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Miloslav Feistauer; Harijs Kalis; Mirko Rokyta

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Mathematical modelling of an electrolysis process

MILOSLAV FEISTAUER, HARIJS KALIS, MIRKO ROKYTA

Dedicated to the memory of Svatopluk Fučík

Abstract. The paper is devoted to the mathematical and numerical study of a problem arising in the investigation of the electrolytical producing of aluminium. The electrolysis process is described by the Poisson equation for the stream function to which we add nonlinear Newton boundary and transmission conditions representing turbulent flows in the boundary and anodes layers. The solvability is proved by the use of the monotone operator theory. The problem is discretized by conforming linear triangular elements and the solvability of the discrete problem and the convergence of approximate solutions to the exact solution is studied.

Keywords: electrolysis, linearized Navier–Stokes equations, elliptic boundary value problem, nonlinear Newton and transmission conditions, weak solution, monotone operator theory, linear conforming triangular elements, convergence

Classification: 35D05,35J65,65N30,76D99,76W05

Introduction.

The electrolysis belongs to modern technologies of obtaining aluminium. The motion of the aluminium metal and the electrolyte induced by the electromagnetic forces is described by the Navier–Stokes equations. In [1] it was shown that provided the forces flux is in the range 200 - 250 kA and the thickness of the aluminium – electrolyte layer (0.05 – 0.3 m) is small in comparison with the horizontal size of the equipment (4 – 10 m), then the nonlinear terms can be neglected and the process can be averaged in the vertical direction. Then we come to a two – dimensional model problem in a domain $\Omega \subset R^2$. This domain consists of several subdomains Ω_i, $i = 1, \dots, N$ – for simplicity we shall suppose that $N = 2$ – which represent electrolytical tanks and of the common boundary $(\partial\Omega_1 \cap \partial\Omega_2) \cap \Omega$ representing the channel with anodes (see Fig.1). Let us assume that the flow is laminar in Ω_1 and Ω_2. Then the so–called stream function satisfies a linear Poisson equation in $\Omega_1 \cup \Omega_2$. However, in thin layers near the boundary $\partial\Omega$ and in the channel $\partial\Omega_1 \cap \partial\Omega_2$ of anodes we get turbulent flows (see [13]). These flows need not be resolved and their contribution can be included into a boundary condition on $\partial\Omega$ and a transmission condition on $\partial\Omega_1 \cap \partial\Omega_2$.

As a result we get a boundary value problem in the domain Ω for the stream function, which is discontinuous across $\partial\Omega_1 \cap \partial\Omega_2$ in general, satisfies the Poisson equation in $\Omega_i\,(i = 1, 2)$, nonlinear boundary condition on $\partial\Omega$ and nonlinear transmission condition on $\partial\Omega_1 \cap \partial\Omega_2$.

Here we shall deal with the solvability and the finite element approximation of this problem, provided the domains Ω_i ($i = 1,2$) are polygonal. (More general situation with nonpolygonal domains will be studied in a forthcoming paper [4].)

1. Continuous problem.

Let $\Omega, \Omega_1, \Omega_2 \subset R^2$ be bounded polygonal domains with their boundaries $\partial\Omega, \partial\Omega_1$, $\partial\Omega_2$ and closures $\overline{\Omega}, \overline{\Omega}_1, \overline{\Omega}_2$ satisfying the relations $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, $\Omega_1 \cap \Omega_2 = \emptyset$. We denote $\Gamma_3 = \partial\Omega_1 \cap \partial\Omega_2$ and $\Gamma_i = \partial\Omega_i - \Gamma_3$, $i = 1,2$ (see Fig. 1).

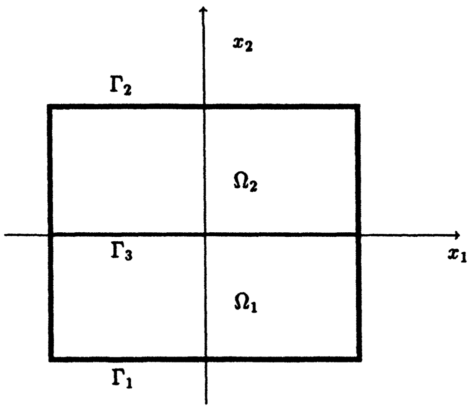

Fig.1

We consider the following boundary value problem: Find $u_i : \overline{\Omega}_i \to R^1$, $i = 1,2$, such that

$$\Delta u_i = \operatorname{div} \vec{f} \quad \text{in } \Omega_i, \quad i = 1,2, \tag{1.1}$$

$$\frac{\partial u_i}{\partial n} + k|u_i|^\alpha u_i = f_n = \vec{f} \cdot \vec{n} \quad \text{on } \Gamma_i, \quad i = 1,2, \tag{1.2}$$

$$\frac{\partial u_1}{\partial n^1} = -\frac{\partial u_2}{\partial n^2} = k|u_2 - u_1|^\alpha (u_2 - u_1) + \vec{f} \cdot \vec{n^1} \quad \text{on } \Gamma_3 \tag{1.3}$$

Here $\vec{f} = (f_1, f_2) : \overline{\Omega} \to R^2$ is a given vector field (determined from Maxwell's equations), $\vec{n} = (n_1, n_2)$ and $\vec{n}^i = (n_1^i, n_2^i)$ denote a unit outer normal to $\partial\Omega$ and to $\partial\Omega_i$, respectively, $k > 0$ and $\alpha \geq 0$ are given constants. (The case $\alpha = 0$ or $\alpha > 0$ corresponds to linear or nonlinear turbulence law, respectively, in the neighbourhood of $\partial\Omega$ and Γ_2.) $\partial/\partial n$ and $\partial/\partial n^i$ denote the derivative in the direction $\vec{n}$ and $\vec{n}^i$, respectively. Of course, $\vec{n}^1 = -\vec{n}^2$ and $\partial/\partial n^1 = -\partial/\partial n^2$ on Γ_3, $\vec{n} = \vec{n}^i$, $\partial/\partial n = \partial/\partial n^i$ on Γ_i, $i = 1,2$.

