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Oscillation theorems for nonlinear second order differential equations with a damping term

S.R. GRACE AND B.S. LALLI

Abstract. Sufficient conditions for the oscillation of nonlinear second order differential equation

$$(a(t)\psi(x(t))\dot{x}(t))^{\cdot} + p(t)\dot{x}(t) + q(t)f(x(t)) = 0$$

are established. These results complement our earlier results.

Keywords: Oscillation, nonlinear, differential equations, damping, nonoscillation.

Classification: 34C10

1. Introduction.

Recently, the present authors [1], [2] studied the oscillatory behaviour of nonlinear second order differential equations of the form

$$(*)\qquad (a(t)\psi(x(t))\dot{x}(t))^{\cdot} + p(t)k(t, x(t)\dot{x}(t))\dot{x}(t) + q(t)f(x(t)) = 0 \quad (^{\cdot} = \frac{d}{dt}),$$

where p and q are nonnegative continuous functions on $[t_0, \infty)$, and proposed an open problem.

The purpose of this note is to consider such a problem concerning equation $(*)$ with $k = 1$ and when both p and q are continuous functions of varying sings. For related results we refer to [1] - [7].

2. Main results.

Consider the nonlinear second order differential equation

$$(1)\qquad (a(t)\psi(x(t))\dot{x}(t))^{\cdot} + p(t)\dot{x}(t) + q(t)f(x(t)) = 0,$$

where $a : [t_0, \infty) \to (0, \infty)$, p, $q : [t_0, \infty) \to R = (-\infty, \infty)$, $\psi : R \to (0, \infty)$ and $f : R \to R$ are continuous.

We are concerned with only continuable solutions of equation (1) which exist on some half line $[t_0, \infty)$. A solution $x(t)$ of equation (1) will be called oscillatory if it has arbitrarily large zeros and it is called nonoscillatory otherwise. We assume that there exist positive constants c, c_1 and k so that

$$(2)\qquad xf(x) > 0 \quad \text{and } f'(x) \geq k > 0 \quad \text{for } x \neq 0, \quad (' = \frac{d}{dx}):$$

$$(3)\qquad 0 < c \leq \psi(x) \leq c_1 \quad \text{for all } x \in R.$$

It will be convenient to make use of the following notations. For all $t \geq t_0$ we let

$$Q(t) = q(t) - \frac{1}{4k}\left(\frac{1}{c} - \frac{1}{c_1}\right)\frac{p^2(t)}{a(t)},$$
$$\gamma(t) = a(t)\dot{\rho}(t) - \frac{1}{c_1}p(t)\rho(t).$$

Theorem 1. *Let conditions (2) and (3) hold, and suppose that*

$$\int^{\infty} \frac{du}{f(u)} < \infty \quad \textit{and} \quad \int^{-\infty} \frac{du}{f(u)} < \infty. \tag{4}$$

Suppose that there exists a differentiable function $\rho : [t_0, \infty) \to (0, \infty)$ *such that*

$$\int^{\infty} \frac{1}{a(s)\rho(s)}\, ds = \infty. \tag{5}$$

Then each of the following conditions ensures the oscillation of continuable solutions of equation (1):

(I) $$\int^{\infty} \frac{\gamma^2(s)}{a(s)\rho(s)}\, ds < \infty, \quad \textit{and}$$

$$\int^{\infty} \rho(s)Q(s)\, ds = \infty; \tag{6}$$

(II) $\gamma(t) \geq 0, \quad \dot{\gamma}(t) \leq 0 \quad$ *for* $\quad t \geq t_0 \quad$ *and condition (6) holds;*

(III) $\gamma(t) > 0, \quad \dot{\gamma}(t) \geq 0 \quad$ *for* $t \geq t_0, \quad$ *and*

$$\lim_{t\to\infty} \frac{1}{\gamma(t)} \int^{t} \rho(s)Q(s)\, ds = \infty. \tag{7}$$

PROOF : Let $x(t)$ be a nonoscillatory solution of (1). Without loss of generality, we assume that $x(t) \neq 0$ for all $t \geq t_0$. Furthermore, we suppose that $x(t) > 0$ for $t \geq t_0$, since the substitution $u = -x$ transforms (1) into an equation of the same form subject to the assumptions of the theorem.

Now, we define

$$w(t) = \rho(t)\frac{a(t)\psi(x(t))\dot{x}(t)}{f(x(t))}, \quad t \geq t_0.$$

Then for every $t \geq t_0$, we obtain

$$\begin{aligned}\dot{w}(t) &= -\rho(t)q(t) - \frac{p(t)}{a(t)}\frac{1}{\psi(x(t))}w(t) + \frac{\dot{\rho}(t)}{\rho(t)}w(t) - \frac{1}{a(t)\rho(t)}\frac{f'(x(t))}{\psi(x(t))}w^2(t)\\ &= -\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)}w(t) - \frac{1}{\psi(x(t))}\left[\frac{f'(x(t))}{a(t)\rho(t)}w^2(t) + \frac{p(t)}{a(t)}w(t)\right].\end{aligned}$$

