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Dedicated to the memory of Zdeněk Frolík

Abstract. A topological space (X,σ) is a Skula space if there is a topology τ on X such that σ is the topology on X generated by $\tau \cup \{X - A : A \in \tau\}$. In 1979 Brümmer asked "Which compact Hausdorff spaces are Skula ?" In 1980, Bonnet began exploring the question "Which superatomic Boolean algebras are canonically good ?". The purpose of this paper is to demonstrate that the questions of Brümmer and Bonnet are identical, to give a simple and useful topological answer and to construct an example (obtained independently by Bonnet, Rubin and Si-Kaddour) related to a classic example of Stone.

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Definition 1. A topological space (X,σ) is a Skula space if there is a topology τ on X such that σ is the topology on X generated by $\tau \cup \{X - A : A \in \tau\}$.

In 1979 Brümmer asked "Which compact Hausdorff spaces are Skula ?" (the reader should reflect on the fact that the Sorgenfrey line *is* Skula).

Definition 2. A Boolean algebra $\mathcal{B}$ is superatomic if every quotient of $\mathcal{B}$ with at least two elements is atomic (that is, has an atom, an element which is greater than no element other than O).

Definition 3. Let $\mathcal{B}$ be a superatomic Boolean algebra. Define, by induction, a sequence $\{I_\alpha, D_\alpha, \tau_\alpha, AT_\alpha : \alpha \in \kappa\}$ by:

- $I_0 = 0$
- $I_{\alpha+1}$ is the ideal generated by $I_\alpha \cup AT_\alpha$
- $I_\alpha = \cup\{I_\beta : \beta < \alpha\}$ when α is limit ordinal
- $D_\alpha = \mathcal{B}/I_\alpha$
- π_α is the canonical homomorphism of $\mathcal{B}$ onto D_α
- $AT_\alpha = (\pi_\alpha^{-1})$(the set of atoms of D_α)

In this construction, each D_α has an atom and so the procedure must eventually lead to a finite D_α. Call the least such α, the rank of $\mathcal{B}$. We say that $G \subset \mathcal{B}$ is a set of representatives for $\mathcal{B}$ if $G = \cup\{G_\alpha : \alpha \leq rank(\mathcal{B})\}$ and, for each $\alpha, \pi_\alpha \restriction G_\alpha$ is a one-to-one function onto the set of atoms of D_α. We say that a Boolean algebra $\mathcal{B}$ is canonically good if it has a set of representatives G which generates $\mathcal{B}$ as a Boolean algebra and which generates a well-founded sublattice of $\mathcal{B}$ as lattice.

In 1980, Bonnet began exploring the question "Which Boolean algebras are canonically good ?". The purpose of this paper is to demonstrate that the questions of

Brümmer and Bonnet are identical, to give a simple and useful topological answer and to construct an example (obtained independently by Bonnet, Rubin and Si-Kaddour) related to a classic example of Stone. We begin with a topological solution to Brümmer's problem:

Theorem 1. *Let (X,σ) be a T_o topological space. If the space (X,σ) is Skula then there is an function which assigns to each $x \in X$ a clopen set $U(x) \subset X$ such that $x \in U(x)$ and*

$$(\forall x, y \in X) x \notin U(y) \vee y \notin U(x) \tag{1}$$

$$x \in U(y) \Rightarrow U(x) \subset U(y) \tag{2}$$

Furthermore, the converse is true for any compact Hausdorff space.

PROOF : Suppose that (X,σ) is Skula. This means that there is a topology τ on X such that σ is the topology an X generated by $\tau \cup \{X - A : A \in \tau\}$. For each $x \in X$, let $U(x)$ be the closure, in τ,of the set $\{x\}$. Clearly each $U(x)$ is clopen and $(\forall x, y \in X) x \in U(y) \Rightarrow U(x) \subset U(y)$.

We shall show that $\{U(x) : x \in X\}$ satisfies (1) and (2). Suppose that $\{x,y\} \subset U(x) \cap U(y)$ and thus that x is in the closure of $\{y\}$ in τ and that y is in the closure of$\{x\}$ in τ. Thus τ does not distinguish between x and y and so since σ is generated by the elements of τ and their complements, σ does not distinguish between x and y either. This contradicts the fact σ is T_0.

Conversely, suppose that (X,σ) has a clopen family $\{U(x) : x \in X\}$ satisfying (1) and (2). Let τ be the topology on X whose subbase is $\{X - U(x) : x \in X\}$. Letρ be the topology on X generated by the elements of τ and their complements.