1.4. Definition. Let $\vec{f} \in [C^1(\overline{\Omega})]^2$. We say that $u = (u_1, u_2)$ is a *classical solution* of the problem (1.1) – (1.3), if $u_i \in C^2(\overline{\Omega}_i)(i = 1, 2)$ satisfy equations (1.1), boundary conditions (1.2) and transmission condition (1.3).

Let us notice that provided $u = (u_1, u_2)$ is a classical solution and we define $\widetilde{u} : \Omega_1 \cup \Omega_2 \to R^1$ by $\widetilde{u} \mid \Omega_i = u_i$, $i = 1, 2$, then in general, $\widetilde{u}$ has a discontinuity across Γ_3 defined together with u by equation (1.1) and conditions (1.2), (1.3). On the other hand, the derivative $\frac{\partial \widetilde{u}}{\partial n^1}$ is "continuous" across Γ_3.

Let $u = (u_1, u_2)$ be a classical solution. If we multiply equation (1.1) by an arbitrary $v_i \in C^\infty(\overline{\Omega}_i)(i = 1, 2)$, integrate (1.1) over Ω_i, apply Green's theorem and use conditions (1.2), (1.3), we get

$$\sum_{i=1}^{2} \int_{\Omega_i} \nabla u_i \cdot \nabla v_i \, dx + \sum_{i=1}^{2} \int_{\Gamma_i} k|u_i|^\alpha u_i v_i \, dS +$$

$$(1.5) \qquad + \int_{\Gamma_3} k|u_2 - u_1|^\alpha (u_2 - u_1)(v_2 - v_1) \, dS = \sum_{i=1}^{2} \int_{\Omega_i} \vec{f} \cdot \nabla v_i \, dx,$$

$$(v_1, v_2) \in C^\infty(\overline{\Omega}_1) \times C^\infty(\overline{\Omega}_2).$$

(Here $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$, $x = (x_1, x_2)$.) Identity (1.5) leads us to the concept of a weak solution of the problem.

We shall deal with the well – known Lebesgue and Sobolev spaces $L^p(\Omega)$, $L^p(\Omega_i), L^p(\partial\Omega)$, $W^{k,p}(\Omega)$, $W^{k,p}(\Omega_i)$ (etc.) $(1 \le p \le \infty, 1 \le k < \infty$, k is an integer), equipped with the norms $\|\cdot\|_{0,p,\Omega}$, $\|\cdot\|_{0,p,\Omega_i}$, $\|\cdot\|_{0,p,\partial\Omega}$, $\|\cdot\|_{k,p,\Omega}$, $\|\cdot\|_{k,p,\Omega_i}$ (etc.), respectively. (See e.g. [10],[11], [14].) By $|\cdot|_{k,p,\Omega}$ we denote the seminorm in $W^{k,p}(\Omega)$:

$$(1.6) \qquad |u|_{k,p,\Omega} = \Big(\sum_{\alpha+\beta=k} \Big\| \frac{\partial^k u}{\partial x_1^\alpha \partial x_2^\beta} \Big\|_{0,p,\Omega}^p \Big)^{1/p}, \quad u \in W^{k,p}(\Omega).$$

Let us remind the completely continuous imbedding $W^{1,2}(\Omega_i) \hookrightarrow\hookrightarrow L^q(\partial\Omega_i)$ for all $q \in [1, +\infty)$ – see [11], [14]. Hence, there exists a constant $c_1 = c_1(q) > 0$ such that

$$(1.7) \qquad \|u\|_{0,q,\partial\Omega_i} \le c_1 \|u\|_{1,2,\Omega_i}, \qquad u \in W^{1,2}(\Omega_i)$$

and from each sequence $\{u_n\}$ bounded in $W^{1,2}(\Omega_i)$ we can choose a subsequence strongly convergent in $L^q(\partial\Omega_i)$.