By completing the square we have

$$\dot{w}(t) = -\rho(t)q(t) + \frac{1}{\psi(x(t))}\frac{p^2(t)\rho(t)}{4a(t)f'(x(t))} + \frac{\dot{\rho}(t)}{\rho(t)}w(t) - \frac{1}{\psi(x(t))}\left[\sqrt{\frac{f'(x(t))}{a(t)\rho(t)}}w(t) + \frac{p(t)\sqrt{\rho(t)}}{2\sqrt{a(t)f'(x(t))}}\right]^2.$$

Using conditions (2) and (3), we have

$$\dot{w}(t) \le -\rho(t)Q(t) + \gamma(t)\frac{\psi(x(t))\dot{x}(t)}{f(x(t))} - \frac{1}{c_1}a(t)\rho(t)f'(x(t))\left(\frac{\psi(x(t))\dot{x}(t)}{f(x(t))}\right)^2. \tag{9}$$

Thus,

$$w(t) \le w(t_0) - \int_{t_0}^{t} \rho(s)Q(s)\,ds + \int_{t_0}^{t} \gamma(s)\frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\,ds - \frac{1}{c_1}\int_{t_0}^{t} a(s)\rho(s)f'(x(s))\left(\frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\right)^2 ds. \tag{10}$$

We consider the following cases:

Case 1. Let (I) hold. It follows from the Schwarz inequality that

$$\left|\int_{t_0}^{t} \gamma(s)\frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\,ds\right| \le \left(\int_{t_0}^{t} \frac{\gamma^2(s)}{a(s)\rho(s)}\,ds\right)^{\frac{1}{2}} \left(\int_{t_0}^{t} a(s)\rho(s)\left(\frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\right)^2 ds\right)^{\frac{1}{2}}$$
$$\le K\left(\int_{t_0}^{t} a(s)\rho(s)\left(\frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\right)^2 ds\right)^{\frac{1}{2}},$$

where $K = \left(\int_{t_0}^{\infty} \frac{\gamma^2(s)}{a(s)\rho(s)}\,ds\right)^{\frac{1}{2}}$ is finite. Thus (10) gives

$$w(t) \le w(t_0) - \int_{t_0}^{t} \rho(s)Q(s)\,ds + K\left(\int_{t_0}^{t} a(s)\rho(s)\left(\frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\right)^2 ds\right)^{\frac{1}{2}} - \frac{k}{c_1}\int_{t_0}^{t} a(s)\rho(s)\left(\frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\right)^2 ds.$$

Clearly, the sum of the last two integrals in the right hand side of the above inequality remains bounded above as $t \to \infty$. Thus, in view of (6),

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} \frac{a(t)\rho(t)\psi(x(t))\dot{x}(t)}{f(x(t))} = -\infty.$$

Consequently there exists a $t_1 \ge t_0$ such that

$$\dot{x}(t) < 0 \quad \text{for } t \ge t_1.$$

This means that there exists a $t_2 \ge t_1$ such that

$$(11) \quad 1 + k_1 \int_{t_2}^{t} a(s)\rho(s)f'(x(s))\psi(x(s))(\frac{\dot{x}(s)}{f(x(s))})^2\, ds \le a(t)\rho(t)\frac{\psi(x(t))(-\dot{x}(t))}{f(x(t))},$$

where $k_1 = kc/c_1$.

Case 2. If (II) holds, then by the Bonnet theorem, for any $t \ge t_0$, there exists a $\xi \in [t_0, t]$ so that

$$\int_{t_0}^{t} \gamma(s)\frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\, ds = \gamma(t_0)\int_{t_0}^{\xi} \frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\, ds = \gamma(t_0)\int_{x(t_0)}^{x(\xi)} \frac{\psi(u)}{f(u)}\, du$$
$$\le c_1\gamma(t_0)\int_{x(t_0)}^{\infty} \frac{du}{f(u)} = M < \infty.$$

As in Case 1, there exists a $t_2 \ge t_0$ so that (11) holds.

Case 3. Let (III) hold. Once again, by Bonnet's theorem, for some $M_1 > 0$, $t \ge t_0$, we have

$$\left|\int_{t_0}^{t} \gamma(s0\frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\, ds\right| \le M_1\gamma(t),$$

and as in case 2 of Theorem 7 in [1], we obtain inequality (11).

The rest of the proof is similar to that of Theorem 7 in [1] and hence is omitted. ■

Remark 1. Condition (3) of Theorem 1 can be replaced with $\psi(x) \ge c > 0$ for all $x \in R$, only if we are concerned with the bounded solutions of equation (1), and in general it seems not to be true. This is illustrated by the following example.

