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A contribution to the theory of countably modulared spaces of double sequences

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Abstract. For the sequences of φ–functions (φ_j) and (Φ_i) we may define two sequences of pseudomodulars (ρ_j) and (v_i), which are generated by variation and sequential modulus. In the following, we define new modulars and respective countably modulared spaces, and next some properties of these spaces and connections between them are considered.

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1. Notation.

In order to built up a general theory of modular spaces it is advisable to investigate concrete examples of modular spaces which may be applied in various problems of mathematical analysis. The theory of countably modulared spaces was started by [1] and next was developed in [4], [5] and also for instance in [6], [9] and [10]. In this paper we consider countably modulared spaces of double sequences, which are generated by sequential modulus and variation.

1.1. Sequences. Let X be the space of all real, bounded double sequences. Sequences belonging to X will be denote by

$$x = (t_{\mu\nu}) = (t_{\mu\nu})_{\mu,\nu=0}^{\infty} = ((x)_{\mu\nu}) = ((x)_{\mu\nu})_{\mu,\nu=0}^{\infty}, \; |x| = (|t_{\mu}\nu|), \; x^p = (t_{\mu\nu}^p).$$

By a convergent sequence we shall mean double sequence converging in the sense of Pringsheim.

The translation operator τ_{mn}, (m,n=0,1,2, ...) of the sequence $x \in X$ is defined by the formula $\tau_{mn}x = ((\tau_{mn}x)_{\mu\nu})$, where

$$(\tau_{mn}x)_{\mu\nu} = \begin{cases} t_{\mu,\nu} & \text{for } \mu < m \text{ and } \nu < n, \\ t_{\mu+m,\nu} & \text{for } \mu \geq m \text{ and } \nu < n, \\ t_{\mu,\nu+n} & \text{for } \mu < m \text{ and } \nu \geq n, \\ t_{\mu+m,\nu+n} & \text{for } \mu \geq m \text{ and } \nu \geq n. \end{cases}$$

The sequence $((\tau_{mn}x)_{\mu\nu})_{\mu,\nu=0}^{\infty}$ is called the (m,n)-translation of the sequence $x \in X$.

1.2.Functions. Let $(\varphi_j)_{j=1}^{\infty}$ and $(\Phi_i)_{i=1}^{\infty}$ be two sequences of φ-functions, and let Ψ be a nonnegative, nondecreasing function of $u \geq 0$ such that $\Psi(u) \to 0$ as $u \to 0_+$.

In the sequel the following hypotheses will be used from time to time:

(1⁰) There exist positive constants K', c, u' and an index i_0 such that $\Phi_i(cu) \leq K'\Phi_{i_0}(u)$ for $0 \leq u \leq u'$ and $i \geq i_0$.

(2⁰) There exist positive constants K, c, u_0 and an index j_0 such that $\varphi_j(cu) \le K\varphi_{j_0}(u)$ for all $j \ge j_0$ and $0 \le u \le u_0$.

(3⁰) There exists a $u_0 > 0$ such that for every $\delta > 0$ there is an $\eta > 0$ satisfying the inequality $\Psi(\eta u) \le \delta\Psi(u)$ for $0 \le u \le u_0$.

(4⁰) For any $u_1 > 0$ and $\delta_1 > 0$ there is an $\eta_1 > 0$ such that $\Psi(\eta u) \le \delta_1\Psi(u)$ for all $0 \le u \le u_1$ and $0 < \eta \le \eta_1$.

1.3. Variation and sequential modulus. The Φ_i-variation of the sequence $x \in X$ is defined as

$$v_{\Phi_i}(x) = v_i(x) = \sup_{(m_\mu),(n_\nu)} \sum_{\mu,\nu=1}^{\infty} \Phi_i(|t_{m_{\mu-1},n_{\nu-1}} - t_{m_{\mu-1},n_\nu} - t_{m_\mu,n_{\nu-1}} + t_{m_\mu,n_\nu}|)$$

where the supremum runs through all increasing subsequences (m_μ) and (n_ν) of indices.

