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On graphs with prescribed edge neighbourhoods

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Abstract. Let G be a graph and let f be its edge. Then $N_G^e(f)$ is the subgraph of G induced by the set of all vertices adjacent to at least one of the end vertices of f.
In the paper some classes of graphs with the prescribed properties of $N_G^e(f)$ are studied.

Keywords: Local properties, neighbourhood of an edge, neighbourhood of a vertex

Classification: 05C99

1. Introduction.

All graphs considered in this article are finite undirected graphs without loops and multiple edges.

Let G be a graph, let f be its edge with end vertices x, y. By the symbol $N_G^e(f)$ or $N_G^e(xy)$ we denote the subgraph of G induced by the set of all vertices of G which are not incident to f and are adjacent to at least one end vertex of f. The graph $N_G^e(f)$ will be called the edge-neighbourhood (or e-neighbourhood) of f in G. By the symbol $\overline{N_G^e(f)}$ or $\overline{N_G^e(xy)}$ we denote a closed e-neighbourhood, i.e. the subgraph of G induced by the vertices x, y and the set of vertices of G which are adjacent to at least one vertex of the pair x, y. A graph induced by the vertex se M is denoted $\langle M \rangle$.

Zelinka [6] proposed the e-neighbourhood version of the well-known Zykov's problem [7] (concerning vertex-neighbourhoods) :

Characterize the graphs H with the property that there exists a graph G such that $N_G^e(f) \cong H$ for each edge f of G.

The graph H with the above mentioned property will be called an e-realizable graph and G will be called the e-realization of H. The se of all e-realizations of H will be denoted $\Re_e(H)$, the class of all e-realizable graphs will be denoted $\mathcal{N}_e$.

Let P be some prescribed property of graphs. If the e-neighbourhood of each edge f of G has the property P then G will be called a graph with prescribed e-neighbourhood.

Zelinka [6], Nedela [5] and the author [2], [3], [4] studied some classes of e-realizable graphs and also showed some graphs which are not e-realizable.

In this article we study some properties of graphs with prescribed e-neighbourhoods and show some e-realizable and non-e-realizable graphs.

2. Some properties of graphs with prescribed e-neighbourhoods.

At first we shall prove some simple lemmas.

Lemma 1. *Let G have no triangles. Then both $N_G^e(f)$ and $\overline{N_G^e(f)}$ are bipartite graphs for each edge f of G.*

PROOF : Suppose that $N_G^e(y_1y_2)$ of an edge $f = y_1y_2$ is not a bipartite graph. Then $N_G^e(y_1y_2)$ contains an odd cycle $C_{2n+1} = \langle x_1, x_2, \ldots, x_{2n+1}\rangle$ and there exists a pair of adjacent vertices x_i and x_{i+1} which are both adjacent either to y_1 or to y_2. Thus G contains a triangle, which is a contradiction.

Hence, $N_G^e(f)$ of each edge f of G is a bipartite graph. Now suppose that the closed e-neighbourhood $\overline{N_G^e(y_1y_2)}$ of an edge y_1y_2 is not a bipartite graph. As G contains no triangles, no vertex of $N_G^e(y_1y_2)$ is adjacent to both y_1 and y_2. Let $N_G^e(y_1y_2)$ have the parts $B_1 = \{x_1, x_2, \ldots, x_r\}$ and $B_2 = \{z_1, z_2, \ldots, z_s\}$. Then at least one of the vertices y_1, y_2 is adjacent to the vertices x_i and z_j belonging to this same component of $N_G^e(y_1y_2)$. Without loss of generality we can suppose that y_1 is adjacent to x_1 and simultaneously to z_j, which are the end vertices of the path $P_{2j} = \langle x_1, z_1, x_2, \ldots, x_j, z_j\rangle$. As G contains no triangles, and x_1 is adjacent to y_1, then z_1 has to be adjacent to y_2 and x_2 again to y_1. Analogously for each i, $1 \leqq i \leqq j$, if x_i is adjacent to y_1 then z_i is adjacent to y_2. But z_j is adjacent to y_1, which is a contradiction and then neither y_1 nor y_2 can be adjacent to vertices belonging to both parts of any component of $N_G^e(y_1y_2)$.

Thus $\overline{N_G^e(y_1y_2)}$ is a bipartite graph. ■

If x is a vertex of G, then by vertex-neighbourhood (or v-neighbourhood) of x in G we mean the subgraph of G induced by the set of all vertices of G which are adjacent to x and we denote it by $N_G^v(x)$.

