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On minimal dynamical systems on Boolean algebras

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Dedicated to the memory of Zdeněk Frolík

Abstract. Let B be an infinite complete Boolean algebra which admits a countable family H of automorphisms such that the dynamical system (B, H) is minimal. Then B is isomorphic to the completion of the free Boolean algebra of size $\pi(B)$.

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Let B be a Boolean algebra (BA for short) and H a group of automorphisms of B. The pair (B, H) is called a minimal dynamical system provided that for every $u \in B^+$ there exist $h_1, \dots, h_n \in H$ such that $h_1(u) \vee \dots \vee h_n(u) = 1$. By the Stone duality, this means that the Stone space of B together with homeomorphisms dual to elements of H forms a minimal system in the sense of topological dynamics. For the notions concerning topological dynamics see [**1**] or [**2**].

Note that if B is an arbitrary BA and H is a group of automorphisms of B, then for every ideal $I \subset B$ which is maximal with respect to the property

$$(*) \qquad h[I] \subseteq I \text{ for every } h \in H,$$

the quotient Boolean algebra $B \upharpoonright I$ together with the group of induced automorphisms forms a minimal dynamical system. Indeed, if $u \in B - I$ then there exist $h_1, \dots, h_n \in H$ such that

$$h_1(u) \vee \dots \vee h_n(u) \in I^c$$

since I is maximal with respect to the condition (*).

A subalgebra A of B is called regular if every subset of A which has the supremum in A has also the supremum in B and they coincide (see e.g. [**3**]). If B is complete, then A is a complete subalgebra of B if it is complete and regular. Dual version of the lemma below is known in topological dynamics (see e.g. [**5**]).

Lemma 1. *Assume H is a group of automorphisms of a* BA *B and (B, H) is a minimal dynamical system. If A is a subalgebra of B such that $h[A] = A$ for any $h \in H$, then A is a regular subalgebra of B.*

PROOF : It suffices to show that for every $X \subset A$ such that $\sup_A X = 1$, also $\sup_B X = 1$. Suppose that there exists some $u \in B^+$ such that $u \wedge x = 0$ for all $x \in X$. By the minimality of (B, H), there exist $h_1, \dots, h_n \in H$ such that

$h_1(u) \vee \cdots \vee h_n(u) = 1$. Clearly $h_i(u) \wedge h_i(x) = 0$ for every $x \in X$ and $i \leq n$. Thus for arbitrary $x_1, \ldots, x_n \in X$ we get

$$\begin{aligned} 0 &= (h_1(u) \wedge h_1(x_1)) \vee \cdots \vee (h_n(u) \wedge h_n(x_n)) \geq \\ &\geq (h_1(u) \vee \cdots \vee h_n(u)) \wedge h_1(x_1) \wedge \cdots \wedge h_n(x_n) = \\ &= 1 \wedge h_1(x_1) \wedge \cdots \wedge h_n(x_n) = h_1(x_1) \wedge \cdots \wedge h_n(x_n). \end{aligned}$$

Every element $h \in H$ is an automorphism of A, so $\sup_A\{h(x) : x \in X\} = 1$. Thus there exist $x_1, \ldots, x_n \in X$ such that $h_1(x_1) \wedge \cdots \wedge h_n(x_n) \neq 0$; we get contradiction.

■

As usual $\pi(B)$ stands for the density of a BA B, i.e. $\pi(B)$ is the minimal cardinality of a dense (in the sense of order) subset of B^+. The well known Vladimirov lemma says that if C is a complete subalgebra of a complete BA B such that $\pi(C) < \pi(B \restriction u)$ for every $u \in B^+$, then there exists $x \in B$ with the property that $u \wedge x \neq 0 \neq u - x$ for every $u \in C^+$. The next lemma is a consequence of the Vladimirov Lemma. In the sequel $< X >$ will denote the subalgebra of B generated by $X \subset B$.

Lemma 2. *Assume C is a complete subalgebra of a complete* BA *B such that $\pi(C) < \pi(B \restriction u)$ for every $u \in B^+$. Then for every $w \in B - C$ there exists $x \in B$ such that $w \in < C \cup \{x\} >$ and for every $u \in C^+$, $u \wedge x \neq 0 \neq u - x$.*

PROOF : For every $w \in B - C$ define:

$$\overline{w} = \inf\{c \in C : w \leq c\}$$

$$\underline{w} = \sup\{c \in C : c \leq w\}.$$

Clearly, $\overline{w}, \underline{w} \in C$ and $\underline{w} < w < \overline{w}$. Take $z = \underline{w} \vee -\overline{w}$. If $z = 0$ it suffices to set $x = w$. Otherwise, by the Vladimirov Lemma, there exists $0 < y \leq z$ with the property that for any $a \in C$ such that $a \wedge z \neq 0$, $a \wedge y \neq 0 \neq a - y$. It is easy to check that for $x = y \vee (w \wedge -\underline{w})$ there is true for all $u \in C^+$ that $u \wedge x \neq 0 \neq u - x$. Clearly, $w \in < C \cup \{x\} >$ since $w = \underline{w} \vee (x \wedge \overline{w})$. ■

Lemma 3. *If (B, H) is an infinite minimal dynamical system, then for every $u \in B^+$, $\pi(B) = \pi(B \restriction u)$.*

PROOF : Clearly, $\pi(B \restriction u) \leq \pi(B)$. By the minimality of (B, H), there exist $h_1, \ldots, h_n \in H$ with $h_1(u) \vee \cdots \vee h_n(u) = 1$. Since every h_i is and automorphism, $\pi(B \restriction h_i(u)) = \pi(B \restriction u)$. This completes the proof.

