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On normal forms of Laplacian and its iterations in harmonic spaces

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Abstract. We give the normal forms of the successive iterations of the Laplacian for harmonic spaces and characterize the particular classes of 2-stein spaces.

Keywords: Iterations of Laplacian, Harmonic spaces, 2-stein spaces.
Classification: 53C20, 58G99

1. Introduction.

The successive iterations Δ^k of the Laplacian Δ on a Riemannian manifold can be calculated at the center of any normal coordinate system by means of the curvature tensor and its covariant derivatives. In [4], O.Kowalski proved that the corresponding *normal forms* for a symmetric space of rank one are certain partial differential operators with constant coefficients.

Our results are stated as follows. We first generalize Kowalski's theorem above to a harmonic space, i.e., the infinite sequence of the conditions $(P)_k$, $k = 2, 3, \dots,$ holds (See Section 2 for the definitions) if and only if the manifold is harmonic (Theorem 1 below). In [4], O.Kowalski also characterized the Einstein and super-Einstein spaces by means of $(P)_2$ and $(P)_2 - (P)_3$ respectively. By the conditions $(P)_2 - (P)_4$, we characterize the particular classes of 2-stein spaces which should be located between the harmonic and the super-Einstein spaces (Theorem 2). We further prove: (1) a 4-dimensional Riemannian manifold satisfying $(P)_2 - (P)_4$ is locally flat or locally isometric to a symmetric space of rank one (Corollary 1); (2) an n-dimensional 3^*-stein space with $3 \leq n \leq 5$ satisfies $(P)_2 - (P)_4$ (Corollary 2).

In Section 2, we state our results precisely; Theorems 1 and 2. In Section 3, we give the proof of Theorem 1. Section 4 is for preparation of the proof of Theorem 2 and its Corollaries 1–2. In Section 5, we give the proof of Theorem 2 and its corollaries. In the final Section 6, we give the normal forms of Δ^k for harmonic spaces by the recurrence formulae.

2. Statement of results.

Let (M, g) be an n-dimensional connected C^∞ Riemannian manifold with $n \geq 2$ and $B_m(r)$ be the geodesic ball in M at center $m \in M$ with small radius $r > 0$ and let $(U; x^1, x^2, \dots, x^n)$ be a normal coordinate system around m. For a function f of class C^∞ near m, we denote by $\tilde{\Delta}_m$ the local differential operator given by

$$\tilde{\Delta}_m f = \sum_{i=1}^{n} \frac{\partial^2 f}{(\partial x^i)^2}.$$

We would like to express our hearty gratitude to Professors O.Kowalski and L.Vanhecke for their valuable comments.

$\widetilde{\Delta}_m$ is independent of the choice of normal coordinate system around m. Due to [2], for each $k = 1, 2, \ldots$, there is a globally defined differential operator on (M, g) which concides with $\widetilde{\Delta}_m^k f$ at m.

In this note we are concerned with the following condition introduced in [4]:

$(P)_k$ There exist constants $A_{k,1}, A_{k,2}, \ldots, A_{k,k-1}$ depending only on (M, g) such that, for each $m \in M$,

$$(\Delta^k f)(m) = (\widetilde{\Delta}_m^k f)(m) + \sum_{i=1}^{k-1} A_{k,i}(\widetilde{\Delta}_m^i f)(m) \tag{2.1}$$

holds for all analytic functions f at m, where k is a natural number.

In (2.1), the condition $(P)_1$ is understood to hold; $(\Delta f)(m) = (\widetilde{\Delta}_m^1 f)(m) \equiv (\widetilde{\Delta}_m f)(m)$.

We call the space (M, g) *harmonic* if, for each $m \in M$, there exist an $r > 0$ and a function $F : (0, r) \to \mathbf{R}$ such that the function $f(n) = F(d(m, n))$ is harmonic in $B_m(r) \setminus \{m\}$, where d is the distance function defined by the Riemannian metric. It is well known that examples of harmonic spaces are those locally isometric to a Euclidean space and a symmetric space of rank one (cf. [1], [7]).