Proposition 1. *Let G be an e-realization of some tree T. Let there exist a vertex x of G such that $N_G^v(x)$ contains a cycle C_r, $r \geqq 4$. Then $N_G^v(x) \cong C_r$.*

PROOF : Suppose that $N_G^v(x)$ contains a cycle C_r with vertices $y_1, y_2, \ldots, y_r$ and the chord y_1y_i, $3 \leqq i \leqq r-1$. Then $N_G^e(xy_r)$ contains C_i with the vertex set $\{y_1, y_2, \ldots, y_i\}$ which is a contradiction.

We can also see that $N_G^v(x)$ contains no other vertex y_{r+1}. In the opposite case $N_G^e(xy_{r+1})$ contains C_r with the vertices $y_1, y_2, \ldots, y_r$, which is again a contradiction. ■

Lemma 2. *Let G be an e-realization of a path P_n, $n \geqq 4$. Let an edge y_1y_2 belong to at least two triangles. Then at least one of the graphs $N_G^v(y_1)$, $N_G^v(y_2)$ is isomorphic to C_r, $r \leqq n+1$.*

PROOF : As G is an e-realization of P_n, $N_G^e(y_1y_2)$ is isomorphic to $P_n = \langle x_1, x_2, \ldots, x_n\rangle$. Let y_1y_2 belong to the triangles $\langle y_1, y_2, x_j\rangle$ and $\langle y_1, y_2, x_k\rangle$, $1 \leqq j < k$.

If $k = j+1$, then either $N_G^e(y_1x_i)$ (for any $i \neq j, j+1$) contains the triangle $\langle y_2, x_j, x_{j+1}\rangle$ or $N_G^e(y_2x_1)$ contains the triangle $\langle y_1, x_j, x_{j+1}\rangle$, which is a contradiction.

If $k = j+2$, then the vertex x_{j+1} is adjacent to at least one of the vertices y_1, y_2. Let it by y_1. Then $N_G^e(x_jy_2)$ contains the triangle $\langle y_1, x_{j+1}, x_{j+2}\rangle$, which is again a contradiction.

Now consider the case $k > j+2$. Suppose that there exists an edge $x_i x_{i+1}$ with the property that $j < i < k-1$ and the vertex x_i is adjacent to y_1 and x_{i+1} is adjacent to y_2. Then $N_G^e(x_i y_1)$ contains a star with the center y_2 and the terminal vertices x_j, x_{i+1}, x_k, which is a contradiction. If x_i is adjacent to y_2 and x_{i+1} is adjacent to y_1, we get a contradiction in the same way. Thus all the vertices $x_{j+1}, \ldots, x_{k-1}$ are adjacent to the only vertex of the pair y_1, y_2. Without losing generality we can suppose that it is y_1. Then $N_G^v(y_1)$ contains C_{k-j+2} with the vertices $y_2, x_j, x_{j+1}, \ldots, x_k$ and according to Proposition 1 $N_G^v(y_1)$ is isomorphic to C_{k-j+2}. The inequality $k - j + 2 = r \leqq n+1$ obviously holds - in the opposite case $N_G^e(y_1 y_2)$ contains at least $n+1$ vertices. ■

Lemma 3. *Let G be an e-realization of a path P_n, $n \geqq 4$. Then G has no triangles.*

PROOF : Suppose that G contains a triangle $\langle x_0, x_1, x_2 \rangle$. Then $N_G^e(x_0 x_1)$ contains some vertex x_3 which is adjacent to x_2 and simultaneously to at least one of the vertices x_0, x_1. Let it be x_0. Thus the edge $x_0 x_2$ belongs to two triangles, and according to the assertion of Lemma 2 the v-neighbourhood of at least one of the vertices x_0, x_2 is isomorphic to a cycle. Without los of generality we can suppose that $N_G^v(x_0) \cong C_r = \langle x_1, x_2, \ldots, x_r \rangle$.

Now we shall distinguish two cases. We begin with the simpler case $r = n+1$. As $G \in \mathcal{R}_e(P_n)$, then $N_G^e(x_1 x_2) \cong P_n$ and thus at least one of the vertices x_3, $x_{n+1} = x_r$ is adjacent to some vertex x_{n+2}. Let it be x_{n+1}. Hence $N_G^e(x_0 x_{n+1})$ contains $n+1$ vertices $x_1, \ldots, x_n, x_{n+2}$, which is a contradiction.