Let us remark that we may introduce more general functional $v'_{\Phi_i}(x) = \Phi_i(|t_{00}|) + v_{\Phi_i}(x)$, but in this case we limit ourselves to the space of all sequences $x \in X$ such that $t_{00} = 0$.

The sequential φ_j-modulus of the sequence $x \in X$ is defined as

$$\omega_{\varphi_j}(x;r,s) \equiv \omega_j(x;r,s) =$$
$$= \sup_{m\ge r}\sup_{n\ge s}\sup_{\mu\ge m}\sup_{\nu\ge n} \varphi_j(|(\tau_{00}x)_{\mu\nu} - (\tau_{m0}x)_{\mu\nu} - (\tau_{0n}x)_{\mu\nu} + (\tau_{mn}x)_{\mu\nu}|)$$

2. Countably modulared spaces. Let Ψ be a given function (defined as in 1.2). For every convex φ-function φ_j (j=1,2, ...) we may define pseudomodulars

$$\rho_{\varphi_j}(x) \equiv \rho_j(x) = \sup_{r,s} rs\Psi(\omega_{\varphi_j}(x;r,s)) \tag{1}$$

and respective modular spaces

$$X_{\rho_{\varphi_j}} \equiv X_{\rho_j} = \{x \in X : \rho_j(\lambda x) \to 0 \quad \text{as} \quad \lambda \to 0_+\}.$$

Moreover, we may introduce an F-norm

$$\|x\|_{\rho_{\varphi_j}} \equiv \|x\|_{\rho_j} = \inf\{\varepsilon > 0 : \rho_j(\frac{x}{\varepsilon}) \le \varepsilon\}$$

and $\bar{s}$-homogeneous norm

$$\|x\|^{\bar{s}}_{\rho_{\varphi_j}} \equiv \|x\|^{\bar{s}}_{\rho_j} = \{\varepsilon > 0 : \rho_j(\frac{x}{\varepsilon^{1/\bar{s}}}) \le 1\} = \sup_{r,s\ge 1}\left(\frac{\omega_{\varphi_j}(x;r,s)}{\Psi_{-1}(\frac{1}{rs})}\right)^{\bar{s}}$$

(if Ψ is an $\bar{s}$-convex function and φ-functions φ_j are convex.)

Moreover, for a given φ-function Φ_i and pseudomodular v_i we define a space

$$X_{v_{\Phi_i}} \equiv X_{v_i} \equiv X_{\Phi_i} = \{x \in X : v_i(\lambda x) \to 0 \text{ as } \lambda \to 0_+\}.$$

By means of the sequences (φ_j) and (Φ_i) we shall define two sequences of pseudomodulars (ρ_j) and (v_i), and the following extended real-valued functionals (which are modulars) in X:

$$(2)\qquad \begin{cases} \rho_0(x) = \sup\limits_j \rho_j(x), & \rho_s(x) = \sum\limits_{j=1}^{\infty} \rho_j(x), \\ \rho_\sigma(x) = \sup\limits_k \frac{1}{k}\sum\limits_{j=1}^{k} \rho_j(x), & \rho_w(x) = \sum\limits_{j=1}^{\infty} \frac{1}{2^j}\frac{\rho_j(x)}{1+\rho_j(x)}, \end{cases}$$

and

$$(3)\qquad \begin{cases} v_0(x) = \sup\limits_i v_i(x), & v_s(x) = \sum\limits_{i=1}^{\infty} v_i(x), \\ v_\sigma(x) = \sup\limits_k \frac{1}{k}\sum\limits_{i=1}^{k} v_i(x), & v_w(x) = \sum\limits_{i=1}^{\infty} \frac{1}{2^i}\frac{v_i(x)}{1+v_i(x)}. \end{cases}$$

In consequence, we may obtain the following countably modulared spaces $X_{\bar{v}}$ and $X_{\bar{\rho}}$, where $\bar{v}$ and $\bar{\rho}$ denote any of the symbols (2) and (3), respectively.

