Commentationes Mathematicae Universitatis Carolinae

Jan Kratochvíl

Perfect codes and two-graphs

Commentationes Mathematicae Universitatis Carolinae, Vol. 30 (1989), No. 4, 755--760

Persistent URL: http://dml.cz/dmlcz/106798

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1989

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

Perfect codes and two–graphs

JAN KRATOCHVÍL

Abstract. A t–perfect code in a graph G is a subset C of its vertices such that every vertex of G is at distance at most t from exactly one code–vertex of C. A 2–graph is an equivalence class of graphs under Seidel's switching. The main result of the paper is a characterization of 2–graphs, all graphs of which contain 1–perfect codes.

Keywords: graph, perfect code, Seidel's switching

Classification: 05C99

Perfect codes in graphs were introduced by Biggs [3] as a generalization of the classical perfect codes in Hamming– and Lee–metrics [1], [2], [13], [10]. Unlike the case of distance–regular graphs where perfect codes are rather rare [2], [12], [13], one can easily construct examples of general (even regular) graphs containing perfect codes [5]. Nevertheless, typical graphs do not contain 1–perfect codes [9], and recognizing graphs that possess 1–perfect codes is NP–complete even when the input graph is regular [8]. One–perfect codes in out–degree–regular digraphs were also studied in [6].

In the sequel, we consider 2–graphs as equivalence classes of graphs under Seidel's switching [11]. We give a characterization of 2–graphs, all graphs of which contain 1–perfect codes, i.e. we characterize all graphs G such that every graph H equivalent to G contains a 1–perfect code. The characterization yields a polynomial recognition algorithm.

The paper is organized as follows: We review the necessary definitions and state the notations is Section 1. In Section 2, we introduce several graph reductions and reveal their connection to perfect codes and 2–graphs. The main result is proved in Section 3 and the concluding remarks are gathered in the last section.

1. Preliminaries.

All graphs considered are finite, undirected and without loops and multiple edges. The vertex set and edge set of a given graph G are denoted by $V(G)$ and $E(G)$, respectively. If there is no danger of confusion, we do not distinguish isomorphic graphs, e.g. any complete graph on n vertices is denoted by K_n, a cycle of length n is denoted by C_n and a path of length n is denoted by P_n. Given two graphs G and H, their disjoint union will be denoted by $G \wedge H$.

Given a graph G, a set $C \subseteq V(G)$ is called a *t–perfect code* in G iff the sets $S_t(u) = \{v | v \in V(G) \ \& \ d(u,v) \leq t\}$, $u \in C$ form a partition of $V(G)$ (i.e. iff for every $v \in V(G)$ there is exactly one $u \in C$ such that $d(u,v) \leq t$). Note that we have $d(u,v) \geq 2t+1$ for any two distinct $ui, v \in C$.

Given a graph G and a subset $A \subset V(G)$, we put $S(G,A) = (V(G), E(G) \div \{uv|u \in A \ \& \ v \notin A\})$ (here "$\div$" stands for the symmetric difference). We write simply $S(G,v)$ instead of $S(G,\{v\})$. We say that $S(G,A)$ is obtained from G by switching the vertices of A. We call graphs G and H equivalent (denoted $G \sim H$) iff H is isomorphic to $S(G,A)$ for some $A \subset V(G)$. A 2–graph (i.e. the equivalence class) determined by a graph G will be denoted by $\langle G \rangle$, i.e. $\langle G \rangle = \{H|H \sim G\}$. (For instance, $\langle C_4 \rangle = \{C_4, D_4, K_{1,3}\}$.)

We say that a 2–graph $\mathcal{G}$ is *t–codeperfect* iff each $H \in \mathcal{G}$ contains a t–perfect code.

2. Graph reductions.

Definition. Let G be a graph and v one of its vertices. We put

$$\rho_1(G,v) = (V(G) \cup \{v'\}, E(G) \cup \{uv'|uv \in E(G)\} \cup \{vv'\}),$$
$$\rho_2(G,v) = (V(G) \cup \{v'\}, E(G) \cup \{uv'|uv \notin E(G), u \neq v\}),$$
$$\rho_3(G,v) = (V(G) \cup \{v'\}, E(G) \cup \{uv'|uv \in E(G)\}),$$
$$\rho_4(G,v) = (V(G) \cup \{v'\}, E(G) \cup \{uv'|uv \notin E(G)\} \cup \{vv'\}).$$

We also write $G = \sigma_i H$, when $H = \rho_i(G,v)$ for some $v \in V(G)$.

For $A \subset \{1,2,3,4\}$, we write $G = \overline{\sigma}_A H$ iff

i) there is a sequence of graphs $G = G_1, G_2, \ldots, G_k = H$ and a sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{k-1}, \varepsilon_i \in A$ such that $G_i = \sigma_{\varepsilon_i} G_{i+1}$ for every $i = 1, 2, \ldots, k-1$;

ii) no σ_i, $i \in A$ can be applied to G.