In the sequel we shall assume that

$$(1.8) \qquad \vec{f} \in [L^2(\Omega)]^2.$$

Let us define the Hilbert space $H(\Omega) = W^{1,2}(\Omega_1) \times W^{1,2}(\Omega_2)$, equipped with the norm

$$(1.9) \qquad \|u\|_{1,2,\Omega} = (\|u_1\|_{1,2,\Omega_1}^2 + \|u_2\|_{1,2,\Omega_2}^2)^{1/2}, \quad u = (u_1, u_2) \in H(\Omega),$$

and define the forms

$$
\begin{aligned}
(1.10)\qquad b(u,v) &= \sum_{i=1}^{2}\int_{\Omega_i} \nabla u_i\cdot\nabla v_i\,dx,\\
c(u,v) &= \sum_{i=1}^{2}\int_{\Gamma_i} k|u_i|^{\alpha}u_i v_i\,dS,\\
d(u,v) &= \int_{\Gamma_3} k|u_2-u_1|^{\alpha}(u_2-u_1)(v_2-v_1)\,dS,\\
L(v) &= \sum_{i=1}^{2}\int_{\Omega_i} \vec{f}\cdot\nabla v_i\,dx,\\
a(u,v) &= b(u,v)+c(u,v)+d(u,v),\\
& u=(u_1,u_2),\quad v=(v_1,v_2)\in H(\Omega).
\end{aligned}
$$

Let us notice that the forms c and d are well–defined in virtue of (1.7).
In $H(\Omega)$ we shall also use a seminorm $|\cdot|_{1,2,\Omega}$:

$$
(1.11)\qquad |u|_{1,2,\Omega} = \left(|u_1|^2_{1,2,\Omega_1} + |u_2|^2_{1,2,\Omega_2}\right)^{1/2},\quad u=(u_1,u_2)\in H(\Omega).
$$

1.12. Definition. We say that $u=(u_1,u_2)$ is a *weak solution* of problem (1.1) – (1.3), if

$$
(1.13)\qquad
\begin{aligned}
&a)\quad u\in H(\Omega),\\
&b)\quad a(u,v)=L(v)\qquad \forall v\in H(\Omega).
\end{aligned}
$$

1.14. Lemma. *The form L is linear and continuous on $H(\Omega)$. For each $u\in H(\Omega)$ the forms $a(u,\cdot)$, $b(u,\cdot)$, $c(u,\cdot)$ and $d(u,\cdot)$ are linear and continuous on $H(\Omega)$. Moreover, b is a continuous bilinear form on $H(\Omega)$.*

From the above considerations it follows that problems (1.1) – (1.3) and (1.13, a-b) are formally equivalent in the following sense: If $u=(u_1,u_2)$ is a classical solution, then it is also a weak solution. On the other hand, provided $u=(u_1,u_2)$ is a weak solution and $u_i\in C^2(\overline{\Omega}_i)$, $i=1,2$, then u is a classical solution.

If $\alpha=0$, then the problem is linear; for $\alpha>0$ we have a nonlinear problem with a similar structure as problems studied in [9] with the use of a variational approach. Here we shall apply the monotone operator method.

2. Solvability.

First, let us prove some auxiliary assertions.

2.1. Lemma. *Let $q\ge 1$. The there exists a constant $c_2=c_2(q)$ such that*

$$
(2.2)\qquad
\begin{aligned}
|u|^2_{1,2,\Omega} + \|u\|^{2-q}_{1,2,\Omega}\left(\sum_{i=1}^{2}\|u_i\|^q_{0,q,\Gamma_i} + \|u_1-u_2\|^q_{0,q,\Gamma_3}\right) &\ge\\
\ge c_2\|u\|^2_{1,2,\Omega}&\\
\forall u=(u_1,u_2)\in H(\Omega),\qquad u\neq 0.&
\end{aligned}
$$

PROOF : a) Let us prove the existence of a constant $c_2 > 0$ such that

$$(2.3)\quad |u|^2_{1,2,\Omega} + \sum_{i=1}^{2} \|u_i\|^q_{0,q,\Gamma_i} + \|u_1 - u_2\|^q_{0,q,\Gamma_s} \geq c_2$$
$$\forall u = (u_1, u_2) \in H(\Omega), \quad \|u\|_{1,2,\Omega} = 1.$$

If (2.3) is not valid, we get a sequence $\{u^n\} \subset H(\Omega)$ such that

$$(2.4)\quad \begin{aligned} &a) \quad \|u^n\|_{1,2,\Omega} = 1 \\ &b) \quad u^n \rightharpoonup u = (u_1, u_2) \quad \text{(weakly) in } H(\Omega), \\ &c) \quad |u^n|^2_{1,2,\Omega} + \sum_{i=1}^{2} \|u^n_i\|^q_{0,q,\Gamma_i} + \|u^n_1 - u^n_2\|^q_{0,q,\Gamma_s} \leq \frac{1}{n}. \end{aligned}$$

From the compact imbedding $W^{1,2}(\Omega_i) \hookrightarrow\hookrightarrow L^q(\partial\Omega_i)$ and (2.4,b) it follows that

$$u^n_i \to u_i \qquad \text{(strongly) in } L^q(\partial\Omega_i),\ i = 1,2.$$

From this, the weak lower semicontinuity of the seminorm $|\cdot|_{1,2,\Omega}$ and (2.4,c) we immediately get

$$|u|^2_{1,2,\Omega} + \sum_{i=1}^{2} \|u_i\|^q_{0,q,\Gamma_i} + \|u_1 - u_2\|^q_{0,q,\Gamma_s} = 0.$$

Thus, $u_i = k_i = \text{const}$ for $i = 1,2$. Of course, also the traces $u_i \mid \partial\Omega_i = k_i$. As $\|u_i\|_{0,q,\Gamma_i} = 0$ we see that $k_i = 0$ for $i = 1,2$ and thus $u = 0$. However, this is a contradiction to (2.4,a).

