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Extensionable topological hulls and topological universe hulls inside the category of pseudotopological spaces

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Dedicated to the memory of Zdeněk Frolík

Abstract. The paper investigates the extensionable topological hulls and topological universe hulls of finally dense subcategories of the category **PsT** of pseudotopological spaces. In case of the extensionable topological hull, upper and lower bounds are given. It is shown that formation of the topological universe hull always leads to the category **PsT**. The results are applied to epireflective subcategories of the pretopological spaces.

Keywords: pseudotopological spaces, pretopological spaces, R_0-spaces; topological universe, topological universe hull, extensionable topological hull

Classification: 54B30, 54A05, 54A20, 18B15, 18B99

Any epireflective subcategory of the category **PrT** of pretopological spaces that is non-trivial (i.e., contains a non-indiscrete space) fails to be cartesian closed [**Sc 82**, 3.3]. Inside the category **Top** of topological spaces, this negative result holds even for reflective subcategories containing a discrete two-point space [**BH 82**, Thm. 1]. However, all these categories possess cartesian closed topological (CCT) hulls (which are contained in **PsT**).

We will investigate the same kind of problem with respect to the much stronger property of being a topological universe [**Ne 84**, 2.0], more precisely: We will determine the topological universe hull, i.e., the least finally dense topological universe extension, of any non-trivial epireflective subcategory of **PrT** . (We assume that subcategories are full and extensions full and concrete.)

Recall that a topological c-construct (i.e., a concrete category over **Set** which is initially complete, small-fibred, and has constants) is a topological universe iff it is cartesian closed and extensionable , where extensionable means that strong partial morphisms are representable [**Pe 77**, Sect. 4]. Moreover, if it exists, the topological universe hull of a category **A** can be obtained by forming the CCT hull of the extensionable topological hull of **A** [**ARS 89**], [**Sc 89**, 3.5]. In order to find the topological universe hull of a category, it is therefore useful and natural to first determine its extensionable topological hull. Surprisingly, both constructions lead to only very few distinct categories when applied to non-trivial epireflective subcategories of **PrT** (Thms. 1, 2), whereas it is well-known that formation of the corresponding CCT hulls produces a host of different categories.

The topological universe **PsT** [**Wy 76**, 4.9] is finally dense extension of every non-trivial epireflective subcategory **A** of **PrT** [**Sc 82**, 4.7]. Consequently, any of the above-mentioned hulls of **A** exists and can be formed inside **PsT**; moreover, function spaces and one-point extensions are formed as in **PsT** [**Sc 86**, Sect. 3].

We use the following abbreviations and notations:
CCTH**A** - cartesian closed topological hull of **A** ;
ETH**A** - extensionable topological hull of **A** ;
TUH**A** - topological universe hull of **A** ;
ERH**A** - epireflective hull of **A** in **PsT** (for $\mathbf{A} \subset \mathbf{PsT}$);
BRH**A** - bireflective hull of **A** in **PsT**;
$[X,Y]$ - function space (underlying set is the set of all **A**-morphisms from X to Y);
$Y^{\#}$ - one-point extension (underlying set is obtained from the underlying set of Y by adjoining one new point ∞_Y);
D_2 - discrete topological space on the set $\{0,1\}$;
2 - Sierpinski space; topological space on $\{0,1\}$ with open sets $\emptyset, \{1\}, \{0,1\}$.
In case of $\mathbf{2}^{\#}$ and $D_2^{\#}$, the point adjoined to $\{0,1\}$ is denoted 2 (rather than ∞).