Now let us investigate the case $r < n+1$. As we supposed that $N_G^v(x_0) \cong C_r = \langle x_1, x_2, \ldots, x_r \rangle$, $N_G^e(x_0 x_r)$ must contain a vertex z_1 which is adjacent to at least one of the vertices x_1, x_{r-1}. Without loss of generality we can suppose that it is x_{r-1}. As z_1 is from $N_G^e(x_0 x_r)$, it has to be adjacent to x_r (if it is adjacent to x_0, then $N_G^e(x_0 z_1)$ contains the cycle $\langle x_1, x_2, \ldots, x_r \rangle$). Thus the edge $x_{r-1} x_r$ belongs to two triangles and the v-neighbourhood of at least one if its end vertices is isomorphic to C_s. Let $N_G^v(x_r) \cong C_s$. This cycle is induced by the vertices $x_1, x_0, x_{r-1}, z_1, \ldots, z_{s-3}$ because the cycle $\langle x_1, x_2, \ldots, x_r \rangle$ has no chord. But hence $N_G^e(x_0 x_r) \cong C_{r+s-4} = \langle x_1, \ldots, x_{r-1}, z_1, \ldots, z_{s-3} \rangle$, which is again a contradiction. ■

Lemma 4. *Let a connected graph G belonging to $\mathcal{R}_e(H)$ for a graph H be not regular. Then G is bipartite if and only if it has no triangles.*

PROOF : (=>) is trivial.

(<=) Let G have no triangles. Then the equality

$$\deg x + \deg y = k \tag{1}$$

holds for each pair of adjacent vertices x, y and k is the number of vertices of $N_G^e(xy) \cong H$.

As we suppose that G is not regular, there exists a pair of adjacent vertices x_0, x_1, such that $\deg x_0 \neq \deg x_1$. It follows from (1) that $\deg x_i = \deg x_1$ for each vertex x_i adjacent to x_0 and, analogously, $\deg x_j = \deg x_0$ for each vertex x_j adjacent to x_1. Because G is connected we can easily see that each vertex of G is either of degree $\deg x_0$ or of degree $\deg x_1$ and no vertices with the same degree can be adjacent to each other.

Thus G is a bipartite graph. ■

Corollary 1. *Let a connected e-realizable graph H have an odd number of vertices, let $G \in \mathcal{R}_e(H)$. Then G is a bipartite graph if and only if it has no triangles.*

PROOF : (=>) is trivial.

(<=) If G has no triangles and H has an odd number of vertices, G is not regular and our assertion follows from Lemma 4. ■

Now we shall present further simple propositions.

Proposition 2. *Let a connected e-realizable graph H have an odd number of vertices, let $G \in \mathcal{R}_e(H)$ be regular. Then G contains a triangle.*

PROOF : Let $N_G^e(y_1y_2) = \langle x_1, x_2, \dots, x_{2n+1}\rangle$. Then somevertex x_i has to be adjacent to both y_1 and y_2 and G contains the triangle $\langle y_1, y_2, x_i\rangle$ and also $\langle y_1, y_2, x_j\rangle$ for a neighbour x_j of x_i. ■

Proofs of the following trivial propositions are left to the reader.

Proposition 3. *Let a regular graph G be an e-realization of H. Let $t(e)$ be the number of triangles which contain an edge e. Then $t(e) = t(f)$ for any edges e, f of G and if H is connected, then $t(e) \geqq 2$.*

Proposition 4. *Let H be a hamiltonian graph and let $G \in \mathcal{R}_e(H)$ have no triangles. Then H is bipartite with parts P_1, P_2; $|P_1| = |P_2| = k$ and G is regular of degree $k+1$.*

Proposition 5. *The graph nK_2 is e-realizable by Q_{n+1}.*

3. Which paths are e-realizable ?

Now we turn our attention to the e-realizability of paths. From Lemma 3, Corollary 1 and simple considerations concerning the number of vertices of P_n we can easily see that the following Theorem holds.

Theorem 1. *Let $G \in \mathcal{R}_e(P_n)$, $n \geqq 4$. Then*

(i) if $n = 2k$, G has no triangles and it is regular of degree $k+1$;

(ii) if $n = 2k+1$, G is bipartite and bi regular of degrees $k+1$ and $k+2$.

Brown and Connelly [1] proved that all paths with the simple exception P_3 are v-realizable and on the other hand Zelinka [6] showed that P_2 and P_3 are e-realizable.

We shall solve the above mentioned problem for $n = 4, 5, 6$ in the next theorem.