We say that G is σ_A–reduced iff $G = \overline{\sigma}_A G$.

Remark. Note that $S(\rho_1(G,v),v') = \rho_2(G,v)$ and $S(\rho_3(G,v),v') = \rho_4(G,v)$ (and of course $S(\rho_2(G,v),v') = \rho_1(G,v)$ and $S(\rho_4(G,v),v') = \rho_3(G,v)$). Hence if G is σ_{12}–reduced, every $H \sim G$ is σ_{12}–reduced as well. Similarly for σ_{34}– and σ_{1234}–reduced graphs. Though for any G, the graphs $\overline{\sigma}_1 G$ and $\overline{\sigma}_3 G$ are uniquely determined, this is not true in general. For instance

$$P_2 = \overline{\sigma}_2 P_4 \qquad \text{and} \qquad D_3 = \overline{\sigma}_2 P_4.$$

Proposition 1. *Let G be a graph and v one of its vertices. Then*

i) *G contains a t–perfect code if and only if $\rho_1(G,v)$ contains a t–perfect code;*

ii) *if $t > 1$, G contains a t–perfect code if and only if $\rho_3(G,v)$ contains a t–perfect code.*

PROOF : i) Suppose $C \subset V(G)$ is a t–perfect code in G. Then C is also a t–perfect code in $\rho_1(G,v)$. If $C \subset V(G) \cup \{v'\}$ is a t–perfect code in $\rho_1(G,v)$, then $\operatorname{card} C \cap \{v,v'\} \leq 1$. Without loss of generality we may suppose that $v' \notin C$, and then C is also a t–perfect code in G. The proof of ii) is similar. ∎

For the sake of simplicity, we are going to use the following notation from now on

$$a(t) = \begin{cases} \{1,2\} & \text{for } t = 1, \\ \{3,4\} & \text{for } t = 2, \\ \{1,2,3,4\} & \text{for } t > 2. \end{cases}$$

Theorem 1. *Let G and H be graphs, t a positive integer and let $H = \overline{\sigma}_{a(t)}G$. Then $\langle G\rangle$ is t-codeperfect if and only if $\langle H\rangle$ is t-codeperfect.*

PROOF : Let $t = 1$ and suppose there is a sequence of graphs satisfying i) of the above definition. The proof goes by induction on k, i.e. on the number of steps in the derivation of H from G. Hence it suffices to prove that for any graph H and $v \in V(H), \langle H\rangle$ is 1-codeperfect iff $\langle \rho_1(H,v)\rangle$ is 1-codeperfect, and $\langle H\rangle$ is 1-codeperfect iff $\langle \rho_2(H,v)\rangle$ is 1-codeperfect. Since $\rho_1(H,v) \sim \rho_2(H,v)$, it is enough to prove the former statement.

Suppose $\langle H\rangle$ is 1-codeperfect and $H' \sim \rho_1(H,v)$, say $H' = S(\rho_1(H,v),A)$. If $\operatorname{card} A \cap \{v,v'\} = 1$, then $C = \{v,v'\}$ is a 1-perfect code in H'. In the opposite case, we may suppose without loss of generality that $A \cap \{v,v'\} = \emptyset$, and hence $H' = \rho_1(S(H,A),v)$ contains a 1-perfect code by Proposition 1.

Conversely, if $\langle \rho_1(H,v)\rangle$ is 1-codeperfect and $H' \sim H$, say $H' = S(H,A)$, we may suppose without loss of generality that $v \notin A$, and then $\rho_1(H',v) = S(\rho_1(H,v),A) \sim \rho_1(H,v)$ and H' contains a 1-perfect code by Proposition 1.

For $t > 1$, the proof is similar. (Note that if $\langle G\rangle$ is 2-codeperfect, $\langle \rho_2(G,v)\rangle$ need not be, e.g. when $G = P_4$.) ∎

Corollary. *For every t there exists a class of graphs $A(t)$ such that for any graph H the following statements are equivalent:*

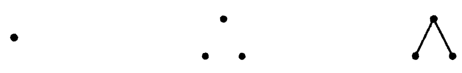
i) *$\langle H\rangle$ is t-codeperfect,*
ii) *every $H' = \overline{\sigma}_{a(t)}H$ is in $A(t)$,*
iii) *at least one $H' = \overline{\sigma}_{a(t)}H$ is in $A(t)$.*

PROOF : Put $A(t) = \{G \mid \langle G\rangle$ is t-codeperfect and G is $\overline{\sigma}_{a(t)}$-reduced$\}$. ∎

3. One-codeperfect two-graphs.

It follows explicitly from Theorem 1 that there is a class $A(1)$ such that for every graph H, $\langle H\rangle$ is 1-codeperfect if and only if there is some $H' = \overline{\sigma}_{12}H$ lying in $A(1)$. We prove that if $A(1)$ is chosen the smallest possible (i.e. if it contains only σ_{12}-reduced graphs), then it is finite. Moreover, we are able to describe it precisely:

Theorem 2. *We have $A(1) = \{K_1, D_3, P_2\}$.*



K_1 D_3 P_2

PROOF : Every graph on at most three vertices determines a 1-codeperfect 2-graph, and K_1, D_3 and P_2 are just all σ_{12}-reduced graphs on at most 3 vertices.

