cap \dots \cap V_{\alpha_k})) = e(\varphi^{-1}_N(W))$$

is N-cylindrical too. (In the first equality we applied our assumption that e is d-regular.)

(ii) Fix an $i \le k$. Since $\mathcal{B}_{N_{\alpha_i}}$ witnesses that $X_{N_{\alpha_i}}$ is e-admissible, $X_{N_{\alpha_i}} \cap \psi_{N_{\alpha_i}}(e(\varphi^{-1}_{N_{\alpha_i}}(U_{\alpha_i}))) \subset \overline{U}_{\alpha_i}$ by Definition 3.8(ii) applied to $X_{N_{\alpha_i}}$ and $\mathcal{B}_{N_{\alpha_i}}$. Therefore,

$$X_N \cap \psi_N(e(\varphi^{-1}_{N_{\alpha_i}}(U_{\alpha_i}))) \subset X_N \cap (\psi^N_{N_{\alpha_i}})^{-1}(\psi_{N_{\alpha_i}}(e(\varphi^{-1}_{N_{\alpha_i}}(U_{\alpha_i})))) \subset$$
$$X_N \cap (\psi^N_{N_{\alpha_i}})^{-1}(\overline{U}_{\alpha_i}) = (\varphi^N_{N_{\alpha_i}})^{-1}(\overline{U}_{\alpha_i}) = \overline{V}_{\alpha_i}.$$

(The last equality holds, since $\varphi^N_{N_{\alpha_i}}$ is open by Lemma 3.11.) Now d-regularity of e yields

$$X_N \cap \psi_N(e(\varphi^{-1}_N(W))) = X_N \cap \psi_N(e(\varphi^{-1}_N(V_{\alpha_1}) \cap \dots \cap \varphi^{-1}_N(V_{\alpha_k}))) =$$
$$X_N \cap \psi_N(e(\varphi^{-1}_N(V_{\alpha_1})) \cap \dots \cap e(\varphi^{-1}_N(V_{\alpha_k}))) \subset$$
$$(X_N \cap \psi_N(e(\varphi^{-1}_{N_{\alpha_1}}(U_{\alpha_1})))) \cap \dots \cap (X_N \cap \psi_N(e(\varphi^{-1}_{N_{\alpha_k}}(U_{\alpha_k})))) \subset$$
$$\overline{V}_{\alpha_1} \cap \dots \cap \overline{V}_{\alpha_k} = \overline{V_{\alpha_1} \cap \dots \cap V_{\alpha_k}} = \overline{W}. \qquad \blacksquare$$

4. κ-metrizable compact spaces and extending of open sets. κ-metrizable spaces were defined by Ščepin [11], [12]. Each Dugundji space is κ-metrizable, but not vice versa [12]. Shirokov [14] proved that a compact space X is κ-metrizable if and only if for some (equivalently, for any) embedding of X into the Tychonoff cube I^τ there exists a regular map $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(I^\tau)$. It turns out that in this characterization I^τ can be replaced by a topological group.

4.1. Theorem. *For any compact space X the following are equivalent:*

(i) *X is κ-metrizable,*

(ii) *there exist a topological group G which contains X as a subspace and a regular map $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(G)$,*

(iii) *there exists a regular map $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(F(X))$,*

(iv) *there exists a regular map $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(A(X))$,*

(v) *for any embedding of X into a topological group G there exists a regular map $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(G)$,*

(vi) *there exist a topological group G, its G_δ-subset B containing X as a subspace and a regular map $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(B)$.*

PROOF : (i) $\Longrightarrow$ (v). Assume that X is a subspace of a topological group G. By the Tychonoff embedding theorem, $G \hookrightarrow I^\tau$ for some cardinal τ. By Shirokov's theorem [14] there is a regular map $e'\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(I^\tau)$. Then $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(G)$ defined by $e(U) = e'(U) \cap G$ for $U \in \mathcal{T}(X)$ is a regular map.

(v) $\Longrightarrow$ (iv) is trivial.

(iv) $\Longrightarrow$ (iii) Let $g\colon F(X) \longrightarrow A(X)$ be the continuous homomorphism extending the identity map of X and $e'\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(A(X))$ be a regular map. Then the map $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(F(X))$ defined by $e(U) = g^{-1}(e'(U))$ for $U \in \mathcal{T}(X)$ is regular.

