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On one class of solvable boundary value problems for ordinary differential equation of n–th order

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Dedicated to the memory of Svatopluk Fučík

Abstract. New sufficient conditions of the existence and uniqueness of the solution of a boundary problem for an ordinary differential equation of n–th order with certain functional boundary conditions are constructed by the method of a priori estimates.

Keywords: boundary value problems with functional conditions, differential equations of n–th order, method of a priori estimates, differential inequalities

Classification: 34B15, 34B10

Introduction.

In the paper we give new sufficient conditions for the existence and uniqueness of the solution to the problem

$$u^{(n)}(t) = f(t, u(t), \dots, u^{(n-1)}(t)), \tag{1}$$

$$\phi_{0i}(u^{(i-1)}) = \phi_i(u, u', \dots, u^{(n-1)}) \qquad (i = 1, \dots, n), \tag{2}$$

where $f : \langle a, b\rangle \times R^n \to R$ satisfies the local Carathéodory condition and for each $i \in \{1, \dots, n\}$ the linear nondecreasing continuous functional ϕ_{0i} on $C(\langle a, b\rangle)$ is concentrated on $\langle a_i, b_i\rangle \subseteq \langle a, b\rangle$ (i.e. the value of ϕ_{0i} depends only on functions restricted to $\langle a_i, b_i\rangle$ and the segment can be degenerated to a point) and ϕ_i is a continuous functional on $[C(\langle a, b\rangle)]^n$. In general $\phi_{0i}(1) = c_i\,(i = 1, \dots, n)$. Without loss of generality we can suppose $\phi_{0i}(1) = 1\,(i = 1, \dots, n)$, which simplifies the notation.

The formulation (1), (2) contains boundary value problems e.g.

$$\phi_{0i}(u^{(i-1)}) = u^{(i-1)}(t_i) \quad (i = 1, \dots, n)$$

where $t_i = a_i = b_i\,(i = 1, \dots, n)$ or the problem

$$\phi_{0i}(u^{(i-1)}) = \int_{a_i}^{b_i} u^{(i-1)}(t)\, d\sigma_i(t) \quad (i = 1, \dots, n).$$

The integral is understood in the Lebesgue–Stieltjes sense, where σ_i is nondecreasing in $\langle a_i, b_i\rangle$ and $\sigma_i(b_i) - \sigma_i(a_i) > 0\,(i = 1, \dots, n)$.

In [2], the equation (1) with the conditions in the special form

$$u^{(i-1)}(t_i) = c_i \quad (c_i \in R) \quad (i = 1, \dots, n) \tag{3}$$

is investigated by means of one–sided estimates, mean while two–sided estimates are needed to treat the problem (1), (4) with

$$\int_{a_i}^{b_i} u^{(i-1)}(t)\, d\sigma_i(t) = c_i \quad (c_i \in R) \quad (i = 1, \ldots, n), \tag{4}$$

as it was done in [1].

In this paper we deal with the more general problem (1), (2) using two–sided estimates. Our results without proofs were communicated in [5].

Main results.

We adopt the following notation:

$\langle a, b\rangle$ – a segment, $-\infty < a \le a_i \le b_i \le b < +\infty$ $(i = 1, \ldots, n)$ R^n – n–dimensional real space with points $x = (x_i)_{i=1}^n$ normed by $\|x\| = \sum_{i=1}^n |x_i|$,

$$R_+^n = \{x \in R^n : x_i \ge 0,\ i = 1, \ldots, n\},$$

$C^{n-1}(\langle a, b\rangle)$ – the space of functions continuous together with their derivatives up to the order $n-1$ on $\langle a, b\rangle$ with the norm

$$\|u\|_{C^{n-1}(\langle a,b\rangle)} = \max\{\sum_{i=1}^{n} |u^{(i-1)}(t)| : a \le t \le b\},$$

$\widetilde{C}^{n-1}(\langle a, b\rangle)$ – a set of all functions absolutely continuous together with their derivatives to the $(n-1)$–order on $\langle a, b\rangle$, the spaces $L^p(\langle a, b\rangle)$ $(p \in \langle 1, \infty))$ are defined in the usual way. According to [3], inequalities between vectors are understood by components, a functional $\phi : [C^0(\langle a, b\rangle)]^n \to R_+$ is said to be homogeneous iff $\phi(\lambda x) = \lambda\phi(x)$ for all $\lambda \in R_+$, $x \in [C^0(\langle a, b\rangle)]^n$ and nondecreasing iff $\phi(x) \le \phi(y)$ for all $x, y \in [C^0(\langle a, b\rangle)]^n, x \le y$. Let us consider the problem (1), (2). Under the solution we understand the function with absolute continuous derivatives up to the order $(n-1)$ on $\langle a, b\rangle$, which satisfies the equation (1) for almost all $t \in \langle a, b\rangle$ and fulfils the boundary conditions (2).