Our first theorem is the following

Theorem 1. *Let (M, g) be an n-dimensional connected C^ω Riemannian manifold with $n \geq 3$. Then the infinite sequence of the conditions $(P)_k$, $k = 2, 3, \ldots$, holds if and only if (M, g) is a harmonic space.*

We denote by (g_{ij}) and $(R_{ijk\ell})$ the metric tensor and the curvature tensor with respect to the normal frame $(\partial/\partial x^1, \partial/\partial x^2, \ldots, \partial/\partial x^n)$. Throughout we exploit Einstein convention as well as the extended one, i.e., the summation convention for repeated indices. The Ricci tensor and the scalar curvature are denoted by (ρ_{ij}) and τ respectively; $\rho_{ij} = R^{u}_{iuj}, \tau = \rho^u_u$. We also denote the length of a tensor $T = (T_{ij})$ by $|T|$, i.e., $|T|^2 = T_{ij}T^{ij}$. Finally, we denote by ∇_i the covariant derivative.

Let $T_m M$ denote the tangent space to M at m. We define the tensor field $\rho^{[k]}(x)$ by

$$\rho^{[k]}(x) = \sum_{p_1, \ldots, p_k = 1}^{n} R_{xp_1xp_2}R_{xp_2xp_3}\ldots R_{xp_kxp_1},$$

for $x \in T_m M$.

We call an Einstein space *k-stein* if there are real valued functions μ_ℓ on M such that $\rho^{[\ell]}(x) = \mu_\ell |x|^{2\ell}$ for all $x \in T_m M$ and $m \in M$ for $2 \leq \ell \leq k$. We further call a k-stein space *k^*-stein* if $|R|^2$ is constant.

We use the following notation:

$$\check{R}_{ij} = R_{iupq}R_{pqrs}R_{rsju}, \qquad \check{R} = \check{R}_{kk}$$
$$\check{\overline{R}}_{ij} = \overline{R}_{iupq}\overline{R}_{pqrs}\overline{R}_{rsju}, \qquad \check{\overline{R}} = \check{\overline{R}}_{kk}$$

Our second theorem is the following

Theorem 2. *Let (M, g) be an n-dimensional connected C^∞ Riemannian manifold with $n \geq 3$. Then the conditions $(P)_2 - (P)_4$ are necessary and sufficient in order that (M, g) be a 2^*-stein space and satisfy*

$$(2.2)\qquad 3\nabla_i R_{abcd}\nabla_j R_{abcd} - 20\check{R}_{ij} + 16\check{\overline{R}}_{ij} = \frac{3|\nabla R|^2 - 20\check{R} + 16\check{\overline{R}}}{n} g_{ij}$$

$$(2.3)\qquad 3|\nabla R|^2 - 20\check{R} + 16\check{\overline{R}} = constant$$

$$(2.4)\qquad \nabla_j(\check{R}_{ij} - 2\check{\overline{R}}_{ij}) = \frac{1}{6}\nabla_j(\check{R} - 2\check{\overline{R}})g_{ij}$$

Corollary 1. *Let (M, g) be an n-dimensional connected C^∞ Riemannian manifold with $3 \leq n \leq 6$. The conditions $(P)_2 - (P)_4$ are necessary and sufficient in order that the following assertions hold:*

(1) *if $n = 3, 4$, then (M, g) is locally flat or locally isometric to a symmetric space of rank one.*

(2) *if $n = 5$, then (M, g) is a 2^*-stein space and, satisfies $|\nabla R|^2 = constant$ and*

$$(2.5)\qquad \nabla_i R_{abcd}\nabla_j R_{abcd} = \frac{|\nabla R|^2}{n} g_{ij}$$

(3) *if $n = 6$, then (M, g) is a 2^*-stein space and, satisfies (2.3) and (2.5).*

Corollary 2. *Let (M, g) be an n-dimensional connected C^∞ 3^*-stein space with $3 \leq n \leq 5$. Then (M, g) satisfies the conditions $(P)_2 - (P)_4$.*

3. Proof of Theorem 1.

For the proof we use the expansions of two geometric mean values.