(iii) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (vi) are trivial.

Now let us check the implication (vi) $\Longrightarrow$ (i) using transfinite induction on the weight of X. If $w(X) = \omega$, then X is metrizable and so κ-metrizable. Suppose that $\tau > \omega$ and that the implication (vi) $\Longrightarrow$ (i) have been checked for any compact space X of weight $< \tau$. Fix a compact subspace X of a G_δ-subset B of a topological group G so that $w(X) = \tau$, and a regular map $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(B)$. Without loss of generality we may assume that X algebraically generates G, so G is σ-compact. In view of Lemma 2.7 we can suppose that B is closed. By Lemma 2.6(i) there is an $N^* \in \mathcal{M}(G)$ so that $B = \pi_{N^*}^{-1}(\pi_{N^*}(B))$. Moreover, by Lemma 2.5 the map e can be chosen to be inclusion–preserving. Now all assumptions of 3.1 are satisfied, and in what follows notations from 3.1 will be used. Since X algebraically generates G, $nw(G) \le nw(X) \le w(X) = \tau$, so one can fix a family $\{U_\alpha\colon \alpha < \tau\} \subset \mathcal{T}(G)$ with $\{e_G\} = \cap\{U_\alpha\colon \alpha < \tau\}$, where e_G is the neutral element of G. By Lemma 2.6(ii), for every $\alpha < \tau$ there exists an $H_\alpha \in \mathcal{M}(G)$ with $H_\alpha \subset U_\alpha$.

By transfinite recursion for each $\alpha < \tau$ we will choose an $N_\alpha \in \mathcal{N}^*(G)$ such that:

(a) $N_\alpha \subset H_\alpha$ for all $\alpha < \tau$,
(b) $w(X_{N_\alpha}) \le |\alpha| \cdot \omega < \tau$ for every $\alpha < \tau$,
(c) N_α is e-admissible for all $\alpha < \tau$,
(d) $N_\beta \subset N_\alpha$ provided that $\alpha < \beta < \tau$.

To start with, use Lemma 2.6(ii),(iii) to find e-admissible $N_0 \in \mathcal{N}^*(G)$ so that $N_0 \subset H_0 \cap N^*$ and $w(X_{N_0}) \le w(X_{H_0 \cap N^*}) = \omega < \tau$. If $\{N_\alpha\colon \alpha < \beta\} \subset \mathcal{N}^*(G)$ satisfying (a) – (d) have already been defined, then let $N_\beta \in \mathcal{N}^*(G)$ be the e-admissible subgroup of G such that $N_\beta \subset \widetilde{H}_\beta = H_\beta \cap \bigcap\{N_\alpha\colon \alpha < \beta\}$ and $w(X_{N_\beta}) \le w(X_{\widetilde{H}_\beta}) \le |\beta| \cdot \sup\{w(X_{N_\alpha})\colon \alpha < \beta\} + w(X_{H_\beta}) \le |\beta| \cdot \omega < \tau^\beta$ (this subgroup exists in view of Lemma 3.13). Since $\bigcap\{N_\alpha\colon \alpha < \tau\} \subset \bigcap\{H_\alpha\colon \alpha < \tau\} = \{e_G\}$, X is homeomorphic to the limit of continuous well–ordered transfinite spectrum $\mathbf{S} = \{X_{N_\alpha}, \varphi_{N_\alpha}^{N_\beta}, \alpha < \beta < \tau\}$. By (c) and Lemma 3.12, for each $\alpha < \tau$ there exists a regular map $e_{N_\alpha}\colon \mathcal{T}(X_{N_\alpha}) \longrightarrow \mathcal{T}(B_{N_\alpha})$. Since $N_\alpha \in N^*$, $B = \pi_{N_\alpha}^{-1}(B_{N_\alpha})$, and so B_{N_α} is a G_δ-subset of the topological group G/N_α. Since $w(X_{N_\alpha}) < \tau$ by (b), our inductive assumption yields that each X_{N_α} is κ-metrizable, and hence, by [**12**, Lattice theorem], X_{N_α} is openly generated in the terminology of Ščepin [**10**, Definition 4]. From (c) and Lemma 3.10 it follows that all limit projections $\varphi_{N_\alpha}\colon X \longrightarrow X_{N_\alpha}$ of the spectrum $\mathbf{S}$ are open, so X is openly generated by [**10**, Theorem 16]. Therefore, X is κ-metrizable by Ivanov's theorem [**6**]. ■

5. Dugundji spaces and extending of open sets. Shirokov [14] showed that a compact space X is Dugundji if and only if for some (or equivalently, for any) embedding of X into the Tychonoff cube I^τ there exists a d-regular map $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(I^\tau)$. Our next result shows that in this characterization I^τ can be replaced by a topological group.