To solve (1), (2), we specify a class of auxiliary functions $g, h_1, \ldots, h_n, \psi_1, \ldots, \psi_n$.

Definition. Let $\psi_i : [C^0(\langle a, b\rangle)]^n \to R_+$ $(i = 1, \ldots, n)$ be the homogeneous continuous nondecreasing functionals and $h_i, g \in L^1(\langle a, b\rangle), h_i \ge 0$ $(i = 1, \ldots, n)$. If the system of differential inequalities

$$\begin{aligned} &|\rho_i'(t)| \le |\rho_{i+1}(t)|, \quad t \in \langle a, b\rangle \quad (i = 1, \ldots, n-1), \\ &|\rho_n'(t) - g(t)\rho_n(t)| \le \sum_{j=1}^{n} h_j(t)|\rho_j(t)|, \quad t \in \langle a, b\rangle \end{aligned} \tag{5}$$

with boundary conditions

$$\min\{|\rho_i(t)| : a_i \le t \le b_i\} \le \psi_i(|\rho_1|, \cdots, |\rho_n|) \quad (i = 1, \ldots, n) \tag{6}$$

has only trivial solution, we say that

(7) $$(g, h_1, \dots, h_n; \psi_1, \dots, \psi_n) \in \mathrm{Nic}(\langle a, b\rangle; a_1, \dots, a_n, b_1, \dots, b_n).$$

Theorem 1. *Let* $(g, h_1, \dots, h_n; \psi_1, \dots, \psi_n) \in \mathrm{Nic}(\langle a, b\rangle; a_1, \dots, a_n, b_1, \dots, b_n)$ *and let the data* $f, \phi_1, \dots, \phi_n$ *of (1), (2) satisfy the inequalities*

(8_1) $$[f(t, x_1, \dots, x_n) - g(t)x_n] \operatorname{sign} x_n \le \sum_{j=1}^{n} h_j(t)|x_j| + \\ +\omega(t, \sum_{i=1}^{n} |x_i|) \quad \textit{for } t \in \langle a_n, b\rangle, x \in R^n$$

(8_2) $$[f(t, x_1, \dots, x_n) - g(t)x_n] \operatorname{sign} x_n \ge -\sum_{j=1}^{n} h_j(t)|x_j| - \\ -\omega(t, \sum_{i=1}^{n} |x_i|) \quad \textit{for } t \in \langle a, b_n\rangle, x \in R^n$$

(9) $$|\phi_i(u, u', \dots, u^{(n-1)})| \le \psi_i(|u|, \dots, |u^{(n-1)}|) + r \\ \textit{for } u \in C^{n-1}(\langle a, b\rangle) \quad (i = 1, \dots, n),$$

where $r \ge 0$ *and* $\omega : \langle a, b\rangle \times R_+ \to R_+$ *is a measurable function nondecreasing in the second variable satisfying*

(10) $$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b \omega(t, \rho)\, dt = 0.$$

Then the problem (1), (2) has a solution.

To prove the theorem (1) we apply the following

Lemma 1. *Let the condition (7) be satisfied. Then there exists a constant* $\rho > 0$ *such that the estimate*

(11) $$\|u\|_{C^{n-1}(\langle a, b\rangle)} \le \rho(r + \|h_0\|_{L^1(\langle a, b\rangle)})$$

holds for each constant $r \ge 0, h_0 \in L^1(\langle a, b\rangle), h_0 \ge 0$ *and for each solution* $u \in \widetilde{C}^{n-1}(\langle a, b\rangle)$ *of the differential inequalities*

(12_1) $$[u^{(n)}(t) - g(t)u^{(n-1)}(t)] \operatorname{sign} u^{(n-1)}(t) \le \\ \le \sum_{j=1}^{n} h_j(t)|u^{(j-1)}(t)| + h_0(t) \quad \textit{if } a_n \le t \le b$$

(12_2) $$[u^{(n)}(t) - g(t)u^{(n-1)}(t)] \operatorname{sign} u^{(n-1)}(t) \ge \\ \le -\sum_{j=1}^{n} h_j(t)|u^{(j-1)}(t)| - h_0(t) \quad \textit{if } a \le t \le b_n$$

with boundary conditions

$$\text{(13)}\quad \min\{|u^{(i-1)}(t)| : a_i \le t \le b_i\} \le \\ \le \psi_i(|u|, |u'|, \dots, |u^{(n-1)}|) + r \quad (i = 1, \dots, n).$$