Example 1. Consider the differential equations

$$(12) \quad (\frac{1}{t}e^{|x(t)|}\dot{x}(t)) - \frac{1}{t^3}\dot{x}(t) + \frac{1+t^2}{t^4 lnt(1+ln^2 t)}(x(t)+x^3(t)) = 0, \qquad t > e^e$$

and

$$(13)\qquad ((\frac{1}{1+\sin^2 t})(1+x^2(t))\dot{x}(t))^{\cdot}+\frac{\sin t}{t^2}\dot{x}(t)+ \\ (1-\frac{\cos t}{t^2})(\frac{1}{1+sin^2 t})(x(t)+x^3(t))=0, \qquad t>0.$$

All conditions of Theorem 1 (I) are satisfied if we take $c=1$, $\rho(t)=t$ in equation (12) and $\rho(t)=1$ in (13) except that the upper bound of the function ψ does not exist. We note that (12) has an unbounded nonoscillatory solution $x(t)=\ln t$ while (13) has the bounded oscillatory solution $x(t)=\sin t$.

Example 2. Consider the differential equation

$$(14)\qquad (t\psi(x(t))\dot{x}(t))^{\cdot}+\frac{\sin t}{t}\dot{x}(t)+(\frac{1}{t}+\sin t)(x(t)+x^3(t))=0, \quad t>0,$$

where $\psi(x)=1+e^{-|x|}$ or $2-\sin x$. The hypotheses of Theorem 1 (I) are satisfied with $\rho(t)=1$ and hence every solution of (14) is oscillatory. We note that some of the oscillation criteria in ([1] - [7]) fail to apply to (14).

Theorem 2. *In addition to conditions (2) and (3) let*

$$(15)\qquad \int\limits_{+0}\frac{du}{f(u)}<\infty \quad \text{and} \int\limits_{-0}\frac{du}{f(u)}<\infty.$$

Suppose that there exists a differentiable function $\rho:[t_0,\infty)\to(0,\infty)$ *such that*

$$(16)\qquad \int^{\infty}\rho(s)Q^*(s)\,ds=\infty$$

and

$$(17)\qquad \int^{\infty}\frac{1}{a(s)\rho(s)}\int\limits_{T}^{s}\rho(\tau)Q^*(\tau)\,d\tau\,ds=\infty, \quad T\geq t_0,$$

where

$$Q^*(t)=q(t)-\frac{1}{4k}\left[\frac{p^2(t)}{ca(t)}-2p(t)\frac{\dot{\rho}(t)}{\rho(t)}+c_1a(t)\left(\frac{\dot{\rho}(t)}{\rho(t)}\right)^2\right].$$

Then every solution of (1) is oscillatory.

PROOF : Let $x(t)$ be a nonoscillatory solution of equation (1). Without loss of generality, we suppose that $x(t)>0$ for $t\geq t_0$. Furthermore, we consider the function w defined in the proof of Theorem 1. Then for every $t\geq t_0$ we obtain

$$\dot{w}(t)\leq-\rho(t)q(t)-\frac{1}{\psi(x(t))}\left[\frac{k}{a(t)\rho(t)}w^2(t)+\left(\frac{p(t)}{a(t)}-\frac{\dot{\rho}(t)}{\rho(t)}\psi(x(t))\right)w(t)\right].$$

Completing the square and using condition (3) we have

$$\dot{w}(t) \le -\rho(t)Q^*(t) \quad t \ge t_0.$$

Thus

$$a(t)\rho(t)\frac{\psi(x(t))\dot{x}(t)}{f(x(t))} \le C - \int_{t_0}^{t} \rho(s)Q^*(s)\,ds, \tag{18}$$

where

$$C = a(t_0)\rho(t_0)\frac{\psi(x(t_0))\dot{x}(t_0)}{f(x(t_0))}.$$

If follows from condition (16) that there exists a $t_1 \ge t_0$ so that

$$\int_{t_0}^{t_1} \rho(s)Q^*(s)\,ds = 0 \quad \text{and} \quad \int_{t_1}^{t} \rho(s)Q^*(s)\,ds \ge 2|C| \qquad \text{for } t \ge t_1.$$

Thus inequality (18) leads to

$$\frac{\dot{x}(t)}{f(x(t))} \le -\frac{1}{2}\frac{1}{\psi(x(t))}\frac{1}{a(t)\rho(t)}\int_{t_1}^{t} \rho(s)Q^*(s)\,ds \le -\frac{1}{2c_1}\frac{1}{a(t)\rho(t)}\int_{t_1}^{t} \rho(s)Q^*(s)\,ds,$$

which implies that

$$G(x(t)) \le G(x(t_1)) - \frac{1}{2c_1}\int_{t_1}^{t}\frac{1}{a(s)\rho(s)}\int_{t_1}^{s}\rho(\tau)Q^*(\tau)\,d\tau\,ds,$$

where

$$G(x(t)) + \int_{0}^{x(t)} \frac{du}{f(u)}.$$

Consequently $G(x(t)) \to -\infty$ as $t \to \infty$, contradicting the fact that $G(x(t)) \ge 0$. This completes the proof. ■

Remarks.

1. In this paper we have put no sign restriction on the function p and we have discarded condition on p of the type (7) required in [1].
2. As illustrated by Example 1, condition (3) cannot be relaxed to $\psi(x) \ge c > 0$ for all $x \in R$.
3. It is easy to check that the results of this paper are related to many of the know oscillation criteria in [1] - [7].

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