5.1. Theorem. *For every compact space X the following conditions are equivalent:*

(i) *X is a Dugundji space,*
(ii) *there exist a topological group G which contains X as a subspace and a d-regular map $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(G)$,*
(iii) *there exists a d-regular map $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(F(X))$,*
(iv) *there exists a d-regular map $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(A(X))$,*
(v) *for every embedding of X into a topological group G there is a d-regular map $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(G)$,*
(vi) *there exist a topological group G, its G_δ-subset B that contains X as a subspace and a d-regular map $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(B)$.*

PROOF : All implications except (vi) $\Longrightarrow$ (i) can be proved in the same way as in the proof of Theorem 4.1. Let us verify (vi) $\Longrightarrow$ (i). As in the proof of Theorem 4.1 we will assume that G is σ-compact, B is closed, $B = \pi_{N^*}^{-1}(\pi_{N^*}(B))$ for some N^* from $\mathcal{M}(G)$, and e preserves inclusions. Let $\mathcal{A} = \{N \in \mathcal{M}^*(G)\colon N \text{ is } e\text{-admissible}\}$. The diagonal map $\varphi = \Delta\{\varphi_N\colon N \in \mathcal{A}\}\colon X \longrightarrow \prod\{X_N\colon N \in \mathcal{A}\}$ is, by Lemmas 3.13 and 2.6(ii), a homeomorphic embedding of X into the product of compact metric spaces. Lemma 3.17 yields that for every $\mathcal{C} \subset \mathcal{A}$ the set $\bigcap\mathcal{C} \in \mathcal{N}^*(G)$ is e-admissible, and so $\Delta\{\varphi_N\colon N \in \mathcal{C}\}$ is an open map (Lemma 3.10). By Theorem 2.1, X is Dugundji. ∎

5.2. Corollary. *Assume that X is a compact subspace of a topological group G, B is a G_δ-subset of G, and there exists a sequence $X = X_0 \subset X_1 \subset \cdots \subset X_n = B$ so that for all $i < n$, X_i is either a retract or a dense subset of X_{i+1}. Then X is Dugundji.*

PROOF : If $r\colon Z \longrightarrow Y$ is a retraction, then the map $e\colon \mathcal{T}(Y) \longrightarrow \mathcal{T}(Z)$ defined by $e(U) = r^{-1}(U)$ for $U \in \mathcal{T}(Y)$ is d-regular. Similarly, if Y is a dense subspace of Z, then the map $e\colon \mathcal{T}(Y) \longrightarrow \mathcal{T}(Z)$ defined by $e(U) = \bigcup\{V \in \mathcal{T}(Z)\colon V \cap Y \subset U\}$ for $U \in \mathcal{T}(Y)$ is d-regular. Therefore, under assumption of our lemma there exists a d-regular map $e\colon \mathcal{T}(X) \longrightarrow \mathcal{T}(B)$, and the result follows from Theorem 5.1. ∎

5.3. Corollary. *If a compact space X is a retract of a dense subspace of some topological group, then X is Dugundji.*

6. Dugundji spaces as set–valued retracts of topological groups. Let X be a subspace of Y. A set–valued map $r\colon Y \longrightarrow X$ which sends each point $y \in Y$ to a subset $r(y)$ of X is said to be a *set–valued retraction* iff $r(x) = \{x\}$ for each $x \in X$. A set–valued retraction $r\colon Y \longrightarrow X$ is *upper semicontinuous* (briefly, *u.s. retraction*) iff $r^{-1}(U) = \{y \in Y : r(y) \subset U\} \in \mathcal{T}(Y)$ for any $U \in \mathcal{T}(X)$.

