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On plane topologies with high sequential order

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Dedicated to the memory of Zdeněk Frolík

Abstract. We prove that the sequential order of the radiolar topology for the plane as well as of a number of related topologies is ω_1.

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There are two distinguished "strange" sequential topologies for the plane, viz. the cross topology and the radiolar topology. Both topologies are homogenous, finer than the usual metric topology and fail to be Fréchet. Observe that while the former is weakly first countable (cf. [**NY**]) and fails to be isotropic (i.e. rotational symmetric), the latter fails to be weakly first countable but it is isotropic. As proved by G.H.Greco ([**GR**]), the sequential order of the cross topology is ω_1. We prove that the sequential order of the radiolar topology, the coarser of the two, is ω_1 as well. Combining the two results, we prove that the sequential order of all sequential topologies for the plane the convergence in which is coarser than the cross convergence and finer than the radiolar convergence is ω_1. As a corollary it follows that the sequential order of all nontrivial radiolar-like topologies in n-dimensional Euclidean spaces, introduced by H.-J.Schmidt (hence of the corresponding Zeeman space-time topologies, cf. [**SC**]), is ω_1 as well.

The sequential order and related ordinal invariants have been introduced and studied by many authors. See, e.g., [**NO**] ,[**AF**], [**KA**], [**RK**], [**NY**], [**GR**], [**FG**], where spaces the sequential order of which is ω_1 and which have various additional properties are described. Note that in [**FG**] the rational torus Q/Z has been equipped with a sequential convergence with unique limits such that it is sequentially compact, $\mathcal{L}$-compatible with the group structure (hence homogenous) and its sequential order is ω_1. To make the paper self-contained, we recall the definitions of the key notions.

Let X be a topological space. For each $A \subset X$, the sequential closure $s\text{-cl}\,A$ of A is the set of all limit points of sequences ranging in A. If $s\text{-cl}\,A = A$, then A is said to be sequentially closed and if each sequentially closed set is closed, then X is said to be a sequential space. For all ordinal numbers α, define by induction

$$0\text{-cl}\,A = A$$
$$\alpha\text{-cl}\,A = s\text{-cl}(\beta\text{-cl}\,A) \quad \text{if } \alpha = \beta + 1,$$
$$\alpha\text{-cl}\,A = s\text{-cl}\Big(\bigcup_{\beta<\alpha} \beta\text{-cl}\,A\Big) \quad \text{if } \alpha \text{ is a limit ordinal.}$$

It is known that ω_1-cl A is the smallest sequentially closed set containing A. The smallest ordinal number $\alpha, \alpha \leq \omega_1$, such that α-cl A is sequentially closed for all subsets A of X is said to be the sequential order of X (or of the underlying topology of X). For a limit ordinal α, the closure operator α-cl is sometimes defined by α-cl $A = \bigcup_{\beta<\alpha} \beta$-cl A; clearly, the fact that a space X has the sequential order ω_1 does not depend on the particular definition of α-cl.

For radiolar topologies in n–dimensional Euclidean spaces, s-cl is closely related to the algebraic hull of a set, a notion studied in real vector spaces. Interesting results concerning iterations of algebraic hulls of convex sets have been obtained by O.M.Nikodym and V.L.Klee,Jr., see [**KO**]; e.g., for $\alpha < \omega_1$, every infinite dimensional space contains a convex set having α different successive iterations of the algebraic hull. On the other hand, H.-J.Schmidt showed (Remark on page 209 in [**SC**]) that in the n–dimensional Euclidean space the radiolar sequential order of a finite union of locally compact convex sets can be at least n.

For the easier reference, denote by P the plane equipped with the usual Cartesian coordinates. For $p \in P$, by a p-radiolar we understand a subset of P such that each straight line through p intersects the subset in a line segment open in the relative metric topology and containing p. Recall that the radiolar topology σ_r for P is the sequential topology determined by the radiolar convergence; the latter means that a sequence $\langle p_n \rangle$ converges in σ_r to p iff each p-radiolar contains p_n for all but finitely many n ([**FL**]).

For $r > 0$, denote $K(r) = \{(x,y) \in P; 0 \leq x \leq r, 0 \leq y \leq r - \sqrt{r^2 - x^2}\}$. For $(a,b) \in P$ and $\varphi \in [0; 2\pi)$, denote by $K(r,(a,b),\varphi)$ the image of $K(r)$ under the translation of the plane translating the origin $(0,0)$ to (a,b) and the subsequent counterclockwise φ-rotation around (a,b). Sets of the form $K(r,(a,b),\varphi)$ will be called admissible and the image of $\{(x,0); 0 < x < r\}$ will be called the base of $K(r,(a,b),\varphi)$.

