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A note on Baire isomorphism

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Dedicated to the memory of Zdeněk Frolík

Abstract. We give an example of an absolute Baire space which is nor Baire isomorphic to any compact Hausdorff space. This answers a question asked by Z.Frolík (see also [1] Problem N 54)

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Introduction. A set A of a topological space X is called Baire set if A belongs to the smallest σ-algebra of subsets of X which contains all zero-sets of X. If X is metrizable space, then σ-algebras of Baire and Borel sets of X coincide. A mapping $f : X \to Y$ is called a Barie mapping if an inverse image of zero-set of Y is a Baire set of X. A bijection $f : X \to Y$ is a Baire isomorphism, if f and f^{-1} are Baire mappings. A Tychonoff space X is called absolute Baire space if X is a Baire subset of $\beta(X)$. Any Čech-complete Lindelöf spce X is absolute Baire space as X is the intersection of sequence of cozero-sets of $\beta(X)$.

In this paper we give a negative answer to following question: whether every absolute Baire space is Baire isomorphic to some compact Huasdorff space? We think that this was originally formulated by Z.Frolík (see also [1] Problem N 54).

We construct a Čech-complete Lindelöf space M, which is not Baire isomorphic to any compact Hausdorff space.

We use the following notations: $|X|$ is cardinality of X, $w(X)$ is weight of X, $c = |I = [0,1]|$, P is the space of irrational numbers, $\exp_{\aleph_0} X = \{A : A \subseteq X, |A| \leqslant \aleph_0\}$, if γ is a family of subsets of $X, A \subseteq X$, then $\gamma(A)$ is a star of A, i.e. $\gamma(A) = \cup\{U \in \gamma : U \cap A \neq \phi\}$.

We shall use the following results

I) [2] If X and Y are metrizable compact spaces and $f : X \to Y$ is a continuous mapping, then $\mathcal{Z}(f) = \{x : |f^{-1}f(x)| > 1\}$ is F_σ-set of X.

II) (that is special case of the result from [3]). Let A be a Borel set in P. If A is not a subset of any σ-compact set in P, then A contains a subset that is closed in P and homeomorphic to P.

III) Let$f : X \to P$ be a mapping onto P and X be a σ-compact separable metric space. Then P contains a closed F, homeomorphic to P, so that $f^{-1}(F)$ is not F_σ in X.

This result easily follows from [4].

We recall that a spectrum $(X_\alpha, P^\alpha_\beta, \mathcal{U})$ is sigma-spectrum [5] if $w(X_\alpha) \leqslant \aleph_0, \alpha \in \mathcal{U}$, $\mathcal{U}$ is $\aleph_0$-complete i.e. every countable chain $\beta \subseteq \mathcal{U}$ has the least upper bound $\gamma = \gamma(\beta)$, and X_γ is naturally homeomorphic to $\lim(X_\alpha, P^\alpha_\beta, \beta)$.

IV) Let $X = \lim S_1$, $Y = \lim S_2$, $S_1 = (X_\alpha, \pi^\alpha_\beta, \mathcal{U})$, $S_2 = (Y_\alpha, \mu^\alpha_b, \mathcal{U})$, where $S_i, i = 1,2$ are sigma-spectrum, and all $\pi^\alpha_\beta, \mu^\alpha_\beta$ are perfect mappings. If X and Y are Baire isomorphic spaces then there are closed and cofinal subset $\mathcal{U}' \subseteq \mathcal{U}$, and Baire isomorphisms $f_\alpha : X_\alpha \to Y_\alpha$, $\alpha \in \mathcal{U}'$, such that $\mu^\alpha_\beta \circ f_\alpha = f_\beta \circ \pi^\alpha_\beta$ for every $\alpha, \beta \in \mathcal{U}'$, $\alpha \geqslant \beta$. In fact, that is proved in [6].

Construction of M. We shall consider a generalization of the Alexandrov duplicate construction, which is similar to the construction from [7].