3.Properties of countably modulared spaces.

Theorem 1. *Let (φ_j) be a given sequence of φ-functions which satisfy the condition (2^0), and let Ψ be a function (defined as in 1.2) which satisfies the property (Δ_2) for small u. The spaces X_{ρ_0}, X_{ρ_σ}, S_{ρ_w} are identical.*

PROOF : If $x \in X_{\rho_w} = \bigcap_{j=1}^{\infty} X_{\rho_j}$, then $\rho_j(\lambda x) \to 0$ as $\lambda \to 0_+$ for each j separately. In consequence

$$rs\Psi(\omega_{\varphi_j}(\lambda x; r, s,)) \to 0 \quad \text{as } \lambda \to 0_+$$

for each j separately and for all r and s. Applying properties of Ψ and definition of $\omega_{\varphi_j}(x; r, s)$ we have

$$\sup_{m\geq r}\sup_{n\geq s}\sup_{\mu\geq m}\sup_{\nu\geq n} \varphi_j(\lambda|t_{m+\mu,n+\nu} - t_{m+\mu,\nu} - t_{\mu,n+\nu} + t_{\mu,\nu}|) \to 0$$

as $\lambda \to 0_+$, for all r and s and for each j separately. Thus

$$\varphi_j(\lambda|t_{m+\mu,n+\nu} - t_{m+\mu,\nu} - t_{\mu,n+\nu} + t_{\mu,\nu}|) \to 0$$

as $\lambda \to 0_+$, for $\mu \geq m \geq r,\ \ \nu \geq n \geq s$, where r and s are arbitrary and for each j separately. Hence, by assumptions

$$\lambda|t_{m+\mu,n+\nu} - t_{m+\mu,\nu} - t_{\mu,n+\nu} + t_{\mu,\nu}| \leq u_0$$

and

$$\varphi_j(\lambda|t_{m+\mu,n+\nu} - t_{m+\mu,\nu} - t_{\mu,n+\nu} + t_{\mu,\nu}|) \leq$$
$$\leq K\varphi_{j_0}(\frac{\lambda}{c}|t_{m+\mu,n+\nu} - t_{m+\mu,\nu} - t_{\mu,n+\nu} + t_{\mu,\nu}|)$$

for m, n, μ, ν as previously and for all $j \geq j_0$, and for sufficiently small $\lambda > 0$. In consequence for $j \geq j_0$ we have the following inequalities

$$\omega_{\varphi_j}(\lambda x; r, s) \leq K^{\omega_{\varphi_{j0}}}(\frac{\lambda}{c}x; r, s)$$

and

$$rs\Psi(\omega_{\varphi_j}(\lambda x; r, s)) \leq \bar{K}rs\Psi(\omega_{\varphi_{j_0}}(\frac{\lambda}{c}x; r, s)),$$

where $\bar{K}$ denotes a certain constant defined by the condition (Δ_2). Finally

$$\rho_j(\lambda x) \leq \bar{K}\varphi_{j_0}(\frac{\lambda}{c}x)$$

for all $j \geq j_0$ and for sufficiently small $\lambda > 0$. Thus $x \in X_{\rho_0}$. By conditions $X_{\rho_0} \not\subset X_{\rho_\sigma} \subset X_{\rho_w}$ and $X_{\rho_w} \subset X_{\rho_0}$ we have $X_{\rho_w} = X_{\rho_\sigma} = X_{\rho_0}$. ∎

Theorem 2. *Let us suppose that φ-functions $\varphi_j(u)$, (j=1,2, ...) satisfy the condition (2^0), and the function Ψ satisfies the condition (Δ_2) for small u. If $x^p \in X_{\rho_w}$ then the condition $x^p \xrightarrow{\rho_w} 0$ implies $x^p \xrightarrow{\rho_0} 0$.*