■

A subset X of a BA B is called independent if for every two disjoint finite subsets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_m\}$ of X, $x_1 \wedge \cdots \wedge x_n \wedge -y_1 \wedge \cdots \wedge -y_m \neq 0$. We call an element $x \in B$ independent w.r.t. C, where $C \subset B$, if $c \wedge x \neq 0 \neq c \wedge -x$ for all $c \in C$. Clearly, if a complete Boolean algebra B contains a dense subalgebra generated by an infinite independent subset of B, then B is isomorphic to the completion of the free Boolean algebra of the power equal to the power of the independent set.

Theorem 1. *Let (B, H) be a minimal dynamical system, where B is an infinite* BA *and H is a countable group of automorphisms. Then the completion of B is isomorphic to the completion of the free* BA *of power $\pi(B)$.*

PROOF : Since (B, H) is minimal, B has to be atomless. Thus, we can assume that $\pi(B) = \kappa > \omega$, because every atomless countable BA has free set of generators. We can also assume that B is complete since the completion of B together with the group of automorphisms induced by H forms a minimal dynamical system. Fix a dense subset $\{u_\alpha : \alpha < \kappa\} \subset B$. We shall construct an increasing chain $\{X_\alpha : \alpha < \kappa\}$ of independent subsets of B and an increasing chain $\{C_\alpha : \alpha < \kappa\}$ of the following properties:

(1) $X_0 = 0$ and $C_0 = \{0, 1\}$,
(2) $X_\alpha \subset C_\alpha$ and $< X_\alpha >$ is dense in C_α for $\alpha < \kappa$,
(3) $|X_\alpha| \leq |\alpha| + \omega$ for all $\alpha < \kappa$,
(4) $u_\alpha \in C_\alpha$ for all $\alpha < \kappa$,
(5) $h[C_\alpha] = C_\alpha$ for all $h \in H$ and all $\alpha < \kappa$.

Assume X_β and C_β are defined for all $\beta < \alpha$, where $\alpha < \kappa$ is fixed. Then $X = \bigcup\{X_\beta : \beta < \alpha\}$ is an independent set and $< X >$ is a dense subalgebra of the BA $\overline{C} = \bigcup\{C_\beta : \beta < \alpha\}$. By the condition (5) and Lemma 1, $\overline{C}$ is a regular subalgebra of B. By the Sikovski's Extension Theorem there exists a homomorphism $g : C \longrightarrow B$, where C is the completion of $\overline{C}$, extending the embedding of $\overline{C}$ into B. Clearly, g has to be a monomorphism because $\overline{C}$ is dense in C. Since $\overline{C}$ is regular subalgebra of B, C has this property too. This can be argued using again the density of $\overline{C}$ in C. Thus we can assume that C is a complete subalgebra of B. Clearly, $h[C] = C$ for all $h \in H$ since $h[\overline{C}] = \overline{C}$ for all $h \in H$. Now, if $u_\alpha \in C$, we can set $X_\alpha = X$ and $C_\alpha = C$. So, assume that $u_\alpha \notin C$. Since $< X >$ is dense in C, by the condition (3) we have $\pi(C) < \kappa$. Thus, by Lemma 3 and Lemma 2, there exists $w \in B$ such that $u_\alpha \in < C \cup \{w\} >$ and w is independent w.r.t. C. Clearly, $< C \cup \{w\} >$ is a regular subalgebra of B.

It remains to take care on the validity of (5). To this end, let $\{h_n : n \in \omega\}$ be an enumeration of H. By induction we define an increasing sequence $\{F_n : n < \omega\}$ of finite independent subsets of B such that:

(a) $F_0 = \{w\}$,
(b) for every $n < \omega$ and every $x \in F_n$, x is independent w.r.t. C,
(c) for every $n < \omega$ and every $\kappa \leq 2n + 1$, $h_k(F_{2n}) \subset < C \cup F_{2n+1} >$,
(d) for every $n < \omega$ and every $\kappa \leq 2n + 2$, $h_k^{-1}(F_{2n+1}) \subset < C \cup F_{2n+2} >$.