If $\gamma = \{F_\alpha : \alpha \in A\}$ is a pair-wise disjoint collection of closed sets of P and $B \subseteq A$, we may definea topology on $P(B) = P \times \{0\} \cup \cup\{F\alpha \times \{1\} : \alpha \in B\}$ as follows: $\cup\{F_\alpha \times \{1\} : \alpha \in B\}$ is open and has the topology of direct sum $\oplus\{F_\alpha : \alpha \in B\}$, and neighborhoods of a point $(x, 0)$ in $P \times \{0\}$ are have the form $U(\alpha) = (U \times \{0\}) \cup ((U \setminus F_\alpha) \times \{1\})$, where $x \in U$, $U \subseteq P$ is open and $\alpha \in B$. Then $P(B)$ is a regular Lindelöf space and $P \times \{0\}$ has original topology. If $B_1 \supseteq B_2$, then $P(B_2)$ is a closed subsed of $P(B_1)$. For $B_1 \supseteq B_2$ we may define a mapping $\mu^{B_1}_{B_2} : P(B_1) \to P(B_2)$ as follows: $\mu^{B_1}_{B_2}(x) = x$ if $x \in P(B_2)$ and $\mu^{B_1}_{B_2}(x) = \pi_p(x)$ if $x \in P(B_1) \setminus P(B_2)$. A mapping $\mu^{B_1}_{B_2}$ is a retraction and, consequently, a quotient mapping. Let us show that $\mu^{B_1}_{B_2}$ is closed (and hence perfect). First of all, $(\mu^{B_1}_{B_2})^{-1}\mu^{B_1}_{B_2}(U(\alpha)) = U(\alpha)$ and $(\mu^{B_1}_{B_2})^{-1}\mu^{B_1}_{B_2}(V \times \{1\}) = V \times \{1\}$, where $V \subseteq F_\alpha$ and V is open in F_α, $\alpha \in B_2$ and the sets $U(\alpha)$, $V \times \{1\}$ form a base for $P(B_2)$. Let $p \in F_\alpha$, $\alpha \in B_1 \setminus B_2$. Then any neighborhood of a set $p \times \{0, 1\}$ contains a neighborhood of the form $U(\alpha) \cup (V \times \{1\})$, where $p \in U$, $U \subseteq F_\alpha$ and a set V is open in F_α. Let W be an open set of P and $W \cap F_\alpha = V$. If $U' = U \cap W$, then $U'(\alpha) \cup (V \times \{1\})$ is an open set of $P(B_1)$ and $(\mu^{B_1}_{B_2})^{-1}\mu^{B_1}_{B_2}$ $(U'(\alpha) \cup (V \times \{1\}) = U'(\alpha) \cup (V \times \{1\})$. Consequently, $\mu^{B_1}_{B_2}$ is a perfect mapping.

So we have constructed the spectrum $S_1 = \{P(B_1) : \mu^{B_1}_{B_2},\ B_1 \supseteq B_2,\ \exp_{\aleph_0} A\}$. Since all the mappings $\mu^{B_1}_{B_2}$ are perfect, $A \supseteq B_1 \supseteq B_2$, it easy follows that $\lim S_1 = P(A)$ and S_1 is sigma-spectrum.

The result space M we shall construct like $P(A)$, choosing a suitable $\gamma = \{F_\alpha : \alpha \in A\}$.