Theorem 2. *The paths P_4, P_5 and P_6 are not e-realizable.*

PROOF : Suppose that $G \in \mathcal{R}_e(P_4)$. Then the e-neighbourhood of an edge y_1y_2 is isomorphic to $P_4 = \langle x_1, \dots, x_4\rangle$ andwithout losing generality we can suppose that y_1 is adjacent just to the vertices x_1, x_3, while y_2 is adjacent just to x_2, x_4. Now investigate $N_G^e(x_3x_4)$. As it contains $P_3 = \langle y_1, y_2, x_2\rangle$, there exists a vertex z of

G such that it is adjacent to x_2 (z cannot be adjacent to y_1 because it does not belong to $N^e_G(y_1y_2)$) and simultaneously to x_4 (z cannot be adjacent to x_3 because in this case G contains the triangle $\langle x_2, x_3, z\rangle$). But in this case $\deg x_2 \geqq 4$, which is a contradiction to the assertion (i) of Theorem 1. Thus P_4 is not e-realizable.

Now suppose that $G \in \mathcal{R}_e(P_5)$. Then $N^e_G(y_1y_2)$ is isomorphic to $P_5 = \langle x_1, x_2, \ldots, x_5\rangle$ and without loss of generality we can suppose that y_1 is adjacent just to x_1, x_3, x_5 and y_2 just to x_2, x_4. Note that y_2, x_1, x_3, x_5 are of degree 3 and y_1, x_2, x_4 are of degree 4. Now we shall investigate $N^e_G(y_1x_1)$ which contains the vertices y_2, x_2, x_3, x_5 and some other vertex z. Because $G \in \mathcal{R}_e(P_5)$, z is adjacent to x_1 and thus $N^e_G(y_1x_1) \cong P_5 = \langle y_2, x_2, x_3, x_5, z\rangle$. As $\langle y_2, x_2, x_3\rangle \cong P_3$ and x_5 is adjacent neither to y_2 nor to x_3, z has to be adjacent to x_5 and to just one of the vertices x_3, y_2. But z cannot be adjacent to x_3 (in the opposite case $\deg x_3 \geqq 4$) and as z does not belong to $N^e_G(y_1y_2)$, it also cannot be adjacent to y_2. Hence P_5 is not e-realizable.

At last suppose that G is an e-realization of P_6 and $N^e_G(y_1y_2) \cong P_6 = \langle x_1, x_2, \ldots, x_6\rangle$. Again without losing generality let y_1 be adjacent just to x_1, x_3, x_5 and y_2 to x_2, x_4, x_6. $N^e_G(y_1x_3)$ contains $P_5 = \langle x_1, x_2, y_2, x_4, x_5\rangle$ and thus G has to contain a vertex z adjacent to x_3 and to exactly one of the vertices x_1, x_5 which are terminal of the path P_5 mentioned above.

If z is adjacent to x_5, then $N^e_G(x_5y_1)$ contains $P_5 = \langle z, x_3, x_4, y_2, x_6\rangle$ and the vertex x_1. Because x_1 cannot be adjacent to x_6 (in the opposite case $N^e_G(y_1y_2) \cong C_6$) it has to be adjacent to z. But hence $N^e_G(x_3y_1)$ contains C_6 with the vertices $x_1, x_2, y_2, x_4, x_5, z$, which is a contradiction and thus z is not adjacent to x_5.

Therefore we shall suppose that z is adjacent to x_1. Then $N^e_G(y_1x_1)$ contains $P_4 = \langle z, x_3, x_2, y_2\rangle$ and the vertex x_5. As y_2 is of degree 4, according to Theorem 1 it cannot be adjacent to any other vertex of G and thus y_2 is the terminal vertex of $P_6 \cong N^e_G(y_1x_1)$. Hence z has to be adjacent either to x_5 (but in this case $N^e_G(y_1x_3)$ contains C_6 with the vertices $x_5, z, x_1, x_2, y_2, x_4$), or to another vertex v of $N^e_G(y_1x_1)$. Because v does not belong to $N^e_G(y_1y_2)$ it has to be adjacent to x_1. But we supposed that z is also adjacent to x_1 and then the vertices x_1, z, v induce the triangle. Therefore, according to the assertion of Lemma 3, G is not an e-realization of P_6.

As for $n \geqq 7$ no e-realizability of P_n is known and we are not able to prove its non-e-realizability by the methods used above, we propose our problem.

Problem. Which paths P_n for $n \geqq 7$ are e-realizable? ■

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