Recall that function spaces in **PsT** are equipped with the structure of continuous convergence, i.e. a filter $\mathfrak{Y}$ on $[X,Y]$ converges to $f \in [X,Y]$ iff $\mathfrak{F} \underset{X}{\longrightarrow} x$ implies $\mathrm{ev}(\mathfrak{Y} \times \mathfrak{F}) \underset{Y}{\longrightarrow} f(x)$ (where ev denotes the usual evaluation map). The one-point extensions $Y^{\#}$ in **PsT** can be described as follows: All filters on $Y^{\#}$ converge to ∞_Y, while for $x \in Y, \mathfrak{F} \underset{Y^{\#}}{\longrightarrow} x$ iff $\mathfrak{F} \supset \overline{\mathfrak{G}} \cap \dot{\infty}_Y$ for some $\mathfrak{G} \underset{Y}{\longrightarrow} x$ (where $\overline{\mathfrak{G}}$ denotes the filter on $Y^{\#}$ generated by $\mathfrak{G}$, and $\dot{z}$ the principal ultrafilter generated by $\{z\}$). **PrT** is closed under one-point extensions in **PsT**, and consequently extensionable [**He 88**].

Following [**Ad 89**], we call a pseudotopological space an R_0-space iff $\dot{x} \longrightarrow y$ implies $\dot{y} \longrightarrow x$. This axiom was already investigated in [**Gä 77**, Sect. 3.7] under the name weak first separation axiom or T_{1w}.

Remark 1.

(1) It is easily seen that for $X \in \mathbf{PsT}$, the following are equivalent:
  (a) X fulfils the R_0-axiom.
  (b) X does not have a Sierpinski subspace.
  (c) $x \in \overline{y}$ implies $y \in \overline{x}$.
  (Here $\overline{z}$ denotes the pretopological closure of $\{z\}$.)
(2) If $X \in \mathbf{Top}$, then X is an R_0-space iff the convergence of $\mathfrak{F} \cap \dot{x}$ implies $\mathfrak{F} \longrightarrow x$ [**Ro 75**, 6.10]. In general, however, this property, which has also been called R_0 in the literature, is stronger than the R_0-axiom given above; the pretopological space $D_2^{\#}$ is an R_0-space violating the condition.
(3) Denote by R_0**PsT** (R_0**PrT**) the category of pseudotopological (pretopological) spaces satisfying the R_0-axiom. It is easy to see that the categories R_0**PsT** and R_0**PrT** are bireflective in **PsT**, and that they are extensionable. Theorem 2 will show that they fail to be cartesian closed. In contrast, the bireflective subcategory of **PsT** consisting of the spaces fulfilling the condition in (2) is cartesian closed [**Ro 75**, 7.32], but not extensionable (by (2)).

After these preparations, we are able to formulate the following result about extensionable topological hulls:

Theorem 1. *Let* **A** *be a finally dense subcategory of* **PsT**. *Then* **PrT** $\subset$ ETHA $\subset$ **PsT** *if* **2** $\in$ ERHA, *and* R_0**PrT** $\subset$ ETH **A** $\subset$ R_0**PsT** *if* **2** $\notin$ ERHA. *In case* **A** $\subset$ **PrT**, *we have* $\text{ETHA} = \begin{cases} \mathbf{PrT} & \text{if } \mathbf{2} \in \text{ERHA}, \\ R_0\mathbf{PrT} & \text{if } \mathbf{2} \notin \text{ERHA}. \end{cases}$

Since every non-trivial epireflective subcategory of **PrT** is finally dense in **PsT**, we obtain:

Corollary. *If* **A** *is a non-trivial epireflective subcategory of* **PrT** , *then*
$$\text{ETHA} = \begin{cases} \mathbf{PrT} & \text{if } \mathbf{2} \in \mathbf{A}, \\ R_0\mathbf{PrT} & \text{if } \mathbf{2} \notin \mathbf{A}. \end{cases}$$

In particular, the only bireflective subcategories of **PrT** which are extensionable are **PrT** , R_0**PrT** and the category **Ind** of indiscrete spaces. Inside **Top** , a stronger result is known: the only extensionable topological subcategories of **Top** are the discrete and the indiscrete spaces [**He 83**, Thm. 2].

To prove Theorem 1, we need the following two lemmas.