Dranishnikov [2] characterized Dugundji spaces as compact u.s. retracts of the Tychonoff cube I^τ. In this characterization I^τ can also be replaced by a topological group.

6.1. Theorem. *For each compact space X these are equivalent:*

(i) *X is Dugundji,*
(ii) *there exists a u.s. retraction $r: G \longrightarrow X$ of some topological group G onto X,*
(iii) *there exists a u.s. retraction $r: F(X) \longrightarrow X$,*
(iv) *there exists a u.s. retraction $r: A(X) \longrightarrow X$,*
(v) *X is a u.s. retract of every topological group G that contains X as a subspace,*
(vi) *there exist a topological group G, its G_δ-subset B and a u.s. retraction $r: B \longrightarrow X$.*

PROOF : Combine Theorem 5.1 and the following lemma which was implicitly proved (but not stated explicitly) in [17]. ■

6.2. Lemma. *Let X be a compact subspace of Y.*

(i) *If $r: Y \longrightarrow X$ is a u.s. retraction, then the map $e: \mathcal{T}(X) \longrightarrow \mathcal{T}(Y)$ so that $e(U) = r^{-1}(U)$ for $U \in \mathcal{T}(X)$, is d-regular.*
(ii) *If $e: \mathcal{T}(X) \longrightarrow \mathcal{T}(Y)$ is a d-regular map, then $r: Y \longrightarrow X$ defined by $r(y) = \bigcap\{\overline{U}: U \in \mathcal{T}(X), y \in e(U)\}$ is a u.s. retraction.*

Observe that the retraction r from items (iii)–(v) of Theorem 6.1 cannot be chosen to be single–valued as the following result shows.

6.3. Corollary. *For $n \notin \{0,1,3,7\}$ the n-dimensional sphere S^n is not a (single-valued) retract of a topological group, but S^n is a u.s. (set–valued) retract of any topological group G which contains S^n as a subspace.*

The first part of this corollary is due to Uspenskii [16, Proposition 16], and the second part follows from Theorem 6.1.

7. Dugundji spaces as compact P-valued retracts of topological groups.

Let **Comp** denotes the category of compact spaces and their (continuous) maps. Let $\mathcal{F}$: **Comp** $\longrightarrow$ **Comp** be arbitrary normal functor [10, Definition 14]. Recall that for each compact space X there is a homeomorphic embedding $i_X: X \longrightarrow \mathcal{F}(X)$, [10, Proposition 3.10]. If X is a compact subspace of Y, then a map $r: Y \longrightarrow \mathcal{F}(X)$ is said to be an *$\mathcal{F}$-valued retraction* of Y onto X provided that $r|_X = i_X$.

Let $C(X)$ be the Banach space of all real–valued continuous functions defined on a compact space X with the norm $\|f\| = \sup_{x\in X} |f(x)|$, and let $(C(X))^*$ be the set of all linear functionals on $C(X)$ equipped with the topology inherited by $(C(X))^*$ from $\mathbf{R}^{C(X)}$. For $f \in C(X)$ we write $f \geq 0$ iff $f(x) \geq 0$ for all $x \in X$. Define $1_X \in C(X)$ by $1_X(x) = 1$ for each $x \in X$. Let $P(X) = \{\lambda \in (C(X))^*: \lambda(1_X) = 1$ and if $f \geq 0$, then $\lambda(f) \geq 0\}$. If X is compact, then so is $P(X)$. If X and Y are compact spaces, then each map $\varphi: X \longrightarrow Y$ induces the map $P(\varphi): P(X) \longrightarrow P(Y)$, defined by $P(\varphi)(\lambda)(f) = \lambda(f \circ \varphi)$ for $\lambda \in P(X)$ and $f \in C(Y)$. This defines the functor P: **Comp** $\longrightarrow$ **Comp**, the so–called *functor of probability measures*. The name is justified by the fact that each $\lambda \in P(X)$ can be interpreted as a regular probability measure on X.

Ščepin proved that for every normal functor $\mathcal{F}$: **Comp** $\longrightarrow$ **Comp** compact $\mathcal{F}$-valued retracts of the Tychonoff cube I^τ are Dugundji [10, Theorem 4.2] and characterized Dugundji compact spaces as compact P-valued retracts of I^τ (see Introduction of [10]). We will show that in these results I^τ can also be replaced by a topological group.