b) Now, if $u \in H(\Omega)$, $u \neq 0$, we put $w = u/\|u\|_{1,2,\Omega}$ and, by (2.3), we have

$$\frac{|u|^2_{1,2,\Omega}}{\|u\|^2_{1,2,\Omega}} + \frac{1}{\|u\|^q_{1,2,\Omega}} \left(\sum_{i=1}^{2} \|u_i\|^q_{0,q,\Gamma_i} + \|u_1 - u_2\|^q_{0,q,\Gamma_s} \right) \geq c_2$$

If we multiply this inequality by $\|u\|^2_{1,2,\Omega}$, we get (2.2). ∎

2.5. Lemma. *The form a is coercive in the following sense: there exists a constant* $c_3 > 0$ *such that*

$$(2.6)\quad a(u,u) \geq c_3 \|u\|^2_{1,2,\Omega} \quad \textit{for all} \quad u \in H(\Omega) \quad \textit{with} \quad \|u\|_{1,2,\Omega} \geq 1.$$

PROOF : If $u = (u_1, u_2) \in H(\Omega)$, then by (1.10),

$$(2.7)\quad a(u,u) = |u|^2_{1,2,\Omega} + k \sum_{i=1}^{2} \int_{\Gamma_i} |u_i|^{\alpha+2}\, dS + k \int_{\Gamma_s} |u_1 - u_2|^{\alpha+2}\, dS \geq$$
$$\geq \min(1,k) \Big[|u|^2_{1,2,\Omega} + \sum_{i=1}^{2} \|u_i\|^{\alpha+2}_{0,\alpha+2,\Gamma_i} +$$
$$+ \|u_1 - u_2\|^{\alpha+2}_{0,\alpha+2,\Gamma_s} \Big].$$

Now let us assume that $\|u\|_{1,2,\Omega} \geq 1$ and put $q = \alpha + 2\,(\geq 2)$. Then, from (2.2) and (2.7) we immediately get

$$a(u,u) \geq \min(1,k)c_2\|u\|^2_{1,2,\Omega},$$

which is (2.6). ■

2.8. Corollary. *There exists a constant* $c_4 = \max(1, \|\vec{f}\|_{0,2,\Omega}/c_3)$ *such that*

$$\|u\|_{1,2,\Omega} \leq c_4 \tag{2.9}$$

for each solution u of problem (1.13, a-b).
(We set $\|\vec{f}\|_{0,2,\Omega} = (\sum_{j=1}^{2} \|f_j\|^2_{0,2,\Omega})^{1/2}$.*)*

PROOF : Let $u \in H(\Omega)$ be a solution of (1.13, a-b) and $\|u\|_{1,2,\Omega} \geq 1$. Then, by (2.6), (1.13, b), (1.10) and the Cauchy inequality,

$$c_3\|u\|^2_{1,2,\Omega} \leq a(u,u) = L(u) \leq \|\vec{f}\|_{0,2,\Omega} \cdot \|u\|_{0,2,\Omega}.$$

Hence, each solution of (1.13, a-b) satisfies (2.9). ■

2.10. Lemma. *The form a is strictly monotone:*

$$a(u,u-v) - a(v,u-v) > 0 \qquad \text{for all} \quad u,v \in H(\Omega),\ u \neq v. \tag{2.11}$$

PROOF : By (1.10), for $u,v \in H(\Omega)$ we get

$$\begin{aligned} &a(u,u-v) - a(v,u-v) = \\ &= |u-v|^2_{1,2,\Omega} + k\sum_{i=1}^{2}\int_{\Gamma_i}(|u_i|^\alpha u_i - |v_i|^\alpha v_i)(u_i - v_i)\,dS + \\ &+k\int_{\Gamma_3}[|u_2-u_1|^\alpha(u_2-u_1) - |v_2-v_1|^\alpha(v_2-v_1)]\,[(u_2-u_1)-(v_2-v_1)]\,dS. \end{aligned} \tag{2.12}$$

From this and the fact that the function "$t \in R^1 \to |t|^\alpha t \in R^1$" is increasing we see that $a(u,u-v) - a(v,u-v) \geq 0$.

If $a(u,u-v) - a(v,u-v) = 0$, then all three terms in the right-hand side of (2.12) are equal to zero. This implies that $u_i - v_i = k_i = \text{const}$ almost everywhere in Ω_i and $u_i = v_i$ on Γ_i a.e. for $i = 1,2$. Hence, $k_i = 0$ and $u = v$ almost everywhere. ■

2.13. Lemma. *There exists a constant* $c_5 > 0$ *such that*

$$\begin{aligned} &|a(u,v) - a(w,v)| \leq \\ &\leq c_5(1 + \|u\|^\alpha_{1,2,\Omega} + \|w\|^\alpha_{1,2,\Omega})\|u-w\|_{1,2,\Omega}\|v\|_{1,2,\Omega} \\ &\qquad \forall u,v,w \in H(\Omega). \end{aligned} \tag{2.14}$$

PROOF : From the definition of the form a we get

$$|a(u,v)-a(w,v)| \le |u-w|_{1,2,\Omega}|v|_{1,2,\Omega}+ \\ +k\sum_{i=1}^{2}\int_{\Gamma_i}\big||u_i|^\alpha u_i-|w_i|^\alpha w_i\big|\,|v_i|\,dS+ \\ +k\int_{\Gamma_3}\big||u_2-u_1|^\alpha(u_2-u_1)-|w_2-w_1|^\alpha(w_2-w_1)\big|\,|v_2-v_1|\,dS. \tag{2.15}$$