Let (M, g) be an n-dimensional connected C^∞ Riemannian manifold with $n \geq 2$. The *first mean value* $M_m(r, f)$ for a real valued continuous function f is defined by

$$M_m(r, f) = (\mathrm{vol}(\partial B_m(r)))^{-1} \int_{\partial B_m(r)} f(\omega)\, d\sigma(\omega),$$

where $d\sigma$ stands for the volume element on the geodesic sphere $\partial B_m(r)$. Similarly, the *second mean value* $L_m(r, f)$ for an f is defined by

$$L_m(r, f) = (\mathrm{vol}(S^{n-1}(1)))^{-1} \int_{S^{n-1}(1)} (f \circ \exp_m(ru))\, du,$$

where $\exp_m$ is the exponential map at $m \in M$ and du is the usual volume element on the $(n-1)$-dimensional unit sphere $S^{n-1}(1)$.

In [2], A.Gray and T.J.Willmore obtained the expansion

$$(3.1)\qquad L_m(r, f) = f(m) + \sum_{k=1}^{\infty} \frac{(\widetilde{\Delta}_m^k f)(m)}{2^k k!\, n(n+2)\dots(n+2k-2)} r^{2k} \qquad (r \to 0)$$

for an analytic function f at m, and computed $\widetilde{\Delta}_m^2 f$ and $\widetilde{\Delta}_m^3 f$ explicitly.

Proof of Theorem 1: Suppose first that (M,g) is a harmonic space. Set $r(n) = d(m,n), n \in M$ and $\Omega = r^2/2$. Then it is known that $\Delta\Omega \equiv \chi(\Omega)$ is a function of Ω only and does not depend on the reference point m (cf. [1], [7]). We call χ the *characteristic function* of M. We further have

$$\Delta r = \frac{n-1}{r} + \sum_{k-1}^{\infty} \alpha_{2k-1} r^{2k-1},$$

where $\alpha_{2k-1} = \chi^{(k)}(0)/2^k k!$.

Now due to [6], there exists a sequence of polynomials $p_k, k = 1,2,\ldots,$ without constant terms such that, for each $m \in M$, the expansion

$$M_m(r,f) = f(m) + \sum_{k=1}^{\infty} p_k(\Delta) f(m) r^{2k} \qquad (r \to 0) \tag{3.2}$$

holds for all analytic functions f at m. Further $p_k, k = 1,2,\ldots,$ are defined by:

$$\delta_\lambda(r) = 1 + \sum_{k=1}^{\infty} p_k(\lambda) r^{2k} \qquad (\lambda = \text{constant})$$

is the solution of $\delta''_\lambda(r) + (\Delta r)\delta'_\lambda(r) - \lambda\delta_\lambda(r) = 0$. Hence, setting $\widetilde{p}_k(\lambda) = 2^k k!\, n(n+2)\ldots(n+2k-2)p_k(\lambda), \widetilde{p}_k(\lambda)$ satisfies the recurrence formula

$$\begin{cases} \widetilde{p}_1(\lambda) = \lambda \\ \widetilde{p}_{k+1}(\lambda) - \lambda\widetilde{p}_k(\lambda) + \sum_{j=1}^{k} c_j^k \alpha_j \widetilde{p}_{k-j+1}(\lambda) = 0, \qquad k \geq 1, \end{cases} \tag{3.3}$$

where $c_j^k = \frac{2^j}{n+2k}\Pi_{s=1}^{j}(k-s+1)(n+2k-2s+2)$. From (3.3), $\widetilde{p}_k(\lambda)$ is written as

$$\widetilde{p}_k(\lambda) = \lambda^k + B_k^{k-1}\lambda^{k-1} + \cdots + B_k^1\lambda, \tag{3.4}$$

for some constants $B_k^{k-1},\ldots,B_k^1$. Thus we have

$$p_k(\Delta) = \frac{\Delta^k + B_k^{k-1}\Delta^{k-1} + \cdots + B_k^1\Delta}{2^k k! n(n+2)\ldots(n+2k-2)} \tag{3.5}$$

On the other hand, it follows from [3] that, for each $m \in M$,

$$M_m(r,f) = L_m(r,f) \qquad (r \to 0). \tag{3.6}$$

Hence by (3.6), comparing the coefficients in the expansions (3.1) and (3.2), we have

$$\widetilde{\Delta}_m^k = \Delta^k + B_k^{k-1}\Delta^{k-1} + \cdots + B_k^1\Delta, \tag{3.7}$$

for $k = 1,2,\ldots$. Thus we obtain (2.1) by induction.