PROOF : By assumption $x^p \xrightarrow{\rho_w} 0$, there exists a positive constant λ_0 dependent on the sequence (x^p) such that

$$\sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\rho_j(\lambda_0 x^p)}{1 + \rho_j(\lambda_0 x^p)} \to 0$$

as $p \to \infty$. In consequence $\rho_j(\lambda_0 x^p) \to 0$ as $p \to \infty$ for each j separately, with a constant $\lambda_0 > 0$. In particular, $\rho_{j_0}(\lambda_0 x^p) \to 0$ as $p \to \infty$. Taking$\lambda_0 = \frac{\lambda}{c}$ we find N such that

$$(*) \qquad \rho_{j_0}(\frac{\lambda}{c}x^p) < \frac{\varepsilon}{k}$$

for $p \geq N$, where ε and k are some positive numbers. Let us remark that the condition (2^0) implies that: There exist a positive constant c and an index j_0 such that for every $u' > 0$ there is a k' such that $\varphi_j(u) \leq k'\varphi_{j_0}(\frac{u}{c})$ for all $0 \leq u \leq u'$ and for all $j \geq j_0$. By assumptions we have that the φ-functions $\varphi_j(u)$, $(j = 1, 2, \dots)$ are equicontinuous at $u = 0$, and moreover that $\varphi_j(\lambda x^p) \to 0$ as $\lambda \to 0_+$, for each j separately and for all p. Therefore,

$$\lambda|t^p_{\mu,\nu} - t^p_{\mu+m,\nu} - t^p_{\mu,\nu+n} + t^p_{\mu+m,\nu+n}| \leq u'$$

and

$$\varphi_j(\lambda|t^p_{\mu,\nu} - t^p_{\mu+m,\nu} - t^p_{\mu,\nu+n} + t^p_{\mu+m,\nu+n}|) \leq$$
$$\leq k'\varphi_{j_0}(\frac{\lambda}{c}|t^p_{\mu,\nu} - t^p_{\mu+m,\nu} - t^p_{\mu,\nu+n} + t^p_{\mu+m,\nu+n}|)$$

for sufficiently small $\lambda > 0$ and for $j \geq j_0$, $\mu \geq m \geq r$, $\nu \geq m \geq s$, where r and s are some positive integers. In the following for $j \geq j_0$ we have

$$rs\Psi(\omega_{\varphi_j}(\lambda x^p; r,s)) \leq \bar{K}rs\Psi(\omega_{\varphi_{j_0}}(\frac{\lambda}{c}x^p; r,s))$$

and

$$(**) \qquad \rho_{j_j}(\lambda x^p) \leq \bar{K}\rho_{j_0}(\frac{\lambda}{c}x^p)$$

for all $j \geq j_0$, for all p and for sufficiently small $\lambda > 0$, where $\bar{K}$ is a constant defined by the conditions (Δ_2) and $(?^0)$. The inequalities $(*)$ and $(**)$ lead to the condition $\rho_j(\lambda x^p) \leq \varepsilon$ for $p \geq N$ and $j \geq j_0$. $(K \leq k)$. Finally, if we choose N_1 in such a manner that $\rho_j(\lambda_0 x^p) < \varepsilon$ for $n \geq N_1$ and $j = 1, 2, \ldots, j_0 - 1$, then $\rho_j(\lambda_1 x^p) < \varepsilon$ for all j, $\lambda_1 = \min\{\lambda, \lambda_0\}$ and for all $p \geq \max\{N, N_1\}$. Thus $x^p \xrightarrow{\rho_0} 0$. ∎

Remark. Let Ψ be a function defined as in 1.2, which satisfies the condition (3^0) and let $\varphi_j(u)$, $(j = 1, 2, \ldots)$ be convex φ-functions. The element $x \in X$ belongs to X_{ρ_j} if and only if $\rho_{\varphi_j}(kx) < \infty$ for some constant $k > 0$.