Let $r,s\in R^1$ and $\varphi(t)=|r+t(s-r)|^\alpha(r+t(s-r))$, $t\in[0,1]$. By a simple calculation we find out that

$$\varphi'(t)=(1+\alpha)(s-r)|r+t(s-r)|^\alpha$$

and thus,

$$|s|^\alpha s-|r|^\alpha r=\varphi(1)-\varphi(0)=\int_0^1\varphi'(t)\,dt= \\ =(1+\alpha)(s-r)\int_0^1|r+t(s-r)|^\alpha\,dt. \tag{2.16}$$

From the properties of the function $|x|^\alpha$ we can derive that $|r+t(s-r)|^\alpha\le$ $\le|r|^\alpha+|s|^\alpha$ $\forall t\in[0,1]$, which together with (2.16) imply

$$\big||s|^\alpha s-|r|^\alpha r\big|\le(1+\alpha)|s-r|(|r|^\alpha+|s|^\alpha). \tag{2.17}$$

If we use (2.15) and (2.17), we get

$$|a(u,v)-a(w,v)|\le|u-w|_{1,2,\Omega}|v|_{1,2,\Omega}+ \\ +k(1+\alpha)\sum_{i=1}^{2}\int_{\Gamma_i}|u_i-w_i|(|u_i|^\alpha+|w_i|^\alpha)|v_i|\,dS+ \\ +k(1+\alpha)\int_{\Gamma_3}|(u_2-u_1)-(w_2-w_1)|\,(|u_2-u_1|^\alpha+|w_2-w_1|^\alpha)\,|v_2-v_1|\,dS. \tag{2.18}$$

Further, let $\alpha/(\alpha+2)+1/p=1$, $\varphi,\psi\in L^{2p}(\Gamma_i)$, $\vartheta\in L^{\alpha+2}(\Gamma_i)$. Then

$$\int_{\Gamma_i}|\varphi|\,|\vartheta|^\alpha|\psi|\,dS\le \\ \le\left(\int_{\Gamma_i}|\vartheta|^{\alpha+2}\,dS\right)^{\frac{\alpha}{\alpha+2}}\left(\int_{\Gamma_i}|\varphi|^{2p}\,dS\right)^{\frac{1}{2p}}\left(\int_{\Gamma_i}|\psi|^{2p}\,dS\right)^{\frac{1}{2p}}= \\ =\|\vartheta\|^\alpha_{0,\alpha+2,\Gamma_i}\cdot\|\varphi\|_{0,2p,\Gamma_i}\cdot\|\psi\|_{0,2p,\Gamma_i}. \tag{2.19}$$

Now, in virtue of the continuous imbedding $W^{1,2}(\Omega_i)\hookrightarrow L^q(\partial\Omega_i)$ valid for $q\in[1,\infty)$ (cf. (1.7)), for which we set the values $\alpha+2$ and $2p$, we derive from (2.18) and (2.19) (after some calculations) the estimate (2.14). ∎

In view of Lemma 1.14 let us define the mapping $A : H(\Omega) \to (H(\Omega))^*$ and the functional $\varphi \in (H(\Omega))^*$ by the identities

$$(2.20) \qquad \begin{aligned} \langle A(u), v\rangle &= a(u,v),\\ \langle \varphi, v\rangle &= L(v),\\ & u, v \in H(\Omega). \end{aligned}$$

Here $(H(\Omega))^*$ denotes the dual to $H(\Omega)$ and $\langle\cdot,\cdot\rangle$ is the duality between $(H(\Omega))^*$ and $H(\Omega)$. I.e. $\langle\varphi, v\rangle$ denotes the value of a continuous linear functional φ defined on $H(\Omega)$ at a point $v \in H(\Omega)$.

Under this notation problem (1.13, a-b) can be written as the operator equation

$$(2.21) \qquad A(u) = \varphi$$

for an unknown $u \in H(\Omega)$. From Lemmas 2.5, 2.10 and 2.13 we immediately get

2.22. Lemma. *The operator A is coercive, strictly monotone and locally Lipschitz-continuous on $H(\Omega)$.*

By the straightforward application of the well-known results from the monotone operator theory ([8], [12], [15], [16]) we come to the following

2.23. Theorem. *Problem (1.13, a-b) has exactly one solution.*

3. Discrete problem.

For the discretization of the continuous problem we use the finite element method and proceed similarly as in [5], where a problem with discontinuous coefficients was studied.

Let $\mathcal{T}_h$ and $\mathcal{T}_{ih}$ denote triangulations of the domains Ω and Ω_i ($i = 1,2$), respectively, formed by finite numbers of closed triangules. (Let us remind that Ω and Ω_i are supposed to be polygonal.) We assume that

(3.1) a) $\mathcal{T}_h = \bigcup_{i=1}^2 \mathcal{T}_{ih}$,
b) $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} T$, $\quad \overline{\Omega}_{ih} = \bigcup_{T\in\mathcal{T}_{ih}} T$;

(3.2) if $T_1, T_2 \in \mathcal{T}_h$, $T_1 \neq T_2$, then either $T_1 \cap T_2 = \emptyset$ or $T_1 \cap T_2$ is a common vertex or $T_1 \cap T_2$ is a common side of T_1, T_2;

(3.3) if $T \in \mathcal{T}_{ih}(i = 1,2)$, then at most two vertices of T are lying on $\partial\Omega_i$.