7.1. Theorem. *Assume that $\mathcal{F}$: **Comp** $\longrightarrow$ **Comp** is a normal functor, G is topological group, B its G_δ-subset and X a compact subspace of B. If there exists an $\mathcal{F}$-valued retraction r of B onto X then X is Dugundji.*

7.2. Theorem. *For each compact space X the following conditions are equivalent:*

(i) *X is Dugundji,*
(ii) *there exists a P-valued retraction of some topological group onto X,*
(iii) *X is a P-valued retract of $F(X)$,*
(iv) *X is a P-valued retract of $A(X)$,*
(v) *X is a P-valued retract of every topological group G that contains X as a subspace.*

From Theorem 7.2 and Corollary 6.3 we immediately obtain

7.3. Corollary. *If $n \notin \{0,1,3,7\}$, then the n-dimensional sphere S^n is not a (single-valued) retract of a topological group, while S^n is a P-valued retract of any topological group which contains S^n as a subspace.*

7.4. Definition [10, Definition 18]. Let $\mathcal{F}$: **Comp** $\longrightarrow$ **Comp** be a normal functor. For a compact space X define a set-valued map $\mathrm{supp}_X\colon \mathcal{F}(X) \longrightarrow X$ by $\mathrm{supp}_X(z) = \bigcap\{\Phi \subset X\colon \Phi = \overline{\Phi} \text{ and } z \in \mathcal{F}(\Phi)\}$ for $z \in \mathcal{F}(X)$ (here we identify $\mathcal{F}(\Phi)$ with the image $\mathcal{F}(j_\Phi)(\mathcal{F}(\Phi))$ of $\mathcal{F}(\Phi)$ under the map $\mathcal{F}(j_\Phi)\colon \mathcal{F}(\Phi) \longrightarrow \mathcal{F}(X)$ extending the inclusion map $j_\Phi\colon \Phi \subset X$).

7.5. Lemma. *If $\mathcal{F}$: **Comp** $\longrightarrow$ **Comp** is a normal functor, X and Z are compact spaces and $f\colon X \longrightarrow Z$ is a map, then $f(\mathrm{supp}_X(x)) = \mathrm{supp}_{f(X)}(\mathcal{F}(f)(x))$ for each $x \in \mathcal{F}(X)$.*

PROOF : Since $\mathcal{F}$ is normal, $\mathcal{F}$ preserves intersections and preimages (see [10, definition 14]), and the result follows. ■

7.6. Lemma. *Let X be a compact subspace of Y and $r\colon Y \longrightarrow \mathcal{F}(X)$ be an $\mathcal{F}$-valued retraction of Y onto X. Suppose $f\colon Y \longrightarrow Z$ is an open map so that $(\mathcal{F}(f\,|_X)\circ r)(y) = (i_{f(X)}\circ f)(y)$ for all $y \in f^{-1}(f(X))$. Then the map $f\,|_X\colon X \longrightarrow f(X)$ is open (note that we consider $f\,|_X$ as a map onto its image).*

PROOF : Under additional assumption that Y is compact this lemma was proved by Ščepin [10, Proposition 3,14]. Considering, if necessary, $Y' = f^{-1}(f(X))$ and $f' = f\,|_{Y'}$, we can assume that $f(X) = Z$. Recall that if $\varphi\colon X \longrightarrow Y$ and $\psi\colon Y \longrightarrow Z$ are set-valued maps, then their composition $\psi\circ\varphi\colon X \longrightarrow Z$ is the set-valued map defined by $(\psi\circ\varphi)(x) = \bigcap\{\psi(y)\colon y \in \varphi(x)\}$ for $x \in X$. Define a set-valued map $\varphi\colon Z \longrightarrow \mathcal{F}(X)$ by $\varphi(z) = \{r(y)\colon y \in f^{-1}(z)\}$ for $z \in Z$, and let $\Theta = \mathrm{supp}_X \circ \varphi$.