Conversely, suppose that the infinite sequence of the conditions $(P)_k, k = 1,2,\ldots,$ holds. Then from (2.1) we have (3.7) by induction.Hence, due to [3, Theorem 2] or [6, Theorem 2 (1)], (M,g) is a harmonic space.

4. Preliminaries for proof of Theorem 2.

In this section, we prepare the explicit formula of $\widetilde{\Delta}^4_m f$ for the super-Einstein space and the curvature properties of the super-Einstein, the 2*-stein and the 3*-stein spaces which we use for the proof of Theorem 2 and its corollaries.

We first introduce the following notation:

$$\begin{aligned} \overline{R}_{ijk\ell} &= R_{ikj\ell}, \quad \overline{R}_{(ij)k\ell} = \overline{R}_{ijk\ell} + \overline{R}_{jik\ell}, \\ A_{ijk\ell\|pq} &= \overline{R}_{ijpr}\overline{R}_{(k\ell)qr} + \overline{R}_{ikpr}\overline{R}_{(j\ell)qr} + \overline{R}_{i\ell pr}\overline{R}_{(jk)qr}, \\ A_{ijk\ell} &= A_{ijk\ell\|pp}, \\ E_{ijk\ell} &= g_{ij}g_{k\ell} + g_{ik}g_{j\ell} + g_{i\ell}g_{jk}. \end{aligned}$$

Now, if $\rho^{[2]}(x) = \mu_2|x|^4$ holds for all $x \in T_mM$ and $m \in M$, then

$$A_{ijk\ell} = \mu_2 E_{ijk\ell}, \tag{4.1}$$

where $\mu_2 = (3n|R|^2 + 2\tau^2)/n^2(n+2)$. Also if $\rho^{[3]}(x) = \mu_3|x|^6$ holds for all $x \in T_mM$ and $m \in M$, then

$$\sum_\sigma A_{ijk\ell\|\alpha\beta}\overline{R}_{(pq)\beta\alpha} = 4\mu_3 \sum_\sigma E_{ijk\ell}g_{pq}, \tag{4.2}$$

where σ runs over all permutations.

We call an Einstein space *super-Einstein* if $|R|^2$ is constant and $\dot{R}_{ij} \equiv R_{ipqr}R_{jpqr} = |R|^2 g_{ij}/n$. Note that 2*-stein spaces are super-Einsteinian. Indeed this is obtained by transvecting (4.1) with $g^{k\ell}$.

1° ([5]) *Let (M,g) be an n-dimensional super-Einstein space. Then it holds that*

$$\begin{aligned} \widetilde{\Delta}^4_m f &= \Delta^4 f + \frac{4}{n}\tau\Delta^3 f + \frac{4}{15n}(\frac{21}{n}\tau^2 + 4|R|^2)\Delta^2 f \\ &+ \frac{8}{45}A_{ijk\ell}\nabla^4_{ijk\ell}f + \frac{1}{105n}(\frac{272}{n^2}\tau^3 + \frac{168}{n}\tau|R|^2)\Delta f \\ &- \frac{1}{63}(3\nabla_i R_{abcd}\nabla_j R_{abcd} - 20\check{R}_{ij} + 16\check{\overline{R}}_{ij})\nabla^2_{ij}f \\ &+ \frac{1}{105}\{82\varphi_i - \frac{5}{18}\nabla_i(3|\nabla R|^2 - 20\check{R} + 16\check{\overline{R}})\}\nabla_i f, \end{aligned} \tag{4.3}$$

where $\varphi_i = \nabla_j\{(\check{R}_{ij} - 2\check{\overline{R}}_{ij}) - \frac{1}{6}(\check{R} - 2\check{\overline{R}})g_{ij}\}$.