Lemma 1. *If* **A** *is a finally dense subcategory of* **PsT**, *then* $D_2 \in$ ERHA.

PROOF : Put **B** = ERHA. Assume $D_2 \notin$ **B**, then **2** $\notin$ **B**. Hence every finite subspace of a **B**-object X is indiscrete, i.e., each principal ultrafilter on X converges to every point of X. But then **B** cannot be finally dense in **PsT** (e.g., the total sink from **B** to **2** is not final). ∎

It follows from [**Bo 75**, II.2.1] that the one-point extension of the Sierpinski space is initially dense in **PrT** . In a similar way, one can show:

Lemma 2. $\{D_2^{\#}\}$ *is initially dense in* R_0**PrT**.

PROOF : Let $X \in R_0\mathbf{PrT}$. For each $a \in X$ and $W \in \mathfrak{V}(a) = \bigcap\{\mathfrak{F} | \mathfrak{F} \to a\}$, define a map $f_{a,W}\colon X \to D_2^{\#}$ as follows: $f_{a,W}(a) = 0$, $f_{a,W}(x) = 2$ if $x \in W - \{a\}$, $f_{a,W}(x) = 1$ if $x \in X - W$. We show that $(f_{a,W}\colon X \to D_2^{\#} | a \in X, W \in \mathfrak{V}(a))$ is an initial source. Continuity of $f_{a,W}$ at $x \in W - \{a\}$ is obvious. At $x = a$, we have $f_{a,W}(\mathfrak{V}(a)) \supset \dot{0} \cap \dot{2} \to 0 = f_{a,W}(a)$ since $f_{a,W}(W) \subset \{0,2\}$. If $x \notin W$, then $\dot{x} \not\to a$, hence $\dot{a} \not\to x$ by R_0. Consequently, $f_{a,W}(\mathfrak{V}(x)) \supset \dot{1} \cap \dot{2} \to 1 = f_{a,W}(x)$.- Now let $(f_{a,W} : X' \to D_2^{\#} | a \in X, W \in \mathfrak{V}(a))$ be initial. If $X' \neq X$, then there exists a $W \in \mathfrak{V}(a) - \mathfrak{V}'(a)$ for some $a \in X$. This would imply $V - W \neq \emptyset$ for all $V \in \mathfrak{V}'(a)$, hence $f_{a,W}(\mathfrak{V}'(a)) \subset \dot{1}$, contradicting the continuity of $f_{a,W} : X' \to D_2^{\#}$. ∎

PROOF of Theorem 1: Put **B** = ERHA. If **2** $\in$ **B**, then $\mathbf{2}^{\#} \in$ ETHB = ETHA, hence **PrT** = BRH$\{\mathbf{2}^{\#}\}$ $\subset$ ETHA $\subset$ **PsT**.- If **2** $\notin$ **B**, then **B** $\subset R_0$**PsT**. By Lemma 1, $D_2 \in$ **B**, consequently, $D_2^{\#} \in$ ETHB = ETHA. By Lemma 2, R_0**PrT** = BRH$\{D_2^{\#}\}$ $\subset$ ETHA $\subset R_0$**PsT**. ∎

We are now able to show the surprising fact that the category **PsT** is the topological universe hull of any of its finally dense subcategories **A** . With Theorem 1, the proof reduces to showing this for **A** $= R_0$**PrT**.

Theorem 2. *If* **A** *is a finally dense subcategory of* **PsT**, *then* TUHA = **PsT**.