7.7. Claim. *$\Theta(z) = (f|_X)^{-1}(z)$ for each $z \in Z$.*

PROOF : Since r is an $\mathcal{F}$-valued retraction, $(f|_X)^{-1}(z) \subset \Theta(z)$ for all $z \in Z$. On the other hand, if $x \in \Theta(z)$, then $x \in \mathrm{supp}_X(r(y))$ for some $y \in f^{-1}(z)$. Now, by Lemma 7.5,

$$f|_X(x) \in f|_X(\mathrm{supp}_X r(y)) = \mathrm{supp}_Z(\mathcal{F}(f|_X)(r(y))) = \mathrm{supp}_Z((i_Z \circ f)(y)) = \mathrm{supp}_Z(i_Z(z)) = \{z\},$$

and so $x \in (f|_X)^{-1}(z)$. Therefore $\Theta(z) \subset (f|_X)^{-1}(z)$. ∎

Since f is open and r is continuous, φ is lower semicontinuous. The map supp_X is lower semicontinuous by [10, Proposition 3.11]. Thus Θ is lower semicontinuous as a composition of lower semicontinuous set-valued maps. In view of Claim 7.7 this yields that $f|_X$ is open. ∎

From now on we will adopt notations and conventions of 3.1 with the only exception that instead of map e we fix an $\mathcal{F}$-valued retraction $r: B \longrightarrow \mathcal{F}(X)$.

7.8. Definition. A set $N \in \mathcal{N}^*(G)$ is *r-admissible* provided that

(3) $$(\mathcal{F}(\varphi_N) \circ r)(x) = (i_{X_N} \circ \psi_N)(x) \text{ for every } x \in \psi_N^{-1}(X_N)$$

7.9. Lemma. *The condition (3) is equivalent to*

(4) $$\varphi_N(\mathrm{supp}_X(r(x))) = \{\psi_N(x)\} \textit{ for all } x \in \psi_N^{-1}(X_N).$$

PROOF : If $x \in \psi_N^{-1}(X_N)$ and (3) holds, then Lemma 7.5 implies

$$\varphi_N(\mathrm{supp}_X(r(x))) = \mathrm{supp}_{X_N}(\mathcal{F}(\varphi_N)(r(x))) = \mathrm{supp}_{X_N}((\mathcal{F}(\varphi_N) \circ r)(x)) = \mathrm{supp}_{X_N}((i_{X_N} \circ \psi_N)(x)) = \{\psi_N(x)\}.$$

On the other hand, suppose that $x \in \psi_N^{-1}(X_N)$ and $\varphi_N(\mathrm{supp}_X(r(x))) = \{\psi_N(x)\}$. Since $\mathcal{F}$ preserves preimages,

$$\mathcal{F}(\varphi_N^{-1}(\psi_N(x))) = (\mathcal{F}(\varphi_N))^{-1}(i_{X_N}(\psi_N(x))).$$

Since $\mathrm{supp}_X(r(x)) \subset \varphi_N^{-1}(\psi_N(x))$ and $\mathcal{F}$ is normal, $r(x) \in \mathcal{F}(\mathrm{supp}_X(r(x))) \subset \mathcal{F}(\varphi_N^{-1}(\psi_N(x))) = (\mathcal{F}(\varphi_N))^{-1}(i_{X_N}(\psi_N(x)))$, and so $\mathcal{F}(\varphi_N) \circ r)(x) \in \{(i_{X_N} \circ \psi_N)(x)\}$. Since $\{(i_{X_N} \circ \psi_N)(x)\}$ is the one-point set, (3) follows. ∎

7.10. Lemma. *Let $\{N_\alpha : \alpha \in A\} \subset \mathcal{N}^*(G)$ be a family consisting of r-admissible sets. Then $N = \bigcap\{N_\alpha : \alpha \in A\} \in \mathcal{N}^*(G)$ is r-admissible too.*