2° ([5]) *Let (M,g) be an n-dimensional super-Einstein space. Then it holds that*

$$\check{R}_{ij} - 2\check{\overline{R}}_{ij} = \frac{1}{n}(\check{R} - 2\check{\overline{R}})g_{ij}, \qquad \text{for } n \le 6, \tag{4.4}$$

$$\check{R} - 2\check{\overline{R}} = -\frac{1}{4}\{(1 - \frac{12}{n} + \frac{40}{n^2})\tau^3 + 3(1 - \frac{8}{n})\tau|R|^2\}, \qquad \text{for } n \le 5. \tag{4.5}$$

3° ([5]) *Let* (M,g) *be an* n*-dimensional* 2^**-stein space. Then it holds that*

(4.6)
$$\nabla_i R_{abcd}\nabla_j R_{abcd} = \nabla_p R_{iabc}\nabla_p R_{jabc} = -\frac{2}{n^2}\tau|R|^2 g_{ij} + \check{R}_{ij} + 4\check{\overline{R}}_{ij},$$

(4.7)
$$|\nabla R|^2 = -\frac{2}{n}\tau|R|^2 + \check{R} + 4\check{\overline{R}}.$$

4° *Let* (M,g) *be an* n*-dimensional* 3^**-stein space. Then it holds that*

(4.8)
$$7\check{R}_{ij} - 2\check{\overline{R}}_{ij} = \frac{1}{n}(7\check{R} - 2\check{\overline{R}})g_{ij},$$

where $7\check{R} - 2\check{\overline{R}} + \frac{2}{n^2}\tau^3 + \frac{9}{n}\tau|R|^2 = 2\mu_3 n(n+2)(n+4)$.

(4.8) is obtained by transvecting (4.2) with $g^{k\ell}g^{pq}$.

5. Proof of Theorem 2.

Proof of Theorem 2. Sufficiency.: Suppose that the conditions $(P)_2 - (P)_4$ hold. Then for each $k = 1,2,3,4$, $\tilde{\Delta}^k_m$ is represented as a linear combination (with constant coefficients) of $\Delta^k, \Delta^{k-1}, \ldots, \Delta$. By (3.1) we obtain, for each $m \in M$, the expansion

(5.1)
$$L_m(r,f) = f(m) + \sum_{k=1}^{4} p_k(\Delta)f(m)r^{2k} + O(r^{10}) \qquad (r \to 0),$$

where $p_k, k = 1,2,3,4$, denote the polynomials without constant terms and with constant coefficients. Due to [5, Theorem 1 (2)], (M,g) is a 2^*-stein space and, satisfies (2.2) and

(5.2)
$$\nabla_j\{(\check{R}_{ij} - 2\check{\overline{R}}_{ij}) - \frac{1}{6}(\check{R} - 2\check{\overline{R}})g_{ij}\} = \frac{5}{82\cdot 18}\nabla_i(3|\nabla R|^2 - 20\check{R} + 16\check{\overline{R}}),$$

whence we have the following

(5.3)
$$\tilde{\Delta}^2_m = \Delta^2 + \frac{2\tau}{3n}\Delta,$$

(5.4)
$$\tilde{\Delta}^3_m = \Delta^3 + \frac{2}{n}\tau\Delta^2 + \frac{4}{15n}(\frac{4}{n}\tau^2 + |R|^2)\Delta,$$

(5.5)
$$\begin{aligned}\tilde{\Delta}^4_m = \Delta^4 &+ \frac{4}{n}\tau\Delta^3 + \frac{4}{15n(n+2)}\{\frac{21n+46}{n}\tau^2 \\ &+ 2(2n+7)|R|^2\}\Delta^2 + \frac{1}{105n}\{\frac{16(51n+116)}{3n^2(n+2)}\tau^3 \\ &+ \frac{8(21n+56)}{n(n+2)}\tau|R|^2 - \frac{5}{3}(3|\nabla R|^2 - 20\check{R} + 16\check{\overline{R}})\}\Delta.\end{aligned}$$

Indeed, (5.3)–(5.4) are shown in [4] and (5.5) is obtained from (4.3). Since the coefficients in (5.3) - (5.5) are constants, (2.3) follows. This with (5.2) implies (2.4). Hence the sufficiency of $(P)_2 - (P)_4$ follows.

Necessity. Suppose that (M, g) is a 2^*-stein space and satisfies (2.2)-(2.4). Then, as in the above, the formulae (5.3)-(5.5) hold and the coefficients are constants. Hence $(P)_2 - (P)_4$ follow.

Theorem 2 is proved. ■

Next we prove Corollaries 1-2.