PROOF : Let us take $x \in X$ and let $\rho_{\varphi_j}(kx) < \infty$, i.e. $rs\Psi(\omega_{\varphi_j}(kx; r,s)) \leq M$ for some $k > 0$, $M > 0$ and for all r and s. It is well known that the condition (3^0) implies the condition (4^0). We choose $u_1 = \sup_{r,s}\omega_{\varphi_j}(kx; r,s) < \infty$ and $\delta_1 > 0$. Let $0 < \lambda < k\eta_1$. In the following we have

$$rs\Psi(\omega_{\varphi_j}(\lambda x; r,s)) \leq rs\Psi(\frac{\lambda}{k}\omega_{\varphi_j}(kx; r,s)) \leq \delta_1 rs\Psi(\omega_{\varphi_j}(kx; r,s)) \leq \delta_1 M$$

for all r and s. Finally, $\rho_{\varphi_j}(\lambda x) \leq \delta_1 M$ and $\rho_{\varphi_j}(\lambda x) \to 0$ as $\lambda \to 0_+$. Thus $x \in X_{\rho_j}$. ∎

Remark. It is clear that if φ-function Φ_i is convex, then the element $x \in X$ belongs to X_{v_i} if and only if $v_i(kx) < \infty$ for some constant $k > 0$.

Theorem 3. *If the φ-functions $\Phi_i(u)$,(i=1,2, ...) satisfy the condition (1^0), then the spaces X_{v_w}, X_{v_e} and X_{v_0} are identical.*

The proof runs on the same lines as proof of Theorem 1.

Theorem 4. *If the φ-functions $\Phi_i(u)$, (i=1,2, ...) satisfy the condition (1^0) then, if $x^p \in X_{v_w}$, the condition $x^p \xrightarrow{v_w} 0$ implies the condition $x^p \xrightarrow{v_0} 0$.*

PROOF : First, let us remark that by previous theorem $x^p \in X_{v_0}$. The condition (1^0) may be written in the form: There exists a positive constant c and an index i_0 such that for every $u' > 0$ there is a $k' > 0$ such that $\Phi_i(u) \leq k'\Phi_{i_0}(\frac{u}{c})$ for all $0 \leq u \leq u'$ and $i \geq i_0$.

Hence

$$(*) \qquad v_{\Phi_i}(\lambda x^p) \leq k' v_{\Phi_{i_0}}(\frac{\lambda}{c}x^p)$$

for $i \geq i_0$ and $\lambda > 0$. By assumption $x^p \xrightarrow{v_w} 0$, there exists a positive constant λ_0 dependent on the sequence (x^p) such that $v_w(\lambda_0 x^p) \to 0$ as $p \to \infty$. From the condition

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{v_{\Phi_i}(\lambda_0 x^p)}{1 + v_{\Phi_i}(\lambda_0 x^p)} \to 0 \text{ as } p \to \infty$$

we conclude that $v_{\Phi_i}(\lambda_0 x^p) \to 0$ as $p \to \infty$ for each i separat[illegible] d with a constant $\lambda_0 > 0$: In particular, $v_{\Phi_{i_0}}(\lambda_0 x^p) \to 0$ as $p \to \infty$. Choosing [illegible] $= \frac{\lambda'}{c}$ we may find N' such that

$$(**) \qquad v_{\Phi_{i_0}}(\frac{\lambda'}{c} x^p) \leq \frac{\varepsilon}{k'} \quad .$$

for $p \geq N_1$, where ε is an arbitrary small number. Applying inequalities $(*)$ and $(**)$ we get $v_{\Phi_i}(\lambda x^p) < \varepsilon$ for $p \geq N'$ and $i \geq i_0$. Now, we choose $\bar{N}$ in such a manner that $v_{\Phi_i}(\lambda_0 x^p) < \varepsilon$ for $p \geq \bar{N}$ and $i < i_0$. Taking $\lambda = \min\{\lambda_0, \lambda'\}$ we obtain $v_{\Phi_i}(\lambda x^p) < \varepsilon$ for $p \geq \max\{N', \bar{N}\}$ and for all i. Consequently, $x^p \xrightarrow{v_0} 0$. ■

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