We denote by $\sigma_h = \{P_1, \ldots, P_N\}$ and $\sigma_{ih} = \{P_1^i, \ldots, P_{N^i}^i\}$ ($i = 1,2$) the set of all vertices of $\mathcal{T}_h$ and $\mathcal{T}_{ih}$, respectively. From the above it follows that

$$(3.4) \qquad \begin{aligned} &a) \quad \sigma_h \subset \overline{\Omega}, \qquad \sigma_{ih} \subset \overline{\Omega}_{ih}, \quad i = 1,2;\\ &b) \quad \sigma_h = \bigcup_{i=1}^{2} \sigma_{ih},\\ &c) \quad \Gamma_3 \cap \Gamma_i \subset \sigma_h, \qquad i = 1,2,\\ &d) \quad \sigma_h \cap \Gamma_3 \subset \sigma_{ih}, \qquad i = 1,2. \end{aligned}$$

Let us notice that the vertices from $\sigma_k \cap \Gamma_3$ are considered twice: as elements of $\overline{\Omega}_1$ and of $\overline{\Omega}_2$.

By h_T and ϑ_T we shall denote the length of the maximal side and the magnitude of the minimal angle of $T \in \mathcal{T}_k$, respectively. We set

$$(3.5) \qquad h = \max_{T \in \mathcal{T}_k} h_T, \qquad \vartheta_k = \min_{T \in \mathcal{T}_k} \vartheta_T.$$

Approximate solutions to problem (1.13, a-b) will be sought in a finite–dimensional space of triangular conforming piecewise linear elements $H_k \subset H(\Omega)$:

$$(3.6) \qquad \begin{aligned} H_k &= X_{1k} \times X_{2k}, \\ X_{ik} &= \{v_{ik};\ v_{ik} \in C(\overline{\Omega}_i),\ v_{ik} \mid T \text{ is affine for each} \\ &\qquad T \in \mathcal{T}_{ik}\},\ i = 1, 2. \end{aligned}$$

Test functions $v = (v_1, v_2)$ in (1.13, b) will be approximated by elements $v_k = (v_{1k}, v_{2k}) \in H_k$. It is evident that $\nabla v_{ik} \mid T = \text{const}$ for each $v_{ik} \in X_{ik}$ and $T \in \mathcal{T}_{ik}$.

Since the form of the vector field $\vec{f}$ can be general, it is suitable to use *numerical integration* for evaluating $L(v_k)$ for $v_k \in H_k$. Let us assume that

$$(3.7) \qquad \vec{f} \in [W^{1,\infty}(\Omega)]^2.$$

Then, of course, $\vec{f} \in [C(\overline{\Omega})]^2$. We write

$$(3.8) \qquad a) \quad \int_{\Omega_i} F\, dx = \sum_{T \in \mathcal{T}_{ik}} \int_T F\, dx,$$

$$b) \quad \int_T F\, dx \approx \text{meas}(T) \sum_{k=1}^{k_T} \omega_{T,k} F(x_{T,k}), \quad \text{if } F \in C(T).$$

Here $x_{T,k} \in T$ and $\omega_{T,k} \in R^1$. Let us assume that

(3.9) the degree of precision of formula (3.8,b) is $d \geq 1$.

If we approximate $L(v_k)$ by (3.8, a-b), we get

$$(3.10) \qquad L_k(v_k) = \sum_{i=1}^{2} \sum_{T \in \mathcal{T}_{ik}} \nabla v_{ik} \mid T \cdot \sum_{k=1}^{k_T} \omega_{T,k} \vec{f}(x_{T,k}).$$

Let us deal with the forms c and d: If $u_k = (u_{1k}, u_{2k})$, $v_k = (v_{1k}, v_{2k}) \in H_k$, then we can write

$$(3.11) \qquad c(u_k, v_k) = \sum_{i=1}^{2} \int_{\Gamma_i} k |u_{ik}|^\alpha u_{ik} v_{ik}\, dS = k \sum_{i=1}^{2} \sum_{m=1}^{M_i} \int_{\Gamma_i^m} |u_{ik}|^\alpha u_{ik} v_{ik}\, dS$$

where Γ_i^m, $m = 1, \dots, M_i$ denote all sides of triangles T adjacent to Γ_i such that $\Gamma_i^m \subset \Gamma_i$. From the definition of the space H_h it follows that $u_{ih} \mid \Gamma_i^m$ and $v_{ih} \mid \Gamma_i^m$ are linear and hence, it is possible to calculate the integrals $\int_{\Gamma_i^m} |u_{ih}|^\alpha u_{ih} v_{ih}\, dS$ exactly. Similar holds for the integrals in the form d. Therefore, we shall *suppose that the values $c(u_h, v_h)$ and $d(u_h, v_h)$ are calculated exactly.* (Let us remark that provided $\alpha = 0$ or $\alpha = 1$, the functions $|u_{ih}^\alpha| u_{ih}\, v_{ih}$ are on Γ_i piecewise quadratic or cubic, respectively, and the integrals over Γ_i can be evaluated exactly with the use of suitable numerical quadratures.)