Theorem. *The sequential order of the radiolar topology for P is ω_1.*

PROOF : Denote by α-cl the α-th iteration of the sequential closure in σ_r. Clearly, it suffices to prove that for each ordinal number α, $\alpha < \omega_1$, the following proposition $P(\alpha)$ holds true:

$P(\alpha)$ For each $p \in P$ and for each admissible set $K(r,p,\varphi)$ there is a set $A \subset K(r,p,\varphi)$ such that α-cl $A \subset K(r,p,\varphi)\setminus\{p\}$ and $(\alpha+1)$-cl $A = \{p\} \bigcup \alpha$-cl A.

Let $p \in P$ and let $K(r,p,\varphi)$ be an admissible set. Then there is a one-to-one sequence $\langle p_n \rangle$ of points of the base of $K(r,p,\varphi)$ converging in P to p. Further, there is a sequence $\langle K(r_n,p_n,\varphi_n)\rangle$ of admissible sets such that

(i) $K(r_n,p_n,\varphi_n) \bigcap K(r_m,p_m,\varphi_m) = \emptyset$ for $n \neq m$,
(ii) $K(r_n,p_n,\varphi_n) \setminus \{p_n\}$ is contained in the interior of $K(r,p,\varphi)$, $n = 1,2,\ldots$.

We proceed by transfinite induction.

1. Let $\alpha = 0$. It suffices to put $A = \{p_n; n = 1,2,\ldots\}$. Now, let $\alpha > 0$ and assume that $P(\beta)$ holds for all ordinal numbers β, $\beta < \alpha$.

2. Let $\alpha = \beta + 1$. Now, according to the inductive assumption, there are sets $A_n \subset K(r_n,p_n,\varphi_n)$, $n = 1,2,\ldots$, such that β-cl $A_n \subset K(r_n,p_n,\varphi_n) \setminus \{p_n\}$ and

$(\beta+1)\text{-cl}\, A_n = \{p_n\} \bigcup \beta\text{-cl}\, A_n$. Put $A = \bigcup_{n=1}^{\infty} A_n$. Clearly, for each sequence $\langle q_n \rangle$, $q_n \in K(r_n, p_n, \varphi_n) \setminus \{p_n\}$, no sequence of $\langle q_n \rangle$ converges in P. Hence $\alpha\text{-cl}\, A = (\beta+1)\text{-cl}\, A \subset K(r,p,\varphi) \setminus \{p\}$ and $(\alpha+1)\text{-cl}\, A = \{p\} \bigcup \alpha\text{-cl}\, A$. Thus $P(\alpha)$ holds.

3. Let α be a limit ordinal number. Let $\langle \alpha_n \rangle$ be an increasing sequence of ordinal numbers converging to α. According to the inductive assumption, there are sets $A_n \subset K(r_n, p_n, \varphi_n)$, $n = 1, 2, \ldots$, such that $\alpha_n\text{-cl}\, A_n \subset K(r_n, p_n, \varphi_n) \setminus \{p_n\}$ and $(\alpha_n + 1)\text{-cl}\, A_n = \{p_n\} \bigcup \alpha_n\text{-cl}\, A_n$. Again, $A = \bigcup_{n=1}^{\infty} A_n$ satisfies the condition in $P(\alpha)$. This completes the proof. ∎

For $p = (a,b)$, by a p-<u>cross</u> we understand a subset of the plane of the form $\{(a,y); |y-b| < r\} \bigcup \{(x,b); |x-a| < r\}$, where r is a positive real number, and by a p-<u>disc</u> we understand any closed circle with center (x,y) such that either $y = b$ and $|x-a| > 0$ is the radius of the circle, or $x = a$ and $|y-b| > 0$ is the radius of the circle. Recall that the <u>cross topology</u> σ_c for the plane is the sequential topology determined by <u>cross convergence</u>. I.e., a sequence $\langle p_n \rangle$ converges to p iff each p-cross contains p_n for all but finitely many n; sequentially open sets with respect to the cross convergence are exactly open sets in the cross topology.

Throughout the rest of the paper, $\alpha\text{-cl}_c$ and $\alpha\text{-cl}_r$ denote the α-th iteration of the sequential closure in σ_c and σ_r, respectively.