PROOF : Let $i: B_N \longrightarrow \prod\{B_{N_\alpha} : \alpha \in A\}$ be the natural one-to-one continuous map, $\varphi = \Delta\{\varphi_{N_\alpha} : \alpha \in A\}: X \longrightarrow \prod\{X_{N\alpha} : \alpha \in A\}$ and $\psi = \Delta\{\psi_{N_\alpha} : \alpha \in A\}: B \longrightarrow \prod\{B_{N\alpha} : \alpha \in A\}$ be the diagonal maps. Fix an $x \in \psi_N^{-1}(X_N)$. Then $x \in \psi_{N_\alpha}^{-1}(X_{N_\alpha})$ for each $\alpha \in A$, and so $\varphi_{N_\alpha}(\mathrm{supp}_X(r(x))) = \{\psi_{N_\alpha}(x)\}$ by Lemma 7.9. This implies that $\varphi(\mathrm{supp}_X(r(x))) = \{\psi(x)\}$. Now from $\psi = i \circ \psi_N$ and $\varphi = i|_{X_N} \circ \varphi_N$ it follows that $\varphi_N(\mathrm{supp}_X(r(x))) = \{\psi_N(x)\}$. Therefore N is r-admissible (Lemma 7.9). ∎

7.11. Lemma. *For any $N \in \mathcal{M}^*(G)$ there exists an r-admissible $\widetilde{N} \in \mathcal{M}^*(G)$ so that $\widetilde{N} \subset N$.*

PROOF : By induction we will define a decreasing sequence $\{N_k : k \in \omega\} \subset \mathcal{M}^*(G)$ so that if $k \in \omega$, $x, y \in B$ and $\psi_{N_{k+1}}(x) = \psi_{N_{k+1}}(y)$, then $(\mathcal{F}(\varphi_{N_k}) \circ r)(x) = (\mathcal{F}(\varphi_{N_k}) \circ r)(y)$. Set $N_0 = N$ and assume that $N_k \in \mathcal{M}^*(G)$ have already been defined for $k \leq j$. Since $N_j \in \mathcal{M}^*(G)$ and X_{N_j} is compact, $w(X_{N_j}) = nw(X_{N_j}) \leq nw(B_{N_j}) \leq nw(G/N_j) \leq \omega$. Since $\mathcal{F}$ preserves the weight, $\mathcal{F}(X_{N_j})$ has a countable base $\mathcal{B}_j$. For each $U \in \mathcal{B}_j$ use Lemma 2.6(i) to choose an $N_U \in \mathcal{M}(G)$ so that $(\mathcal{F}(\varphi_{N_j}) \circ r)^{-1}(U)$ is N_U-cylindrical. Set $N_{j+1} = \bigcap\{N_U : U \in \mathcal{B}_j\} \cap N_j$. Lemma 2.6(iii) implies that $N_{j+1} \in \mathcal{M}^*(G)$. Now suppose that $x_i \in B$ and $x'_i = (\mathcal{F}(\varphi_{N_j}) \circ r)(x_i)$ for $i = 1, 2$ and $x'_1 \neq x'_2$. Then there are $U'_i \in \mathcal{B}_j$ with $x'_i \in U'_i$ and $\overline{U'_1} \cap \overline{U'_2} = \emptyset$. Therefore $x_i \in U_i = (\mathcal{F}(\varphi_{N_j}) \circ r)^{-1}(U'_i)$ and $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. In view of Lemma 3.5 both U_i are N_{j+1}-cylindrical, so $\psi_{N_{j+1}}(x_1) \neq \psi_{N_{j+1}}(x_2)$.

Since $\{N_k : k \in \omega\} \subset \mathcal{M}^*(G)$, $\widetilde{N} = \bigcap\{N_k : k \in \omega\} \in \mathcal{M}^*(G)$ (Lemma 2.6(iii)). Now observe that $\widetilde{N}$ is r-admissible. ■

PROOF of Theorem 7.1: As in the proof of Theorem 4.1 we can suppose that X algebraically generates G, B is closed and $B = \pi_{N^*}^{-1}(\pi_{N^*}(B))$ for some $N^* \in \mathcal{M}(G)$. So we will use notations and conventions of 3.1 together with that which was adopted before Definition 7.8. Let $\mathcal{R} = \{N \in \mathcal{M}^*(G) : N \text{ is } r\text{-admissible}\}$. By Lemmas 7.11 and 2.6(ii) the diagonal map $\varphi = \Delta\{\varphi_N : N \in \mathcal{R}\} : X \longrightarrow \prod\{X_N : N \in \mathcal{R}\}$ is a homeomorphic embedding of X into the product of compact metric spaces. Lemma 7.10 implies that for every $\mathcal{C} \subset \mathcal{R}$ the set $\bigcap\mathcal{C} \in \mathcal{N}^*(G)$ is r-admissible, and so the map $\Delta\{\varphi_N : N \in \mathcal{C}\}$ is open (Lemma 7.6). By Theorem 2.1, X is Dugundji. ■

PROOF of Theorem 7.2: (i) $\Longrightarrow$ (v). Assume that X is a subspace of a topological group G. Embed G homeomorphically into I^τ. By Haydon's theorem [5] exists a P-valued retraction $r : I^\tau \longrightarrow P(X)$, and its restriction to G would be as required.