Now, the *discrete problem* can be written quite analogously as continuous problem (1.13, a-b): Find $u_h = (u_{1h}, u_{2h})$, $u_{ih} : \overline{\Omega_i} \to R^1$ such that

$$\text{(3.12)} \qquad a) \quad u_h \in H_h,$$
$$b) \quad a(u_h, v_h) = L_h(v_h) \qquad \forall v_h \in H_h.$$

3.13. Theorem. *Discrete problem (3.12, a-b) has a unique solution u_h.*

PROOF is an easy consequence of Lemmas 1.14, 2.5, 2.10, 2.13 and [12, Chap.1, Lemma 4.3]. ∎

4. Convergence.

Let $\{\mathcal{T}_h\}_{h \in (0,h_0)}$ be a *regular* system of triangulations of $\Omega\,(h_0 > 0)$. I.e., there exists a constant $\vartheta_0 > 0$ such that $\vartheta_h \ge \vartheta_0$ for all $h \in (0, h_0)$. We shall study the behaviour of approximate solutions u_h, if $h \to 0+$.

In virtue of results from [2, Chap. 4] (cf. also [6, Th.2.2.4]) we get the following

4.1. Lemma. *Under assumptions (3.7) and (3.9) there exists a constant $c_6 > 0$ such that*

$$\text{(4.2)} \qquad |L(v_h) - L_h(v_h)| \le c_6 h \|v_h\|_{1,\Omega_h} \qquad \forall v_h \in H_h,\ \forall h \in (0, h_0).$$

4.3. Lemma. *Solutions u_h of discrete problems (3.12, a-b) satisfy the estimate*

$$\text{(4.4)} \qquad \|u_h\|_{1,2,\Omega} \le \hat{c}_4 \qquad \forall h \in (0, h_0),$$

where $\hat{c}_4 > 0$ is a constant independent of h.

PROOF : Let $h \in (0, h_0)$ and u_h be the solution of (3.12, a-b). If $\|u_h\|_{1,2,\Omega} \ge 1$, then by (2.6), (3.12, b) and (4.2), similarly as in the proof of 2.8, we get

$$\begin{aligned} c_3 \|u_h\|^2_{1,2,\Omega} &\le a(u_h, u_h) = L_h(u_h) \le \\ &\le |L_h(u_h) - L(u_h)| + |L(u_h)| \le \\ &\le (c_6 h + \|\vec{f}\|_{0,2,\Omega}) \|u_h\|_{1,2,\Omega}. \end{aligned}$$

Hence, (4.4) is valid with $\hat{c}_4 = \max(1, (c_6 h_0 + \|\vec{f}\|_{0,2,\Omega})/c_3)$. ∎

The *convergence* of approximate solutions to the exact solution, if $h \to 0+$, can be proved with the use of the monotone operator theory. Here we give a very simple proof based on the compactness method ([3], [7]).

Let $\{h_m\}_{m=1}^{\infty} \subset (0, h_0)$, $h_m \to 0+$. On the basis of (4.4) and the reflexivity of the space $H(\Omega)$ we can choose a subsequence $\{h_n\} \subset \{h_m\}$ such that

$$(4.5) \qquad u_{h_n} = (u_{1h_n}, u_{2h_n}) \rightharpoonup u = (u_1, u_2) \quad \text{weakly in } H(\Omega).$$

4.6. Theorem. *If $h_n \to 0+$ and (4.5) is valid, then $u_{h_n} \to u$ (strongly) in $H(\Omega)$ and u is a solution of (1.13, a-b).*

Proof : Let $v = (v_1, v_2) \in C^\infty(\overline{\Omega}_1) \times C^\infty(\overline{\Omega}_2)$. By v_h let us denote the H_h – interpolation of v. I.e., $v_h = (r_{1h}v_1,\ r_{2h}v_2)$, where $r_{ih} : W^{1,2}(\Omega_i) \cap C(\overline{\Omega}_i) \to X_{ih}$ is the Lagrange interpolation operator: if $v_i \in W^{1,2}(\Omega_i) \cap C(\overline{\Omega}_i)$, then

$$(4.7) \qquad \begin{aligned} &r_{ih}v_i \in X_{ih}, \\ &(r_{ih}v_i)(P_j^i) = v_i(P_j^i) \qquad \forall P_j^i \in \sigma_{ih}. \end{aligned}$$

In virtue of the well-known approximation results ([2, Th.3.2.1])

$$(4.8) \qquad v_h \to v \qquad \text{(strongly) in } H(\Omega).$$

From (4.5) and the compact imbedding $W^{1,2}(\Omega_i) \hookrightarrow\hookrightarrow L^q(\Gamma_i \cup \Gamma_3)\,(i = 1, 2,\ q \geq 1)$ we have

$$(4.9) \qquad u_{ih_n} \to u_i \qquad \text{in } L^q(\Gamma_i \cup \Gamma_3).$$

Of course, also

$$(4.10) \qquad v_{ih} \to v_i \qquad \text{in } L^q(\Gamma_i \cup \Gamma_3).$$

Now, for each $h := h_n$ we shall use relation (3.12, b) with v_{h_n} defined above and write it in the form

$$(4.11) \qquad \begin{aligned} &b(u_{h_n}, v_{h_n}) + c(u_{h_n}, v_{h_n}) + d(u_{h_n}, v_{h_n}) = \\ &= (L_{h_n}(v_{h_n}) - L(v_{h_n})) + L(v_{h_n}). \end{aligned}$$