First we need to observe that $(\forall U \in \tau)(\forall x \notin U) U(x) \cap U = \emptyset$. This is true for basic open U in τ since any such U is of the form $\bigcap\{X - U(x_i) : i < n\}$ and so $x \notin U$ implies $x \in U(x_i)$ for some $i < n$. The assumption $x \in U(y) \Rightarrow U(x) \subset U(y)$ now implies that $U(x) \subset U(x_i)$ and thus that $U(x) \cap U = \emptyset$. The observation is true for all open sets U since any such U is the union of basic open subsets, any $x \notin U$ is not an element of any of the basic open subsets and so $U(x)$ is disjoint from each of the basic open subsets and so disjoint from U.

Now we shall show that $\rho = \sigma$.

Suppose that $V \in \rho$ and $x \in V$. Let W be a ρ basic open set such that $x \in W \subset V$. By definition of $\rho, W = W_1 \cap W_2$ where W_1 is open in τ end W_2 is closed in τ. Now by the observation above, since $x \in W_2, U(x) \subset W_2$. Thus there is ρ basic open set $W^* = W_1 \cap U(x)$ such that $x \in W^* \subset W$. We can assume, without loss of generality, that W_1 is τ basic open and so express $W_1 = \bigcap\{X - U(x_i) : i < n\}$. Now W^* has been expressed as the finite intersection of σ clopen sets and so we have shown that x is in the σ interior of V. Since x an arbitrary element of V, we have shown that $V \in \sigma$.

We have shown that $\rho \subset \sigma$. Meanwhile ρ is a Hausdorff topology since each $U(x)$ is ρ clopen and $\{U(x) : x \in X\}$ satisfies (1). Thus, since σ is a compact Hausdorff topology, it has no smaller Hausdorff subtopologies and thus $\sigma = \rho$ and so (X,σ) is Skula and the proof is complete. ∎

Theorem 2 (Brümmer, Kunzi, Fletcher [2]). *Any compact Hausdorff Skula space X is scattered.*

We can use lemma 1 to prove this theorem or we can work directly with the topology τ which generates σ. We shall do both.
PROOF (First): Suppose $\{U(x) : x \in X\}$ has been defined as in theorem 1. Define a relation $\triangleleft$ by letting $x \triangleleft y \Leftrightarrow x \in U(y)$. This relation is antisymmetric by condition 1 and is transitive by condition 2. This means that $\triangleleft$ is a partial order. We shall show that $\triangleleft$ is well-founded. Suppose that $\{x_n : n \in \omega\}$ were a decreasing sequence of points. Let $X^* = \{x \in X : (\forall n \in \omega) x \triangleleft x_n\}$ a decreasing sequence of closed sets with empty intersection and that is a contradiction. ∎

PROOF (Second): We need some simple but useful lemmas:

Lemma 1. *If (X, σ) is a Skula space and $Y \subset X$ then $(Y, \sigma \restriction Y)$ is a Skula space.*

PROOF : Suppose τ is a topology on X whose open sets together with the closed sets generate σ. The open sets together with the closed sets of $(Y, \tau \restriction Y)$ generate $\sigma \restriction Y$. ∎

In the next lemma $\mathbf{S}$ denotes the Sierpiński space $\{0, 1\}$ with isolated point 1.

Lemma 2. *If $X \subset \mathbf{S}^\kappa$ and X generates (by its open sets and its closed sets) a space without isolated points, then X is not closed in the sense of 2^κ..*

PROOF : Suppose X is closed in the sense of 2^κ. Take a maximal $A \subset k$ such that $(\exists x \in X) : x^{-1}(0) = A$. It exist because X is closed. Now $|\bigcap \{\{x \in X : x(a) = 1\} : a \in A\}| = 1$ and so there is a closed point x in X. The space generated by X has x as an isolated point. ∎

Lemma 3. *If $X \subset \mathbf{S}^\kappa$ and X generates a compact space, then X is closed in the sense of 2^κ.*

PROOF : Suppose not, that $r \in 2^\kappa$ is a limit point but that $r \notin X$. The family of closed sets $\{[r \restriction F] : F \in [\kappa]^{<\omega}\}$ is centered in the generated topology but has empty intersection with X. The lemma is proved. ∎

Take the space which generates it and embed it into a power of the Sierpin'ski space. Using the Cantor-Bendixson theorem, X contains a maximal dense-in-itself subset Y which is also a compact Hausdorff space and, by lemma 1, Skula. Lemma 2 and lemma 3 imply that Y must have an isolated point which means that $Y = \emptyset$ and so X is scattered. ∎

Some simple consequences of theorem 1 are:

Corollary 1. *Any compact Hausdorff space of scattered height 3 is Skula.*

PROOF : Apply Hausdorff to separate the finitely many points at level 3 by means of clopen sets. Each point at level 2 lies, without loss of generality, in one of the clopen sets. Choose a clopen neighborhood of that point which contains only one element at level 2 and which lies inside that clopen set. Choose a singleton clopen neighborhood foe each point at level 1. ∎

Corollary 2. *Any hereditarily paracompact scattered compact space is Skula.*

PROOF : Let γ be the minimal scattered height of a counterexample X. If γ is a limit ordinal then X is the free union hereditarily scattered paracompact spaces of smaller height which are thus Skula by hypothesis and so admit a clopen assignment as in theorem 1. The union of these assignments works.