Let $\mathcal{F} = \{\varphi : \varphi : P \to X$ is a Baire isomorphism and $X \subseteq \mathrm{R}^{\aleph_0}\}$ and let $\Gamma = \{\gamma : \gamma$ is pair-wise disjoint collection of closed subsets of P such that 1) for every $F \in \gamma$, F is homeomorphic to P and 2) $|\gamma| = c\}$. Then $|\mathcal{F}| = c$ [2]. For every $\gamma \in \Gamma$, let $\mathcal{F}(\gamma) = \{\varphi \in \mathcal{F}$: there exists a $\gamma(\varphi) \in \Gamma$ such that 1) $\gamma(\varphi)$ is a refinement of γ, 2) every $F \in \gamma$ contains at most own set of $\gamma(\varphi)$, 3) $\gamma(\varphi)$ can be represented as a disjoint union of countable families $\gamma_\alpha(\varphi)$, $\alpha < c$, such that $\varphi(\cup\gamma_\alpha(\varphi))$ is a σ-compact set for every $\alpha < c$.

V) There exists $\gamma \in \Gamma$, such that for every $\varphi \in \mathcal{F}(\gamma)$ $|\{F : F \in \gamma), \varphi(F)$ is not σ-compact set and $\varphi(F) \cap \varphi(F') = \phi$ for every $F' \in \gamma \setminus \{F\}\}| = c$.

Proof : Take an arbitrary $\delta \in \Gamma$. If $\mathcal{F}(\delta) = \phi$ then let $\gamma = \delta$. Assume that $\mathcal{F}(\delta) \neq \phi$. Due to $|\mathcal{F}(\delta)| \leqslant c$ we construct by transfinite induction the families

μ_α, $\alpha < c$, such that 1) for every $\alpha < c$ there exist $\varphi \in \mathcal{F}(\delta)$ and $\beta < c$ such that $\mu_\alpha = \delta_\beta(\varphi)$ 2) $\delta(\cup\mu_\alpha) \cap \delta(\cup\mu_\beta) = \phi$ if $\alpha \neq \beta$, 3) for every $\varphi \in \mathcal{F}(\delta)|T_\varphi = \{\alpha : \mu_\alpha = \delta_\beta(\varphi)$ for some $\beta < c\}| = c$. Let $\alpha \in T_\varphi$. Then the set $\varphi(\cup\mu_\alpha)$ is σ-compact, i.e. $\varphi(\cup\mu_\alpha) = \bigcup_{j=1}^{\infty} B_j$, where B_j is a compact set, $j \in N$. Let $\Phi_\alpha \in \mu_\alpha$. Since Φ_α is not σ-compact set and $\Phi_\alpha = \bigcup_{j=1}^{\infty}(\Phi_\alpha \cap \varphi^{-1}(B_j))$, then for some $\gamma_0 \in N$ there is not a σ-compact set T such that $\Phi_\alpha \cap \varphi^{-1}(B_{j_0}) \subseteq T \subseteq \Phi_\alpha$. The set $\Phi_\alpha \cap \varphi^{-1}(B_{j_0})$ is Borel, consequently, according to II) there exists a closed set Φ'_α which is homeomorphic to P and $\Phi'_\alpha \subseteq \Phi_\alpha \cap \varphi^{-1}(B_{j_0})$. If a set $\varphi(\Phi'_\alpha)$ is not σ-compact, then let $\Phi''_\alpha = \Phi'_\alpha$. Suppose $\varphi(\Phi'_\alpha)$ is σ- compact set, then, according to III) there is a closed set $\Phi \subseteq \Phi'_\alpha$, homeomorphic to P and such that $\varphi(\Phi)$ is not F_σ in $\varphi(\Phi'_\alpha)$, hence, it is not σ-compact. Let $\Phi''_\alpha = \Phi$. The family $\gamma = \{\Phi''_\alpha : \alpha < c\}$ is found. So the V) is proved. ■

Let $M = P(A)$, where $\gamma = \{F_\alpha : \alpha \in A\}$ has all the properties if V) The space M is Čech-complete and Lindelöf, since $\mu^A_\phi : M \to P \times \{0\}$ is a perfect mapping from M onto P.