Let us study particular terms in (4.11), if $h_n \to 0$. As b is a continuous bilinear form on the Hilbert space $H(\Omega)$, from (4.5) and (4.8) we immediately get

$$(4.12) \qquad b(u_{h_n}, v_{h_n}) \to b(u, v).$$

It is evident that

$$(4.13) \qquad L(v_{h_n}) \to L(v)$$

and, in view of Lemma 4.1 and the boundedness of $\{v_{h_n}\}$

$$(4.14) \qquad |L_{h_n}(v_{h_n}) - L(v_{h_n})| \leq ch_n \|v_{h_n}\|_{1,2,\Omega} \to 0.$$

Further, using the same technique as in the proof of Lemma 2.13, we find out that

$$|c(u_{h_n}, v_{h_n}) - c(u,v)| \le k \sum_{i=1}^{2} \int_{\Gamma_i} \Big\{ |u_{ih_n} - u_i|(|u_{ih_n}|^\alpha + |u_i|^\alpha)|v_{ih_n}|(1+\alpha) + $$
$$+ |u_i|^{\alpha+1}|v_{ih_n} - v_i| \Big\} \, dS \le$$

$$\le \text{const} \sum_{i=1}^{2} \Big\{ \|u_{ih_n} - u_i\|_{0,2p,\Gamma_i} \|v_{ih_n}\|_{0,2p,\Gamma_i} \Big(\|u_i\|^\alpha_{0,\alpha+2,\Gamma_i} + \tag{4.15}$$
$$+ \|u_{ih_n}\|^\alpha_{0,\alpha+2,\Gamma_i} \Big) + \|u_i\|^{\alpha+1}_{0,\alpha+2,\Gamma_i} \|v_{ih_n} - v_i\|_{0,q,\Gamma_i} \Big\} \to 0,$$
$$\left(\text{where } \quad \frac{\alpha}{\alpha+2} + \frac{1}{p} = 1, \quad \frac{\alpha+1}{\alpha+2} + \frac{1}{q} = 1\right),$$

as it follows from (4.9), (4.10) and the boundedness of the sequence $\{u_{h_n}\}$. Similarly we prove that

$$d(u_{h_n}, v_{h_n}) \to d(u,v). \tag{4.16}$$

Summarizing (4.12) – (4.16), we see that $a(u,v) = L(v)$. Since $C^\infty(\overline{\Omega}_1) \times C^\infty(\overline{\Omega}_2)$ is dense in $H(\Omega)$, the function u satisfies (1.13, b) and hence, it is a sought weak solution.

Now let us prove the strong convergence $u_{h_n} \to u$ in $H(\Omega)$. By (1.10) and (3.12, b),

$$|u_{h_n} - u|^2_{1,2,\Omega} \le b(u_{h_n} - u, u_{h_n} - u) = \tag{4.17}$$
$$= b(u_{h_n}, u_{h_n}) - b(u_{h_n}, u) - b(u, u_{h_n} - u) =$$
$$= L(u_{h_n}) - c(u_{h_n}, u_{h_n}) - d(u_{h_n}, u_{h_n}) - b(u_{h_n}, u) - b(u, u_{h_n} - u).$$

In virtue of (4.9) ($q \ge 1$), similarly as above, we find out that

$$\begin{aligned} c(u_{h_n}, u_{h_n}) &\to c(u,u) \\ d(u_{h_n}, u_{h_n}) &\to d(u,u) \end{aligned} \tag{4.18}$$

Further, by (4.5),

$$\begin{aligned} &b(u_{h_n}, u) \to b(u,u) \\ &b(u, u_{h_n} - u) \to 0, \qquad L(u_{h_n}) \to L(u). \end{aligned} \tag{4.19}$$

As we have already proved, u is a solution of (1.13, a-b) and hence,

$$0 = L(u) - c(u,u) - d(u,u) - b(u,u). \tag{4.20}$$

Now, from (4.17) – (4.20) it follows that $|u_{h_n} - u|_{1,2,\Omega} \to 0$. Moreover, since $W^{1,2}(\Omega_i) \hookrightarrow\hookrightarrow L^2(\Omega_i)\,(i = 1,2)$, we have $u_{ih_n} \to u_i$ (strongly) in $L^2(\Omega_i)$ and thus, $u_{h_n} \to u$ in $H(\Omega)$. ∎

If we take into account that the solution u of problem (1.13, a-b) is unique, we come to the following convergence result:

4.21. Theorem. *It holds:*

$$\lim_{h\to 0+} u_h = u \qquad in\ H(\Omega).$$

4.22. Remark. Since the operator A is not strongly monotone we are not able to prove the error estimate (even if $u_i \in W^{2,2}(\Omega_i)$, $i = 1,2$). In case of a nonpolygonal domain we get similar results. However, the convergence proof is more technical. This will be contained in a forthcoming paper [4], where also methods for the solution of the discrete problem will be treated.

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M.Feistauer and M.Rokyta: Mathematical Institute, Charles University, Sokolovská 83, 186 00 Praha 8, Czechoslovakia;

H.Kalis: Latvian State University, boulevard Rainis 29, 226 050 Riga, USSR

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