If γ is a successor ordinal, then find a clopen partition of the space, each element of which intersects the top level in exactly one point. Each element of the clopen partition minus its top point is a hereditarily paracompact scattered space of smaller scattered height and so has a clopen assignment as in theorem 1. The union of these clopen assignments (using the partition elements for the points at the top level) satisfies theorem 1.

We now describe the example (obtained independently by Bonnet, Rubin and Si-Kaddour) which is inspired by an example of Stone (counterexample 3 in [3]). The question of the existence of compact Hausdorff scattered space which is not Skula was first asked by Guillaume Brümmer in 1979. ■

Example 1. There is a compact Hausdorff space X scattered height 4 which is not Skula.

PROOF : Start with $(\omega+1)\times\omega$. We shall add a closed discrete set to this space so that the resulting space is locally compact and Hausdorff. List all functions $f:\omega\to\omega$ as $\{f_\alpha : \alpha\in 2^\omega\}$. List a family of almost disjoint subsets of ω by $\{A_\alpha : \alpha\in 2^\omega\}$. Let $B_\alpha = (\{\omega\}\times A_\alpha)\cup\{(i,n) : i > f_\alpha(n), n\in A_\alpha\}\subset(\omega+1)\times\omega$ for each $\alpha\in 2^\omega$. Add the closed discrete set 2^ω to the space $(\omega+1)\times\omega$ by defining a neighborhood of α to be of the form $\{\alpha\}\cup(B_\alpha\cap(\omega+1)\times(\omega-n))$ for $n\in\omega$. The resulting space is locally compact and Hausdorff. Let X be the Alexandroff compactification. Suppose that this space were Skula, that is, that there were a clopen family $\{U(x) : x\in X\}$ satisfying (1) and (2). Let $f:\omega\to\omega$ be defined so that $(\forall i > f(n))(i,n)\in U((\omega,n))$. Now let $f_\alpha > f$ be any function. Clearly $U(\alpha)\supset^*(\{\omega\}\times A_\alpha)$. Thus if $A=\{n\in A_\alpha : (\omega,n)\in U(\alpha)\}$ then $(\forall n\in A)U((\omega,n))\subset U(\alpha)$. Now the set $\{(i,n)\in\omega\times\omega : i > f(n), n\in A_\alpha\}$ has only two possible cluster points X outside $(\omega+1)\times\omega$ and these are α and the Alexandroff infinity ∞. Since α has a neighborhood whose complement in this set is non-compact (since $f_\alpha \gtrsim f$), we deduce that ∞ is indeed a cluster point of this set and so, since U_α is closed, $\infty\in U_\alpha$. This is true, for any α such that $f_\alpha > f$ and there are infinitely many of these. Since neighborhoods of ∞ contain all but finitely many α and $U(\infty)$ is an open neighborhood of ∞ there is an α such that $\alpha\in U(\infty)$. This contradicts (1). ■

To draw the equivalence between the two questions need a simple lemma from [1].

Lemma 4 (Bonnet, Rubin, Si-Kaddour (Lemma 2.4.8)). *Let $\mathcal{B}$ be a Boolean algebra and let X be its Stone space. If $\mathcal{B}$ is a canonically good algebra, then there is an function which assigns to each $x\in X$ a clopen set $U(x)\subset X$ such that condition 1 and 2 in theorem 1 hold true. Conversely, if there is such a function and X is scattered then $\mathcal{B}$ is canonically good.*

A combination of theorem 1 and this lemma yields the main equivalence:

Theorem 3. *The clopen algebra of a compact Hausdorff Skula space is a canonically good Boolean algebra. Conversely, the Stone space of a canonically good Boolean algebra is a compact Hausdorff Skula space.*

This theorem allows us to recognize some of the results of [1] as equivalent to our results above. Recall that if a Boolean algebra is superatomic then its Stone space is scattered and conversely the clopen algebra of a compact Hausdorff scattered space is superatomic. Their theorem 3.2.3 is thus our theorem 2 in the language of Boolean algebras. Similarly, their lemma 4.7.2 is thus our (or rather Stone's) example 1. We thank Guillaume Brümmer and Petr Simon for bringing this problem to our attention.

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