PROOF : By contradiction, let there exist $r_m \in R_+$, $h_{om} \in L^1(\langle a, b\rangle)$ and $u_m \in \widetilde{C}^{n-1}(\langle a, b\rangle)$ for any natural m, such that

$$\text{(14)}\quad \|u_m\|_{C^{n-1}(\langle a,b\rangle)} \ge m(r_m + \|h_{om}\|_{L^1(\langle a,b\rangle)}),$$

$$\text{(15}_1\text{)}\quad [u_m^{(n)}(t) - g(t)u_m^{(n-1)}(t)] \operatorname{sign} u_m^{(n-1)}(t) \le \\ \le \sum_{j=1}^{n} h_j(t)|u_m^{(j-1)}(t)| + h_{0m}(t) \quad \text{if } a_n \le t \le b$$

$$\text{(15}_2\text{)}\quad [u_m^{(n)}(t) - g(t)u_m^{(n-1)}(t)] \operatorname{sign} u_m^{(n-1)}(t) \ge \\ \ge -\sum_{j=1}^{n} h_j(t)|u_m^{(j-1)}(t)| - h_{0m}(t) \quad \text{if } a \le t \le b_n$$

and

$$\text{(16)}\quad \min\{|u_m^{(i-1)}(t)| : a_i \le t \le b_i\} \le \\ \le \psi_i(|u_m|, |u_m'|, \dots, |u_m^{(n-1)}|) + r_m \quad (i = 1, \dots, n).$$

Denoting

$$\widetilde{u}_m(t) = \frac{u_m(t)}{\|u_m\|_{C^{n-1}(\langle a,b\rangle)}}, \widetilde{h}_{0m}(t) = \frac{h_{0m}(t)}{m(r_m + \|h_{om}\|_{L^1(\langle a,b\rangle)})},$$

we get

$$\text{(17)}\quad \|\widetilde{u}_m\|_{C^{n-1}(\langle a,b\rangle)} = 1, \quad \|\widetilde{h}_{0m}\|_{L^1(\langle a,b\rangle)} \le \frac{1}{m}.$$

On the other hand, according to (14) – (16), we have

$$\text{(18}_1\text{)}\quad [\widetilde{u}_m^{(n)}(t) - g(t)\widetilde{u}_m^{(n-1)}(t)] \operatorname{sign} \widetilde{u}_m^{(n-1)}(t) \le \\ \le \sum_{j=1}^{n} h_j(t)|\widetilde{u}_m^{(j-1)}(t)| + \widetilde{h}_{0m}(t) \quad \text{if } a_n \le t \le b$$

$$\text{(18}_2\text{)}\quad [\widetilde{u}_m^{(n)}(t) - g(t)\widetilde{u}_m^{(n-1)}(t)] \operatorname{sign} \widetilde{u}_m^{(n-1)}(t) \ge \\ \ge -\sum_{j=1}^{n} h_j(t)|\widetilde{u}_m^{(j-1)}(t)| - \widetilde{h}_{0m}(t) \quad \text{if } a \le t \le b_n$$

and

(19) $$\min\{|\widetilde{u}_m^{(i-1)}(t)| : a_i \le t \le b_i\} \le$$
$$\le \psi_i(|\widetilde{u}_m|, |\widetilde{u}'_m|, \dots, |\widetilde{u}_m^{(n-1)}|) + r_m \quad (i = 1, \dots, n).$$

For any $i \in \{1, \dots, n\}$ and a natural m we chose a point $t_{im} \in \langle a_i, b_i \rangle$ such that

(20) $$|\widetilde{u}_m^{(i-1)}(t_{im}0| = \min\{|\widetilde{u}_m^{(i-1})(t)| : a_i \le t \le b_i\} \quad (i = 1, \dots, n).$$

Let ρ_{nm} be the solution of the Cauchy problem

(20) $$\rho'_{nm}(t) = g(t)\rho_{nm}(t) +$$
$$+ [\sum_{j=1}^{n} h_j(t)|\widetilde{u}_m^{(j-1)}(t)| + \widetilde{h}_{0m}(t)] \cdot \operatorname{sign}(t - t_{nm})$$

(22) $$\rho_{nm}(t_{nm}) = |\widetilde{u}_m^{(n-1)}(t_{nm})|.$$

Then, according to [2], lemma 4.1 and to the conditions $(18_{1,2})$,

$$\widetilde{u}_m^{(n-1)}(t)| \le \rho_{nm}(t), \quad a \le t \le b.$$

Therefore, if we put

(23) $$\rho_{im}(t) = |\widetilde{u}_m^{(i-1)}(t_{im})| + |\int_{t_{im}}^{t} \rho_{(i+1)m}(\tau)\, d\tau| \quad (i = 1, \dots, n),$$

we shall have

(24) $$|\widetilde{u}_m^{(i-1)}(t)| \le \rho_{im}(t) \quad \text{when } a \le t \le b \quad (i = 1, \dots, n).$$