Lemma 5.1.. *Let (M, g) be as in Corollary 1. Then the following assertions are mutually equivalent, except for the case $n = 6$ in (3):*

(1) *the conditions $(P)_2 - (P)_4$ hold;*

(2) *(M, g) is a 2^*-stein space and satisfies (2.2), (2.3);*

(3) *$(n \le 5)$ (M, g) is a 2^*-stein space and satisfies (2.5), $|\nabla R|^2 = constant$.*

PROOF : Notice that (2.4) holds by (4.4) - (4.5), provided (M, g) is a super-Einstein space with $n \le 6$. Then combining Theorem 2 and [**5**, Proposition 6.3], we obtain the assertions of Lemma 5.1. ■

PROOF of Corollary 1: This is immediate from Lemma 5.1 and [**6**]. ■

PROOF of Corollary 2: Suppose first that $3 \le n \le 6$. Then by (4.4), (4.6) - (4.8), we have (2.5) and

$$\check{R}_{ij} = \frac{\check{R}}{n} g_{ij}, \qquad \check{\bar{R}}_{ij} = \frac{\check{\bar{R}}}{n} g_{ij}. \tag{5.6}$$

Substituting (5.6) into (4.6) and applying ∇_i, we obtain

$$(n + 12)\check{R} + 8(2n - 3)\check{\bar{R}} - 3|\nabla R|^2 = \text{constant}. \tag{5.7}$$

This with (4.5) and (4.7) implies $|\nabla R|^2 = $ constant. Hence the conditions $(P)_2 - (P)_4$ follow from Lemma 5.1. ■

6. Examples.

Let (M, g) be a harmonic space with dim $M = n$. Then from (3.3), we obtain the recurrence formulae for $A_{k,i}$ in (2.1) and B_k^{k-m} in (3.7) ($A_{p,p} = B_p^p = 1, p = 1, 2, \ldots$ by convention):

$$A_{k,i} = -\sum_{m=1}^{k-i} B_k^{k-m} A_{k-m,i} (i = 1, 2, \ldots, k-1), \tag{6.1}$$

$$B_k^{k-m} = -\sum_{s=1}^{k-m} \sum_{\ell=1}^{m} 2^\ell c_\ell^{s+m-1} \alpha_{2\ell-1} B_{s+m-\ell}^s \quad (m = 1, 2, \ldots, k-1). \tag{6.2}$$

For example, from (6.2) we have

$$B_k^{k-1} = -k(k-1)\alpha_1, \tag{6.3}$$

$$B_k^{k-2} = \frac{1}{6} k(k-1)(k-2)\{(3k-1)\alpha_1^2 - 4(2n + 3k - 5)\alpha_3\}, \tag{6.4}$$

$$\begin{aligned} B_k^{k-3} = &-\frac{1}{30} k(k-1)(k-2)(k-3)[5k(k-1)\alpha_1^3 \\ &- 4\{10(k-1)n + 15k^2 - 43k + 22\}\alpha_1\alpha_3 \\ &+ 4\{15n^2 + 6(8k-17)n + 8(5k^2 - 21k + 19)\}\alpha_5]. \end{aligned} \tag{6.5}$$

On the other hand, $\alpha_1, \alpha_3, \alpha_5$ are obtained by [1], [7], [8]:

$$\alpha_1 = -\frac{\tau}{3n}, \qquad \alpha_3 = -\frac{1}{90n(n+2)}\{\frac{2\tau^2}{n} + 3|R|^2\},$$

$$(6.6) \qquad \alpha_5 = \frac{1}{48 \cdot 315n(n+2)(n+4)}\{27|\nabla R|^2 - 32(\frac{\tau^3}{n^2} + \frac{9\tau}{2n}|R|^2 + \frac{7}{2}\overset{\vee}{R} - \overset{\vee}{\bar{R}})\}.$$

Substituting (6.6) into (6.3) - (6.5), we have the formulae for $B_k^{k-m}(m = 1,2,3)$, whence by (6.1) we can write down the formulae for $A_{k,k-\ell}(\ell = 1,2,3)$.In particular (5.3) - (5.5) are obtained and the normal forms of Δ^2, Δ^3, Δ^4 are also computed.

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