PROOF : Denote by **B** the topological universe CCTH(R_0**PrT**). By Theorem 1, we have **PsT** $\supset$ TUHA = CCTH(ETHA) $\supset$ **B**. It remains to be shown that **B** $\supset$ **PsT**. Since **PsT** = TUHTop by [**Wy 76**, 4.9], and **Top** = BRH{**2**}, it suffices to prove **2** $\in$ **B**. Define a (pre)topological space X as follows: The underlying set of X is the set $\mathbb{N}$ of natural numbers; $\mathfrak{V}(0) = \mathfrak{U} \cap \dot{0}$, where $\mathfrak{U}$ is a free ultrafilter on $\mathbb{N}$, and $\mathfrak{V}(x) = \dot{x}$ if $x \neq 0$. Since $X, D_2^{\#} \in R_0\mathbf{PrT}$, we have $[X, D_2^{\#}] \in \mathbf{B}$. The maps $f, g : X \to D_2^{\#}$ defined by $f(0) = 0$, $f(x) = 2$ if $x \neq 0$, and $g(0) = 2$, $g(x) = 1$ if $x \neq 0$, are continuous, i.e., $f, g \in [X, D_2^{\#}]$. It is easy too see that $\dot{f} \to g$, but $\dot{g} \not\to f$. Consequently, $\{f, g\}$ is a Sierpinski subspace of $[X, D_2^{\#}]$. ∎

Remark 2. (1) In particular, TUH(R_0**PsT**) = **PsT**, and consequently, R_0**PsT** is not a topological universe, in contradiction to [**Ad 89**, III.1]. The topological universe hull of all categories **K** fulfilling the assumptions of [**Ad 89**, III.1] is the category **PsT**.

(2) The main result of [**Ad 89**] is a simplification of the construction of the topological universe hull of a category **A** , using saturated collections (structured sinks consisting of certain inclusion maps) instead of partially closed sinks [**ARS 89**]. This simpler method is based on the existence of a very strictly dense subcategory (a finally dense subclass satisfying several quite strong conditions) of **A**. Application of Theorem 2 to [**Ad 89**, II.5] shows that, unfortunately, no finally dense subcategory of **PsT** which is contained in R_0**PsT** has a very strictly dense subcategory; this excludes, in particular, any non-trivial epireflective subcategory of **Top** distinct from **Top** and T_0**Top** from having such a subcategory. It is an interesting question under which conditions a category contains a very strictly dense subcategory.

Remark 3. (1) Theorem 2 applies, in particular, to any noon-trivial epireflective subcategory of **PrT** .

(2) While for epireflective subcategories of **PrT** , being non-trivial is equivalent to being finally dense in **PsT**, there are non-trivial epireflective subcategories of **PsT** which are not finally dense in **PsT** (cf. the proof of Lemma 1): they consist of pseudotopological spaces whose finite subspaces are indiscrete. At least one of these categories is a topological universe, namely the category **ConsPsT** of "constant" pseudotopological spaces, i.e., pseudotopological spaces where the same filters converge to every point. Indeed, **ConsPsT** is the largest of the epireflective subcategories **A** of **PsT** that are not finally dense in **PsT**, but form topological universes: In every space of such a category **A** , each principal ultrafilter converges to all points. Now let $X \in \mathbf{A}$, $x, y \in X$ and $\mathfrak{U}$ an ultrafilter with $\mathfrak{U} \underset{X}{\longrightarrow} x$. The function space $[X, X]_{\mathbf{A}}$ in **A** is natural and carries, consequently, a pseudotopology finer than the structure of continuous convergence. Hence $\dot{1} \xrightarrow[{[X,X]_{\mathbf{A}}}]{} c_y$ implies $\mathfrak{U} = \mathrm{ev}(\dot{1} \times \mathfrak{U}) \to \mathrm{ev}(c_y, x) = c_y(x) = y$ (where 1 denotes the identity map, c_y the constant map with value y from X to X).- Though it could be argued that subcategories of **ConsPsT** are not of much use, similar categories have turned out

to be quite interesting: The topological universes of filter–merotopic spaces [**Ka 65**] and of grill–determined semineamess spaces [**Ro 75**] are isomorphic to the constant convergence spaces [**Sc 79**, Conn. 3]; the bornological spaces can be considered as a bicoreflective subcategory of the topological universe of constant limit spaces; and another interesting aspect is that of considering constant convergence as "global convergence" without reference to points (e.g. in the sense of Cauchy filters).

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