Suppose G is a σ_{12}-reduced graph on $n > 3$ vertices such that $\langle G\rangle$ is 1-codeperfect. Without loss of generality we may suppose that G contains an isolated vertex, say v (otherwise we consider $G' = S(G,\{u \mid uv \in E(G)\}) \sim G$. By Remark after the definition of the reductions, G' is also σ_{12}-reduced). Put $H = G - v$, i.e. $G = K_1 \wedge H$.

For every $w \in V(H)$ consider the graph $G_w = S(G,\{w,v\})$. According to the assumption, G_w contains a 1-perfect code. Since G_w is σ_{12}-reduced, $G_w$is connected and has diameter less than 3. Hence this code contains exactly one code-vertex, and as $wv \notin E(G)$, we see that there is a $\overline{w} \in V(H)$ such that $w\overline{w} \notin E(H)$ and $\overline{w}$ is adjacent to all other vertices of H. Note that $\overline{\overline{w}} = w$ immediately follows. Thus the vertices of H are grouped into pairs $w, \overline{w}$ and H is a complement of a perfect matching. So H does not contain a 1-perfect code, and neither does G, contradicting the assumption. ■

In view of Theorem 2, one can describe directly all 1-codeperfect 2-graphs:

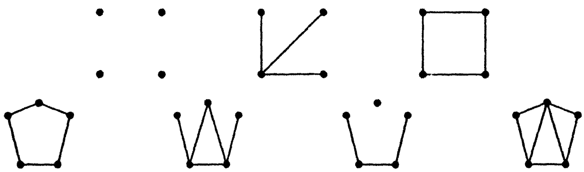
Corollary 2. *A 2-graph is 1-codeperfect if and only if it contains a graph composed of at most three isolated complete graphs.*

It immediately follows that also the class of σ_1-reduced graphs which determine 1-codeperfect 2-graphs is finite:

Corollary 3. *Let $A'(1)$ be the set of all induced subgraphs of C_6 which are σ_1-reduced, i.e. $A'(1) = \{K_1, D_2, D_3, P_2, P_3, P_4, K_1 \wedge P_2\}$. Then, given a graph G, the 2-graph $\langle G \rangle$ is 1-codeperfect if and only if $G' = \overline{\sigma}_1 G$ (which is unique) lies in $A'(1)$.*

By Theorem 1 or by the corollaries, one can very quickly (in a linear time) demonstrate that a given graph does determine a 1-codeperfect 2-graph (it suffices to guess a switching set of vertices or a sequence of reductions). If it does not, this fact can be evidenced even quicker (in a constant time) by using the following theorem.

Theorem 3. *A given graph G determines a 1-codeperfect 2-graph if and only if none of the seven graphs depicted in the figure is contained in G as an induced subgraph.*



PROOF : Let M be one of the graphs in the figure and let $M \leq G$ (M is an induced subgraph of G). Since M is σ_1-reduced, we have $M \leq \overline{\sigma}_1 G$ (note that $\overline{\sigma}_1 G$ is unique). But $M \not\leq C_6$, hence $\overline{\sigma}_1 G \not\leq C_6$ and $\langle G \rangle$ is not 1-codeperfect according to Corollary 3.

The converse implication is the crucial part of this theorem. Its proof is rather technical and we give just a sketch of it here.

Suppose $M \not\leq G$ for any M from the figure. We want to prove that then $\overline{\sigma}_1 G \leq C_6$. Let G contain an induced cycle C_k of length $k > 3$. Then $k = 6$, since $C_4 \not\leq G, C_5 \not\leq G$ and $K_1 \wedge P_3 \leq C_k$ for $k \geq 7$. Denote the vertices of this C_6 by $v_1, v_2, \ldots, v_6$ consecutively. For any $A \subset \{1, 2, \ldots, 6\}$ put $V_A = \{u | u \in V(G) - \{v_1, \ldots, v_6\}$ & $\{i | uv_i \in E(G)\} = A\}$. We have $V_\emptyset = \emptyset$, since $u \in V_\emptyset$ would yield $K_1 \wedge P_3 \leq G$. Similarly, $V_A = \emptyset$ unless card $A = 3$ and A contains three consecutive vertices of C_6. Therefore putting $V(i) = V_{\{i-1,i,i+1\}} \cup \{v_i\}$, $i = 1, 2, \ldots, 6$, we get that $V(G)$ is a disjoint union of $V(i)$, $i = 1, 2, \ldots, 6$. One can show analogously that every $V(i) \cup V(i+1)$ induces a complete subgraph of G, and thus $\overline{\sigma}_1 G \leq C_6$.