Such a construction is possible since at every induction step one can apply Lemma 2 finitely many times. Now one can set $X_\alpha = X \cup \bigcup\{F_n : n < \omega\}$. The algebra C_α is defined to be the completion of the BA $\tilde{C} = < C \cup \bigcup\{F_n : n < \omega\} >$. By the construction, $h_k[\tilde{C}] = \tilde{C}$ for every $\kappa < \omega$. By the condition (a), $u_\alpha \in \tilde{C}$. Since F_n's forms the increasing sequence of independent sets, X_α is independent and $< X_\alpha >$ is dense in $\tilde{C}$. Arguing as above one can see that C_α is a regular subalgebra of B. It is easy to see that both X_α and C_α satisfy conditions (1)–(5).

Now assume that X_α's and C_α's are already constructed for all $\alpha < \kappa$. Then $X = \bigcup\{X_\alpha : \alpha < \kappa\}$ is an independent subset of B and, by the conditions (4) an

(2), the completion of $< X >$ is isomorphic to B. The proof is complete.

■

Remark. If B is a homogeneous Boolean algebra and H is the set of all automorphisms of B, then (B, H) is a minimal system. So, the assumption in Theorem 1 concerning the power of H is essential.

Remind that the Gleason space of a compact space X is the Stone space of Boolean algebra of all regular open subsets of X.

Theorem 2. *Let f be a one-to-one mapping of ω into itself which has at most finitely many finite orbits. Then every closed non-empty subset of $\beta\omega - \omega$ which is minimal invariant with respect to the extension βf of f over $\beta\omega$ is homeomorphic to the Gleason space of* $\{0,1\}^c$.

PROOF : Denote by F the closed minimal invariant subset of $\beta\omega$. F is clearly separable by the minimality and $F \subseteq \beta\omega - \omega$. Since $\beta\omega - \omega$ is an F-space, F is extremally disconnected. Equivalently, the Boolean algebra of clopen subsets of F is complete. By Theorem 1, F is homeomorphic to the Gleason space of $\{0,1\}^\kappa$ for some κ.

It remains to show that $\kappa = 2^\omega$. Clearly, $\kappa \leq 2^\omega$ since F is separable. We shall use a familiar Kronecker dynamical system. Let $X = \prod\{T_i : i \in I\}$, where $|I| = 2^\omega$ and for every $i \in I$ T_i is the circle. Define a mapping $g : X \longrightarrow X$ as a product of rotations, i.e. $g = \prod\{g_i : i \in I\}$, where g_i is a rotation of T_i by an angle r_i such that the family $\{r_i : i \in I\}$ is independent over the set of rationals. If is well-known that (X, g) is a minimal dynamical system, moreover $\pi w(X) = 2^\omega$.

In order to show that $\kappa \geq 2^\omega$ we need to find a homomorphism from $(F, \beta f \upharpoonright F)$ onto (X, g). Without loss of generality one can assume that for every $n \in \omega$, the orbit $O(n) = \{f^k(n) : k \in \mathbb{Z}\}$ is infinite.

Pick $\{z_i : i \in J\} \subseteq \omega$ in such a way that $\bigcup\{O(z_i) : i \in J\} = \omega$, $O(z_i) \cap O(z_j) = \emptyset$ for $i \neq j$. Next, choose points x_i for $i \in J$ in X with disjoint orbits in (X, g) and define $\varphi : \omega \longrightarrow X$ by

$$\varphi(f^k(z_i)) = g^k(x_i)$$

for every $i \in J$, $k \in \mathbb{Z}$. Notice that $\varphi \circ f = g \circ \varphi$ holds immediately from the definition.

Clearly the Čech-Stone extension $\beta\varphi$ maps $\beta\omega$ onto X by the minimality of X. The restriction $\beta\varphi \upharpoonright F$ maps F onto X as well, which is again a consequence of the minimality of (X, g). Since we have $\beta\varphi \circ \beta f = g \circ \beta\varphi$, we have found the desired homomorphism of dynamical systems.

Finally, let us show that $\pi w(F) \geq \pi w(X) = 2^\omega$. If is enough to prove that $\beta\varphi[U]$ has a non-empty interior whenever U is non-empty open in F. Since $(F, \beta f \upharpoonright F)$ is minimal, there is an $m \in \omega$ such that

$$U \cup \beta f[U] \cup \cdots \cup \beta f^m[U] = F$$

We obtain:

$$X = \beta\varphi[F] = \beta\varphi[U] \cup \beta\varphi[\beta f[U]] \cup \ldots$$

$$\cup\beta\varphi[\beta f^m[U]] =$$
$$= \beta\varphi[U] \cup g[\beta\varphi[U]] \cup \cdots \cup g^m[\beta\varphi[U]].$$

So $\beta\varphi[U]$ cannot be nowhere dense. ∎

Corollary. *Consider the shift σ on the set $\mathbb{Z}$ of integers, where $\sigma(n) = n + 1$. The extension $\beta\sigma$ over $\beta\mathbb{Z}$ defines a compact dynamical system $(\beta\mathbb{Z}, \beta\sigma)$. Then the phase space of every minimal dynamical subsystem of $(\beta\mathbb{Z}, \beta\sigma)$ is homeomorphic to the Gleason space of $\{0,1\}^c$.*

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