Let α be an ordinal number, $\alpha < \omega_1$. Denote by $Q(\alpha)$ the following proposition:

$Q(\alpha)$ For each point p and for each p-disc D there exists a set $A \subset D$ such that
- (i) $\zeta\text{-cl}_c\, A = \zeta\text{-cl}_r\, A \subset D \setminus \{p\}$ for all $\zeta \le \alpha$; and
- (ii) $(\alpha+1)\text{-cl}_c\, A = (\alpha+1)\text{-cl}_r\, A = \{p\} \bigcup \alpha\text{-cl}_r\, A$.

Lemma. *$Q(\alpha)$ holds for all α, $\alpha < \omega_1$.*

PROOF : Given a point p, let D be a p-disc with radius r and center q. Let $K(r,p,\varphi)$ be the admissible set the base of which is a subset of the line segment with p and q as its endpoints. Let $\langle p_n \rangle$ be a one–to–one sequence of points of the base of $K(r,p,\varphi)$ converging in the cross topology to p. Then there are mutually disjoint p_n-discs $D_n \subset K(r,p,\varphi)$. We proceed by transfinite induction.

1. Let $\alpha = 0$. It suffices to put $A = \{p_n; n = 1, 2, \ldots\}$. Now, let $\alpha > 0$ and assume that $Q(\beta)$ holds for all ordinal numbers β, $\beta < \alpha$.

2. Let $\alpha = \beta + 1$. According to the inductive assumption., there are sets $A_n \subset D_n$, $n = 1, 2, \ldots$, such that $\zeta\text{-cl}_c\, A_n = \zeta\text{-cl}_r\, A \subset D_n \setminus \{p_n\}$ for all $\zeta \le \beta$ and $(\beta+1)\text{-cl}_c\, A_n = (\beta+1)\text{-cl}_r\, A_n = \{p_n\} \bigcup \beta\text{-cl}_r\, A_n$. Put $A = \bigcup_{n=1}^{\infty} A_n$. Clearly, for each sequence $\langle q_n \rangle$, $q_n \in D_n \setminus \{p_n\}$, no subsequence of $\langle q_n \rangle$, converges in the radiolar convergence, and hence in the cross convergence. Thus A satisfies the condition stated in $Q(\alpha)$.

3. Let α be a limit ordinal number, $\alpha < \omega_1$, and let $\langle \alpha_n \rangle$ be an increasing sequence of ordinal numbers converging to α. According to the inductive assumption there are sets $A_n \subset D_n$, $n = 1, 2, 3, \ldots$, such that $\zeta\text{-cl}_c\, A_n = \zeta\text{-cl}_r\, A_n \subset D_n \setminus \{p_n\}$ for all $\zeta \le \alpha_n$ and $(\alpha_n+1)\text{-cl}_c\, A_n = (\alpha_n+1)\text{-cl}_r\, A_n = \{p_n\} \bigcup \alpha_n\text{-cl}_r\, A_n$. Again, the set $A = \bigcup_{n=1}^{\infty} A_n$ satisfies the condition stated in $Q(\alpha)$. This completes the proof. ∎

Corollary 1. *Let σ be a sequential topology for the plane the convergence in which is coarser than the cross convergence and finer than the radiolar convergence. Then the sequential order of σ is ω_1.*

Remark 1. Each of the previous constructions can be easily adopted for the rational plane $Q^2 = \{(a, b) \in P; a, b \in Q\}$. Thus the radiolar topology for Q^2 (hence each sequential topology derived from a convergence between the cross convergence and the radiolar convergence) has the sequential order ω_1.

Remark 2. Consider the following modifications of the cross topology. Instead of "orthogonal" crosses we can use either "lean" crosses or "simply curved" crosses (e.g. with one arm shaped like $\{(x, x^3) \in P; |x| < \epsilon\}$ and the other arm shaped like its $\pi/2$-rotation). It is easy to see that the sequential order of the modified cross topologies and also of all "simply curved" radiolar topologies, defined in the obvious way, is ω_1 as well. The radiolar topology for the plane can be further generalized in two directions. First, instead of the plane we consider Euclidean spaces of higher dimensions. Secondly, we fix a set A of straight lines and, in the definition of p-radiolar, we restrict ourselves to parallels (through p) to straight lines in A; the corresponding topology is said to be an A-radiolar topology (cf. [**SC**]).

Corollary 2. *The sequential order of an A-radiolar topology is* ω_1 *whenever A contains at least two straight lines.*

Remark 3. M. Dolcher in [**DO**] proved that the sequential order of the space of all real continuous functions equipped with the pointwise convergence is at least 2. It might be interesting to find out its (exact) sequential order.

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