(v) $\Longrightarrow$ (iv) and (iii) $\Longrightarrow$ (ii) are trivial.

(iv) $\Longrightarrow$ (iii). Let $f : F(X) \longrightarrow A(X)$ be the homomorphism whose restriction to X coincides with the identity map of X, and let $r_A : A(X) \longrightarrow P(X)$ be a P-valued retraction. Then $r_F : F(X) \longrightarrow P(X)$ defined by $r_F(y) = (r_A \circ f)(y)$ for $y \in F(X)$ is a P-valued retraction of $F(X)$ onto X.

Since functor P is normal, implication (ii) $\Longrightarrow$ (i) follows from Theorem 7.1. ■

8. Dugundji spaces and extending of functions. Denote by $C(X)$ the linear space of all real–valued (continuous) functions defined on a space X. Let $1_X \in C(X)$ be the function defined by $1_X(x) = 1$ for each $x \in X$. Let X be a subspace of Y. Following Pełczyński [9], we say that a linear operator $u : C(X) \longrightarrow C(Y)$ is *regular* if the following conditions hold:

(i) for every $f \in C(X)$ the restriction of $u(f)$ to X coincides with f,
(ii) if $f \geq 0$, then $u(f) \geq 0$,
(iii) $u(1_X) = 1_Y$.

In 1968, Pełczyński [9] defined Dugundji spaces as those compact spaces X for which there exists a homeomorphic embedding of X into the Tychonoff cube I^τ

admitting a regular operator $u: C(X) \longrightarrow C(I^\tau)$. Our last result shows that in this definition I^τ can be replaced by a topological group.

8.1. Theorem. *For every compact space X the following are equivalent*

(i) *X is Dugundji,*
(ii) *there exists a homeomorphic embedding of X into a topological group G admitting a regular operator $u: C(X) \longrightarrow C(G)$,*
(iii) *there exists a regular operator $u: C(X) \longrightarrow C(F(X))$,*
(iv) *there exists a regular operator $u: C(X) \longrightarrow C(A(X))$,*
(v) *for every embedding of X into a topological group G there exists a regular operator $u: C(X) \longrightarrow C(G)$,*
(vi) *there exist a topological group G, its G_δ-subset B which contains X as a subspace and a regular operator $u: C(X) \longrightarrow C(B)$.*

This theorem immediately follows from Theorem 7.1, 7.2 and the following folklore

8.2. Lemma. *Suppose that X is a subspace of Y.*

(i) *If $u: C(X) \longrightarrow C(Y)$ is a regular operator, then the map $r: Y \longrightarrow P(X)$ defined by $r(y)(f) = u(f)(y)$ for $y \in Y$ and $f \in C(X)$, is a P-valued retraction of Y onto X,*
(ii) *If $r: Y \longrightarrow P(X)$ is a P-valued retraction of Y onto X, then the map $u: C(X) \longrightarrow C(Y)$, defined by $u(f)(y) = r(y)(f)$ for $f \in C(X)$ and $y \in Y$, is a regular operator.*

Historical remark. In 1986 V. V. Uspenskii obtained a series of results concerning Dugundji spaces. In particular, he proved Theorems 1 and 4 of [16] and also the following statement: a compact retract of a dense subspace of some Lindelöf Σ-group G is Dugundji. He asked whether the requirement that G is a Lindelöf Σ-space can be dropped in the statement above. The author showed that this requirement is superfluous there (Corollary 5.3). To prove that, the author introduced the machinery of extending of open sets into the context of topological groups (Sections 3–5). All results of this paper were obtained in July 1987 and were announced in [13]. At the same time the author communicated them to V.V. Uspenskii. After that, attempting to generalize the results of the author, V. V. Uspenskii proved Theorems 5, 7 and Propositions 10, 11 of [16].

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