Formulae (21), (22) and (24) yield

(25) $$\rho_{nm}(t) \le \exp(\int_{t_{nm}}^{t} g(\tau)\, d\tau |\widetilde{u}_m^{(n-1)}(t_{nm})| +$$
$$+ |\int_{t_{nm}}^{t} [\exp(\int_{\tau}^{t} g(s)\, ds] [\sum_{j=1}^{n} h_j(\tau)|\widetilde{u}_m^{(j-1)}(\tau)| +$$
$$\widetilde{h}_{0m}(\tau)]\, d\tau]$$

and

(26) $$\rho_{nm}(t) \le \exp(\int_{t_{nm}}^{t} g(\tau)\, d\tau \rho_{nm}(t_{nm}) +$$
$$+ |\int_{t_{nm}}^{t} [\exp(\int_{\tau}^{t} g(s)\, ds] [\sum_{j=1}^{n} h_j(\tau)\rho_{jm}(\tau) + \widetilde{h}_{0m}(\tau)]\, d\tau|.$$

According to (17), (21) and (25) we obtain

$$(27)\qquad |\rho_{nm}(t)| \le r_0 \quad \text{if } a \le t \le b \quad (m = 1, 2, \dots)$$

and

$$(28)\qquad |\rho'_{nm}(t)| \le \widetilde{h}_{0m}(t) + \widetilde{h}(t) \quad \text{if } a \le t \le b \quad (m = 1, 2, \dots)$$

where

$$r_0 = (2 + \sum_{j=1}^{n} \int_a^b h_j(\tau)\, d\tau) \exp(\int_a^b |g(\tau)|\, d\tau)$$

and

$$\widetilde{h}(t) = r_0 |g(t)| + \sum_{j=1}^{n} h_j(t).$$

Formulae (17), (19), (20), (23) and (24) imply, that

$$(29)\qquad \sum_{i=1}^{n} \|\rho_{im}\|_{C^0(\langle a,b\rangle)} \ge 1$$

$$(30)\qquad |\rho_{im}(t_{im})| \le \psi_i(\rho_{1m}, \dots, \rho_{nm}) + \frac{1}{m} \quad (1 = 1, \dots, n; m = 1, 2, \dots)$$

and

$$(31)\qquad |\rho_{im}(t_{im})| \le 1 \quad (i = 1, \dots, n; m = 1, 2, \dots).$$

From (17), (23), (27), (28) and (31) it follows that the sequences $\{\rho_{im}\}_{m=1}^{+\infty} (i = 1, \dots, n)$ are uniformly bounded and uniformly continuous. According to the lemma of Arzela–Ascoli we can suppose without loss of generality that these sequences uniformly converge. The sequences of points $\{t_{im}\}_{m=1}^{+\infty} (i = 1, \dots, n)$ can be taken convergent as well. Denote

$$t_{i0} = \lim_{m \to +\infty} t_{im} \quad (i = 1, \dots, n)$$

and

$$\rho_{i0}(t) = \lim_{m \to +\infty} \rho_{im}(t) \quad \text{if } a \le t \le b \quad (i = 1, \dots, n).$$

Clearly,

$$(32)\qquad t_{i0} \in \langle a_i, b_i \rangle \quad (i = 1, \dots, n).$$

Passing to the limit in the equations (23) and in the inequalities (26), (30), using (17) we obtain

$$(33)\qquad \rho_{i0}(t) = \rho_{i0}(t_{i0}) + |\int_{t_0}^{t} \rho_{(i+1)0}(\tau)\, d\tau| \quad (i = 1, \dots, n),$$

$$(34)\qquad \rho_{n0}(t) \le \rho_n(t) \quad \text{if } a \le t \le b,$$

where

$$\rho_n(t) = \exp(\int_{t_{n0}}^{t} g(\tau)\,d\tau)\rho_{n0}(t_{n0}) + \tag{35}$$
$$+ \left| \int_{t_{n0}}^{t} [\exp(\int_{\tau}^{t} g(s)\,ds)][\sum_{j=1}^{n} h_j(\tau)\rho_{j0}(\tau)]\,d\tau \right|,$$

and

$$|\rho_{i0}(t_{i0})| \le \psi_i(\rho_{10},\dots,\rho_{n0}) \quad (i = 1,\dots,n). \tag{36}$$

Let us introduce the functions

$$\rho_i(t) = \rho_{i0}(t_{i0}) + \left| \int_{t_{i0}}^{t} \rho_{i+1}(\tau)\,d\tau \right| \quad (i = 1,\dots,n-1). \tag{37}$$

This together with (33) and (34) yields

$$\rho_{i0}(t) \le \rho_i(t),\ \rho_i(t_{i0}) = \rho_{i0}(t_{i0}) \quad (i = 1,\dots,n). \tag{38}$$

Formula (35) gives

$$\rho'(t) = g(t)\rho_n(t) + [\sum_{j=1}^{n} h_j(t)\rho_{j0}(t)]\,\mathrm{sign}(t - t_{n0}). \tag{39}$$