Suppose G does not contain an induced cycle of length > 3 (i.e. G is chordal). Since $D_4 \not\leq G$, G has at most three connected components. If it has exactly three of them, $D_3 = \overline{\sigma_1} G$, while if it has two of them, either $D_2 = \overline{\sigma}_1 G$ or $K_1 \wedge P_2 = \overline{\sigma}_1 G$. If G is connected and k is the length of a longest induced path in G, one can show similarly as above that $\overline{\sigma}_1 G = P_k \leq C_6$. ∎

4. A note on the computational complexity.

It is known that recognizing graphs that contain t-perfect codes is NP-complete for every t [7]. For $t = 1$, this problem was proved to be NP-complete even when restricted to regular graphs [8]. It turns out that asking not only "does this particular graph possess a 1-perfect code" but "does every graph equivalent to this particular one contain a 1-perfect code" makes the problem considerably easier. This is a direct consequence of Theorem 2 or 3:

Theorem 4. *Recognizing graphs that determine 1-codeperfect 2-graphs is polynomial.*

The situation is not so clear for $t > 1$. Here we have

Proposition 2. *For $t > 1$, the class $A(t)$ is infinite.*

PROOF : Consider a graph $H_{n,n} = (\{1, 2, \ldots, 2n\}, \{ij | 1 \leq i \leq n, n+1 \leq j \leq 2n \text{ and } j - i \neq n\})$ (sometimes it is called a Hiraguchi graph) and put $G_n = K_1 \wedge H_{n,n}$. Then G_n is σ_{1234}-reduced, and one can check that $\langle G_n \rangle$ is a t-codeperfect 2-graph for any $t \geq 2$. ∎

However, Proposition 2 does not say anything about the complexity of recognizing t-codeperfect 2-graphs. Hence we are left with the following open problem:

Problem. Given $t \geq 2$, what is the computational complexity of deciding whether a given graph determines a t-codeperfect 2-graph or not?

REFERENCES

[1] Astola J., *The theory of Lee-codes*, (research report), Lappeenranta, Lappeenranta Univ. of Technology, 1982.

[2] Best M.R., *A contribution to the nonexistence of perfect codes*, (academisch proefschrift), Amsterdam, Math.Centrum, 1982.

[3] Biggs N., *Perfect codes in graphs*, J.Combin.Theory Ser.B **15** (1973), 289-296.

[4] Cameron P.J., Thas J.A., Payne S.E., *Polarities of generalized hexagons and perfect codes*, Geometriae Dedicata **5** (1976), 525-528.

[5] Dvořáková-Ruličová I., *Perfect codes in regular graphs*, Comment.Math.Univ.Carolinae **29** (1988), 79-83.

[6] Etienne G., *Perfect codes and regular partitions in graphs and groups*, Europ.J.Combinatorics **8** (1987), 139-144.

[7] Kratochvíl J., *Perfect codes in general graphs*, in Proceedings 7th Hungarian colloqium on Combinatorics, Eger 1987, Colloquia Math. Soc. J. Bolyai 52, pp.357-364.

[8] Kratochvíl J., Křivánek M., *On the computational complexity of codes in graphs*, to appear in Proceedings MFCS'88, Karlovy Vary 1988, Lecture Notes in Comp.Sci. 324, Springer Verlag, Berlin 1988, pp. 396-404.

[9] Kratochvíl J., Malý J., Matoušek J., *On the existence of perfect codes in a random graph*, to appear in Proceedings Random Graphs'87, Poznań 1987.

[10] Post K.A., *Nonexistence theorem on perfect Lee codes over large alphabets*, Information and Control **29** (1975), 369-380.

[11] Seidel J.J., *Graphs and 2-graphs*, 5-th Southeastern Confer. on Combin., Graphs, Computing, pp. 125-143, Utilitas Math. Publ.Inc.Winnipeg, 1974.

[12] Smith D.H., *Perfect codes in the graphs O_k and $L(O_k)$*, Glasgow Math.J. **21** (1980), 169-172.

[13] Tietäväinen A., *On the nonexistence of perfect codes over finite fields*, SIAM J.Appl.Math. **24** (1973), 88-96.

Fac.of Math.and Phys., Charles University, Sokolovská 83, 186 00 Prague, Czechoslovakia

(*Received* May 5,1989)