The space M is not Baire isomorphic to any compact Hausdorff space. Suppose on the contrary, that M is Baire isomorphic to some sompact Hausdorff space Y. Then, according to [8], $w(M) = w(Y)$. Hence, we can assume that $Y \subseteq I^A$. Let $Y_B = \pi_B(Y) \subseteq I^B$, where $B \subseteq A$, and if $A \supseteq B_1 \supseteq B_2$ then $\pi^{B_1}_{B_2} = \pi_{B_2}|Y_{B_1} : Y_{B_1} \to Y_{B_2}$. Clearly Y is the limit space of the sigma-spectrum $S_2 = \{Y_{B_1}, \pi^{B_1}_{B_2}, B_1 \supseteq B_2, \exp_{\aleph_0} A\}$. According to IV) there is $\mathcal{U}' \subseteq \exp_{\aleph_0} A$, closed and cofinal subset of $\exp_{\aleph_0} A$ and Baire isomorphisms $f_B : P(B) \to Y_B$, $B \in \mathcal{U}'$, so that $f_{B_2} \circ \mu^{B_1}_{B_2} = \pi^{B_1}_{B_2} \circ f_{B_1}$ for all $B_1 \supseteq B_2, B_1, B_2 \in \mathcal{U}'$. Without loss of generality we can assume that $\mathcal{U} \ni B_0$, where B_0 is minimal.

VI) There is a family of pairs $\{(B_\alpha, B'_\alpha) : 1 \leqslant \alpha < c\}$ so that $B_\alpha \supsetneq B'_\alpha$, $B_\alpha, B'_\alpha \in \mathcal{U}'$ and $(B_\alpha \setminus B'_\alpha) \cap (B_\beta \setminus B'_\beta) = \phi$ if $\alpha \neq \beta$.

We shall construct the family by induction. Let (B_1, B'_1) be an arbitrary pair, where $B_1 \supsetneq B'_1$, $B_1, B'_1 \in \mathcal{U}'$ (this pair does exist, because of cofinality of $\mathcal{U}'$). Suppose there are pairs $(B_\alpha, B'_\alpha) : 1 \leqslant \alpha < \beta < c$. Let us set $\mathcal{B}_\beta = \{B_\alpha : \alpha < \beta\} \cup \{B'_\alpha : \alpha < \beta\}$. Then $|\mathcal{B}_\beta| < c$ and since $\mathcal{U}'$ is cofinal there is $\mathcal{B}'_\beta \supseteq \mathcal{B}_\beta$, such that $\mathcal{B}'_\beta \subseteq \mathcal{U}', |\mathcal{B}'_\beta| = |\mathcal{B}_\beta|$ and $\mathcal{B}'_\beta$ is directed by inclusion. Let $x_\beta \in A \setminus \cup\mathcal{B}'_\beta$ and $T_0 \in \mathcal{U}'$, $x_\beta \in T_0$. Let us construct by induction the sets $T_i, T_i \subseteq T_{i+1}$, $T_i \in \mathcal{U}'$, $i \geqslant 0$, $T_i \cap (\cup\mathcal{B}'_\beta) = \{y_{im}\}_{m=1}^{\infty}$ and $S_i, S_i \subseteq S_{i+1}$, $S_i \in \mathcal{B}_\beta$, $i \geqslant 1$, $i \in N$ and so that $T_{i+1} \supseteq S_i$, $S_{i+1} \supseteq \{y_{km} : k, m \leqslant i\}$. Then both $T = \bigcup_{i=0}^{\infty} T_i$ and $S = \bigcup_{i=1}^{\infty} S_i$ belong to $\mathcal{U}'$, since $\mathcal{U}'$ is closed, $T \cap (\cup\mathcal{B}'_\beta) = S$, $x_\beta \in T \setminus S \neq \phi$, $T \supseteq S$. In addition $(T \setminus S) \cap B_\alpha = \phi$ for every $\alpha < \beta$. So a pair $B_\beta = T$, $B'_\beta = S$ is found. The VI) is proved.