From (32) and (36)–(39) it follows that $(\rho_i)_{i=1}^n$ is a solution of the problem (5), (6). Therefore, according to the condition (7)

$$\rho_i(t) \equiv 0 \quad (i = 1,\dots,n).$$

On the other hand, (29) and (38) imply

$$\sum_{i=1}^{n} \|\rho_i\|_{C^0((a,b))} \ge 1,$$

which is a contradiction and the lemma is proved. ■

PROOF of Theorem 1: Let ρ be a constant from Lemma 1. By (10) there exists $\rho_0 > 0$ such that

$$\rho(r + \int_a^b \omega(t,\rho_0)\,dt) \le \rho_0. \tag{40}$$

Putting

$$\chi(s) = \begin{cases} 1 & \text{if } |s| \le \rho_0 \\ 2 - s/\rho_0 & \text{if } \rho_0 < |s| < 2\rho_0 \\ 0 & \text{if } |s| \ge 2\rho_0 \end{cases}$$

$$\widetilde{f}(t,x_1,\dots,x_n)=\chi(\|x\|)[f(t,x_1,\dots,x_n)-g(t)x_n], \tag{41}$$

$$\widetilde{\phi}_i(u,u',\dots,u^{(n-1)})=\chi(\|u\|_{C^{n-1}(\langle a,b\rangle)})\phi_i(u,u',\dots,u^{(n-1)}), \tag{42}$$
$$(i=1,\dots,n)$$

We consider the problem

$$u^{(n)}(t)=g(t)u^{(n-1)}+\widetilde{f}(t,u,\dots,u^{(n-1)}), \tag{43}$$

$$\phi_{0i}(u^{(i-1)})=\widetilde{\phi}_i(u,u',\dots,u^{(n-1)}) \quad (i=1,\dots,n). \tag{44}$$

From (41) and (42) it follows immediately that $\widetilde{f}:\langle a,b\rangle\times R^n\to R$ satisfies the local Carathéodory conditions, $\widetilde{\phi}_i: C^{n-1}(\langle a,b\rangle)\to R(i=1,\dots,n)$ are continuous functionals,

$$f_0(t)=\sup\{|\widetilde{f}(t,x_1,\dots,x_n)|:(x_i)_{i=1}^n\in R^n\}\in L^1(\langle a,b\rangle) \tag{45}$$

and

$$r_i=\sup\{|\widetilde{\phi}_i(u,u',\dots,u^{(n-1)})|:u\in C^{n-1}(\langle a,b\rangle)\}<+\infty \tag{46}$$
$$(i=1,\dots,n).$$

We want to show that the homogeneous problem

$$v^{(n)}(t)=g(t)v^{(n-1)}(t) \tag{43_0}$$

$$\phi_{0i}(v^{(i-1)})=0 \quad (i=1,\dots,n) \tag{44_0}$$

has only trivial solution. Let v be an arbitrary solution of this problem. Then

$$v^{(n-1)}(t)=c\cdot w(t), \quad \text{where } c=\text{const}.$$

and $w(t)=\exp[\int_a^t g(\tau)\,d\tau]$. According to ($44_0$)

$$c\phi_{0n}(w)=0.$$

However, if ϕ_{0n} is a nondecreasing functional and $\phi_{0n}(1)=1$, we have

$$\phi_{0n}(w)\geq\exp[-\int_a^b|g(t)|\,dt]\phi_{0n}(1)>0.$$

Consequently, $v^{(n-1)}(t)\equiv 0$. Referring to the equation (44_0) and $\phi_{0i}(1)=1\,(i=1,\dots,n-1)$, we come to the conclusion that $v(t)\equiv 0$.

Using 2.1 from [3], we obtain that the condition (45), (46) and the unicity of trivial solution of the problem (43_0), (44_0) guarantee the existence of solutions of the problem (43), (44). Let u be the solution of the problem (43), (44). Then for $t \in \langle a, b\rangle$

$$[u^{(n)}(t) - g(t)u^{(n-1)}(t)] \operatorname{sign} u^{(n-1)}(t) =$$
$$= \widetilde{f}(t, u(t), \dots, u^{(n-1)}(t)) \operatorname{sign} u^{(n-1)}(t) =$$
$$= \chi\Big(\sum_{i=1}^{n} |u^{(i-1)}(t)|\Big)[f(t, u(t), \dots, u^{(n-1)}(t)) - g(t)u^{(n-1)}(t)] \operatorname{sign} u^{(n-1)}(t) \leq$$
$$\leq [f(t, u(t), \dots, u^{(n-1)}(t)) - g(t)u^{(n-1)}(t)] \cdot \operatorname{sign} u^{(n-1)}(t).$$

From here, taking in consideration ($8_{1,2}$) and (9), we obtain inequalities ($12_{1,2}$) and (13), where

$$h_0(t) = \chi\Big(\sum_{i=1}^{n} |u^{(i-1)}(t)|\Big)\omega\Big(t, \Big(\sum_{i=1}^{n} |u^{(i-1)}(t)|\Big) \leq \omega(t, 2\rho_0).$$

Therefore, by Lemma 1 and the inequality (40), we get

$$\|u\|_{C^{n-1}(\langle a,b\rangle)} \leq \rho[r + \int_a^b \omega(t, 2\rho_0)\, dt] < \rho_0.$$

Consequently

$$\chi\Big(\sum_{i=1}^{n} |u^{(i-1)}(t)|\Big) = 1 \quad \text{when } a \leq t \leq b \text{ and}$$
$$\chi(\|u\|_{C^{n-1}(\langle a,b\rangle)}) = 1.$$

Putting these equalities into (41), (42), we obtain that u is a solution of the problem (1), (2). ■

Theorem 2. *Let $(g, h_1, \dots, h_n, \psi_1, \dots, \psi_n) \in \mathrm{Nic}(\langle a, b\rangle; a_1, \dots, a_n, b_1, \dots, b_n)$ and let the data $f, \phi_1, \dots, \phi_n$ of (1), (2) satisfy the inequalities*

(47_1)
$$\{[f(t, x_{11}, \dots, x_{1n}) - f(t, x_{21}, \dots, x_{2n})] - g(t)[x_{1n} - x_{2n}]\}\cdot$$
$$\cdot \operatorname{sign}[x_{1n} - x_{2n}] \leq \sum_{j=1}^{n} h_j(t)|x_{1j} - x_{2j}|$$
for $t \in \langle a_n, b\rangle$, $x_1, x_2 \in R^n$

(47_2)
$$\{[f(t, x_{11}, \dots, x_{1n}) - f(t, x_{21}, \dots, x_{2n})] - g(t)[x_{1n} - x_{2n}]\}\cdot$$
$$\cdot \operatorname{sign}[x_{1n} - x_{2n}] \geq -\sum_{j=1}^{n} h_j(t)|x_{1j} - x_{2j}|$$
for $t \in \langle a_n, b\rangle$, $x_1, x_2 \in R^n$

(48)
$$|\phi_i(u, u', \dots, u^{n-1}) - \phi_i(v, v', \dots, v^{(n-1)})| \leq$$
$$\leq \psi_i(|u - v|, |u' - v'|, \dots, |u^{(n-1)} - v^{(n-1)}|)$$
for $u, v \in C^{n-1}(\langle a, b\rangle)$ $(i = 1, \dots, n)$.

Then the problem (1), (2) has unique solution.

PROOF : From ($47_{1,2}$) and (48) the conditions ($8_{1,2}$) and (9) follow, where $\omega(t,\rho) = |f(t,0,\dots,0)|$ and $r = \max\{|\phi_i(0,\dots,0)| : i = 1,\dots,n\}$. Therefore by Theorem 1 the problem (1), (2) has a solution. We shall prove its uniqueness.

Let u and v be arbitrary solutions of the problem (1), (2). Put

$$\rho_i(t) = u^{(i-1)}(t) - v^{(i-1)}(t) \quad (i = 1,\dots,n).$$

The assumptions ($47_{1,2}$), (48) guarantee that the vector function $(\rho_1,\dots,\rho_n)$ is a solution of the system of the differential inequalities 95) satisfying the conditions

$$|\phi_{0i}(\rho_i)| \le \psi_i(|\rho_1|,\dots,|\rho_n|) \quad (i = 1,\dots,n).$$

However,

$$|\phi_{0i}(\rho_i)| \ge \phi_{0i}(1)\min\{|\rho_i(t)| : a_i \le t \le b_i\} = \min\{|\rho_i(t)| : a_i \le t \le b_i\}.$$

Thus, the inequalities (6) are satisfied and according to the condition (7) the equalities

$$\rho_i(t) \equiv 0 \quad (i = 1,\dots,n)$$

hold, i.e. $u(t) \equiv v(t)$. ∎

Effective criteria.