We note, that $f_{B_0}|P \times \{0\} : P \times \{0\} \to f_{B_0}(P \times \{0\})$ is Baire isomorphism and $f_{B_0}|P \times \{0\} \in \mathcal{F}$. Let $f_{B_0}|P \times \{0\} = \varphi_0$. We shall show that $\varphi_0 \in \mathcal{F}(\gamma)$.

The following diagram is comutative

$$\begin{array}{ccc} P(B_\alpha) & \xrightarrow{f_{b_\alpha}} & Y_{B_\alpha} \\ \mu^{B_\alpha}_{B'_\alpha}\big\downarrow & & \big\downarrow \pi^{B_\alpha}_{B'_\alpha} \\ P(B'_\alpha) & \xrightarrow{f_{B'_\alpha}} & Y_{B'_\alpha} \\ \mu^{B'_\alpha}_{B_0}\big\downarrow & & \big\downarrow \pi^{B'_\alpha}_{B_0} \\ P(B_0) & \xrightarrow{f_{B_0}} & Y_{B_0} \end{array}$$

Since $B_\alpha \supseteq B'_\alpha \supseteq B_0$ and diagram is comutative, then $f_{B_\alpha}(\mathcal{Z}(\mu^{B_\alpha}_{B'_\alpha})) = \mathcal{Z}(\pi^{B_\alpha}_{B'_\alpha})$ and $f_{B_0}(\mu^{B_\alpha}_{B_0}(\mathcal{Z}(\mu^{B_\alpha}_{B'_\alpha}))) = \pi^{B_\alpha}_{B_0}(\mathcal{Z}(\pi^{B_\alpha}_{B'_\alpha}))$. According to I) $\pi^{B_\alpha}_{B_0}(\mathcal{Z}(\pi^{B_\alpha}_{B'_\alpha}))$ is σ- compact and $\mu^{B_\alpha}_{B_0}(\mathcal{Z}(\mu^{B_\alpha}_{B'_\alpha})) = \cup\{F_\alpha \times 0 : \alpha \in B_\alpha \setminus B'_\alpha\}$. Hence $\varphi_0(\cup\{F_\alpha \times \{0\} : \alpha \in B_\alpha \setminus B'_\alpha\})$ is σ- compact for every $1 \leqslant \alpha < c$. Since $|B_\alpha| \leqslant \aleph_0$ and the family $\{B_\alpha \setminus B'_\alpha\}_{1\leqslant\alpha<c}$ is pair-wise disjoint, we have $\varphi_0 \in \mathcal{F}(\gamma)$. Since $\varphi_0 \in \mathcal{F}(\gamma)$, then $|\mathcal{A}(\varphi_0) = \{\alpha : \varphi_0(F_\alpha)$ is not σ - compact and $\overline{\varphi_0(F_\alpha)} \cap \varphi_0(F_\beta) = \phi$ for every $\alpha \neq \beta$, $F_\beta \in \gamma\}| = c$. Let us choose $\alpha_0 \in \mathcal{A}(\varphi_0) \setminus B_0$. According to cofinality of $\mathcal{U}'$ there is $\tilde{B} \in \mathcal{U}'$, such that $\tilde{B} \supseteq B_0 \cup \{\alpha_0\}$. Then $\varphi_0(\cup\{F_\alpha : \alpha \in \tilde{B} \setminus B_0\})$ is σ -compact. But since $\overline{\varphi_0(F_{\alpha_0})} \cap \varphi_0(\cup\{F_\alpha : \alpha \in \tilde{B} \setminus B_0\}) = \varphi_0(F_{\alpha_0})$, then $\varphi_0(F_{\alpha_0})$ is closed in σ -compact set $\varphi_0(\cup\{F_\alpha : \alpha \in \tilde{B} \setminus B_0\})$. Thus $\varphi_0(F_{\alpha_0})$ is σ -compact. But this contradicts to $\alpha_0 \in \mathcal{A}(\varphi_0)$ and so we complete the proof. ∎

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