Theorem 3. *Let the inequalities*

$$(49_1) \qquad f(t,x_1,\dots,x_n)\operatorname{sign} x_n \le \sum_{j=1}^{n} h_j(t)|x_j| + \omega(t, \sum_{i=1}^{n}|x_i|)$$

$$\text{for } t \in \langle a_n, b\rangle, \quad x \in R^n$$

$$(49_2) \qquad f(t,x_1,\dots,x_n)\operatorname{sign} x_n \ge -\sum_{j=1}^{n} h_j(t)|x_j| - \omega(t, \sum_{i=1}^{n}|x_i|)$$

$$\text{for } t \in \langle a, b_n\rangle, \quad x \in R^n$$

$$(50) \qquad |\phi_i(u,u',\dots,u^{(n-1)})| \le \sum_{j=1}^{n} r_{ij}\|u^{(j-1)}\|_{L^q(\langle a,b\rangle)} + r$$

$$\text{for } u \in C^{n-1}(\langle a,b\rangle), \quad (i = 1,\dots,n),$$

hold, where $r, r_{ij} \in R^+ (i,j = 1,\dots,n), \omega : \langle a,b\rangle \times R_+ \to R_+$ is a measurable function nondecreasing in the second variable satisfying (10), $h_i \in L^p(\langle a,b\rangle)$, $h_i \ge 0, p \ge 1$, $1/p + 2/q = 1$ and

$$(51) \qquad s_i = \sum_{m=1}^{n}\{(b-a)^{1/q}\sum_{j=i}^{n}[\frac{2(b-a)}{\pi}]^{\frac{2}{q}(j-i)}(\prod_{k=i}^{j-1}\Delta_k)r_{jm} + $$

$$+ [\frac{2(b-a)}{\pi}]^{\frac{2}{q}(n+1-i)}(\prod_{k=i}^{n-1}\Delta_k)h_{0m}\} < 1 \quad (i = 1,\dots,n),$$

where $\Delta_k \max\{(b-a_k)^{1-\frac{2}{q}}, (b_k-a)^{1-\frac{2}{q}}\}\ (k=1,\dots,n-1)$,

$$h_{0m} = \max\{\|h_m\|_{L^p((a,b_m))}, \|h_m\|_{L^p((a_m,b))}\}\ (m=1,\dots,n).$$

Then the problem (1), (2) has a solution.

Theorem 4. *Let the inequalities*

$$(52_1)\qquad [f(t,x_{11},\dots,x_{1n}) - f(t,x_{21},\dots,x_{2n})]\,\mathrm{sign}[x_{1n}-x_{2n}] \le \le \sum_{j=1}^{n} h_j(t)|x_{1j}-x_{2j}| \quad \textit{for } t\in\langle a_n,b\rangle,\quad x_1,x_2\in R^n$$

$$(52_2)\qquad [f(t,x_{11},\dots,x_{1n}) - f(t,x_{21},\dots,x_{2n})]\,\mathrm{sign}[x_{1n}-x_{2n}] \ge \ge -\sum_{j=1}^{n} h_j(t)|x_{1j}-x_{2j}| \quad \textit{for } t\in\langle a_n,b\rangle,\quad x_1,x_2\in R^n$$

$$(53)\qquad |\phi_i(u,u'\,u^{(n-1)}) - \phi_i(v,v',\dots,v^{(n-1)})| \le \le \sum_{j=1}^{n} r_{ij}\|u^{(j-1)}\|_{L^q((a,b))}$$

$$\textit{for } u,v\in C^{n-1}(\langle a,b\rangle \quad (i=1,\dots,n)$$

hold, where the functions h_i and constant r_{ij} and s_i satisfy the assumptions of Theorem 3.

Then the problem (1), (2) has unique solution.

Proofs of Theorems 3 and 4 are based on the following assertion.

Lemma 2. *Let $g(t)\equiv 0$, $h_i,h_0\ge 0$, $h_i\in L^p((a,b))$, $(i=1,\dots,n)$, $p\ge 1$, $1/p+2/q=1$,*

$$(54)\qquad \psi_i(|u|,|u'|,\dots,|u^{(n-1)}|) = \sum_{j=1}^{n} r_{ij}\|u^{(j-1)}\|_{L^q((a,b))} \quad (i=1,\dots,n),$$

where $r_{ij}\in R^+\ (i,j=1,\dots,n)$ and the condition (51) is satisfied. Then (7) holds.

PROOF : Let the assumptions of the lemma be satisfied. It is clear that the data $(g,h_1,\dots,h_n;\psi_1,\dots,\psi_n)$ are of the class $\mathrm{Nic}(\langle a,b\rangle;a_1,\dots,a_n,b_1,\dots,b_n)$.

Let the vector function $(\rho_1(t),\dots,\rho_n(t))$ be the solution of the problem (5), (6). We shall prove that this solution is zero.

Let us choose $t_i\in\langle a_i,b_i\rangle$ so that

$$(55)\qquad |\rho_i(t_i)| = \min\{|\rho_i(t)| : a_i\le t\le b_i\} \quad (i=1,\dots,n).$$

Then integrating relations (5) and using Hölder inequality we obtain

$$|\rho_i(t)| \le |\rho_i(t_i)| + |\int_{t_i}^{t} |\rho_{i+1}(\tau)|\, d\tau| \le$$
$$\le |\rho_i(t_i)| + |t - t_i|^{1-2/q} |\int_{t_i}^{t} |\rho_{i+1}(\theta)|^{q/2}\, d\tau|^{2/q}$$
$$(i = 1, \dots, n-1),$$

and

$$|\rho_n(t)| \le |\rho_n(t_n)| + \sum_{j=1}^{n} |\int_{t_n}^{t} h_j(\tau)|\rho_j(\tau)|\, d\tau| \le$$
$$\le |\rho_n(t_n)| + \sum_{j=1}^{n} |\int_{t_n}^{t} |h_j(\tau)|^p\, d\tau|^{1/p} \cdot |\int_{t_n}^{t} |\rho_j(\tau)|^{q/2}\, d\tau|^{2/q}.$$

Consequently, using Wirtinger inequality (see e.g. [4], p.409), we obtain

$$\|\rho_i\|_{L^q(\langle a,b\rangle)} \le (b-a)^{1/q} |\rho_i(t_i)| +$$
$$+ [\frac{2(b-a)}{\pi}]^{2/q} \Delta_i \|\rho_{i+1}\|_{L^q(\langle a,b\rangle)} \quad (i = 1, \dots, n-1),$$

$$\|\rho_i\|_{L^q(\langle a,b\rangle)} \le (b-a)^{1/q} \sum_{j=1}^{n-1} [\frac{2(b-a)}{\pi}]^{\frac{2}{q}(j-i)} \prod_{k=i}^{j-1} \Delta_k |\rho_j(t_j)| + \tag{56}$$
$$+ [\frac{2(b-a)}{\pi}]^{\frac{2}{q}(n-i)} \prod_{k=i}^{n-1} \Delta_k \|\rho_n\|_{L^q(\langle a,b\rangle)} \quad (i = 1, \dots, n-1)$$

and

$$\|\rho_n\|_{L^q(\langle a,b\rangle)} \le (b-a)^{1/q} |\rho_n(t_n)| + \tag{57}$$
$$+ \sum_{j=1}^{n} h_{0j} [\int_a^b |\int_{t_n}^{t} |\rho_j(\tau)|^{q/2}\, d\tau|^2\, d\tau]^{1/q} \le$$
$$\le (b-a)^{1/q} |\rho_n(t_n)| + [\frac{2(b-a)}{\pi}]^{2/q} \sum_{j=1}^{n} h_{0j} \|\rho_j\|_{L^q(\langle a,b\rangle)}.$$

Substituting the inequality (57) into (56) we have

$$\|\rho_i\|_{L^q(\langle a,b\rangle)} \le (b-a)^{1/q} \sum_{j=i}^{n} [\frac{2(b-a)}{\pi}]^{\frac{2}{q}(j-i)} (\prod_{k=i}^{j-1} \Delta_k) |\rho_j(t_j)| + \tag{58}$$
$$+ [\frac{2(b-a)}{\pi}]^{\frac{2}{q}(n+1-i)} (\prod_{k=i}^{n-1} \Delta_k) \cdot \sum_{j=1}^{n} h_{0j} \|\rho_j\|_{L^q(\langle a,b\rangle)}$$
$$(i = 1, \dots, n).$$

Applying (6) and (54), we get

$$\|\rho_i\|_{L^q((a,b))} \leq \sum_{m=1}^{n}\{(b-a)^{1/q}\sum_{j=i}^{n}[\frac{2(b-a)}{\pi}]^{\frac{2}{q}(j-i)}(\prod_{k=i}^{j-1}\Delta_k)r_{jm}+ \tag{59}$$

$$+[\frac{2(b-a)}{\pi}]^{\frac{2}{q}(n+1-i)}(\prod_{k=i}^{n-1}\Delta_k)\cdot h_{0m}\}\|\rho_m\|_{L^q((a,b))}$$

$$(i=1,\dots n).$$

Denoting $\rho_0 = \max\{\|\rho_i\|_{L^q((a,b))} : i = 1,\dots,n\}$, we obtain

$$\rho_0 \leq \rho_0 \cdot \max\{s_i : i = 1,\dots,n\}.$$

Since $s_i < 1 (i = 1,\dots,n)$, it follows that $\rho_0 = 0$. Consequently, $\rho_i(t) \equiv 0 (i = 1,\dots,n)$. ■

Proof of Theorem 3 and 4: The assertions of Theorems 3 and 4 immediately follow from Lemma 2 and Theorems 1 and 2. ■

Remark. If $h_j \in L^{p_j}((a,b)), p_j \geq 1, \frac{1}{p_j} + \frac{1}{q_j} = 1, q_j \leq q, g \in L^{p_0}((a,b)), p_0 \geq 1, \frac{1}{p_0} + \frac{1}{q_0} = 1, q_0 \leq q$, then the corresponding conditions for s_i generalizing the inequalities (51) can be derived by means of the inequalities of Levin (see e.g. [2], Lemma 4.7).

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