Jan Slovák

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Commentationes Mathematicae Universitatis Carolinae, Vol. 30 (1989), No. 2, 389--393

Persistent URL: http://dml.cz/dmlcz/106757

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JAN SLOVÁK

Abstract. The aim of this note is to present a simple proof of the uniqueness of the Levi-Cività connection among all natural first order connections on Riemannian manifolds, without any additional assumptions. We also consider naturality with respect to orientation preserving isometries and we show that in this case the dimensions three and one are exceptional.

Keywords: natural bundles, natural operators, natural connections

Classification: 58A20, 53A55

The naturality of the Levi-Cività connection is a well known topic discussed by several authors, cf. [1], [2]. In [1], a natural connection is a rule associating a connection $\Gamma_{(M,g)}$ to each Riemannian manifold (M,g) such that for any local isometry $f:(M,g)\to(M',g')$, $f^*\Gamma_{(M',g')}=\Gamma_{(M,g)}$. One of the results in [1] reads, that the Levi-Cività connection is the only first order natural connection under some polynomiality assumptions. An equivalent, and probably more convenient, setting is the framework of natural bundles and natural operators due to A. Nijenhuis,[4]. This is pointed out in [2] and the uniqueness result is presented there without any polynomial assumption, but the authors assume the symmetry of the resulting connection. Their proof is based on the so called method of differential equations and involves rather technical manipulation with partial differential equations.

Let us recall that non-degenerate symmetric 2-forms of signature $0\le s\le m$ on an m-dimensional manifold M are sections of the value R_sM of a first order natural subbundle of the natural vector bundle $T^{(0,2)}$, and the linear connections on M are sections of the value QM of a second order natural affine bundle Q. By definition, a natural operator $\Gamma:R_s\to Q$ is a system of operators $\Gamma_M:C^\infty(R_sM)\to C^\infty(QM)$ transforming pseudometrics into linear connections, such that for all manifolds M, M', for every local diffeomorphism $f:M\to M'$, and for every pseudometric g it holds $\Gamma_M(f^*g)=f^*(\Gamma_{M'}g)$. In order to describe all natural first order operators $\Gamma:R_s\to Q$ we need to find all G^2_m-equivariant maps between the standard fibres of the first jet prolongation $J^1R_s\mathbf{R}^m$ and of $Q\mathbf{R}^m$, where $G^2_m=\operatorname{inv}J^2_0(\mathbf{R}^m,\mathbf{R}^m)_0$. But remember, that if g is a pseudo-metric of signature s on $\mathbf{R}^m$, then in every normal coordinates at a point $x\in\mathbf{R}^m$

$$j^1g(x)=j^1g_s(x)\ ,$$

where $g_s=\sum_{i=1}^{m-s}(dx^i)^2-\sum_{m-s+1}^{m}(dx^i)^2$ is the canonical metric of signature s. Therefore by the definition of naturality we have

Lemma 1. *If two first order natural operators on pseudometrics of signature s have the same value on the first order jet of the canonical pseudometric on $\mathbf{R}^m$ at $0 \in \mathbf{R}^m$, then they are equal.*

In our considerations, we shall use only the induced G^1_m-equivariancy of the mappings between the standard fibres. So let us recall that the standard fibres of $R_s M$, or QM, with the induced action of the linear group $GL(m, \mathbf{R}) = G^1_m$, $m = \dim M$, are the non-degenerate bilinear forms of signature s in $\mathbf{R}^{m*} \otimes \mathbf{R}^{m*}$, or the affine space $\mathbf{R}^m \otimes \mathbf{R}^{m*} \otimes \mathbf{R}^{m*}$, respectively.

Proposition 1. *Let $T^{(p,q)}$ be the natural vector bundle of p-contravariant and q-covariant tensors and let $\Gamma : R_s \to T^{(p,q)}$ be a first order natural operator. If $p-q$ is odd, then Γ is the zero operator, i.e. every pseudometric is mapped to the identically zero tensor.*

PROOF : Let $\gamma^{i_1\dots i_p}_{j_1\dots j_q}(j^1_0 g)$ be the coordinate expression of the G^2_m-equivariant map corresponding to Γ, g_s be the canonical metric with signature s. By Lemma 1, it suffices to prove that $\gamma^{i_1,\dots,i_p}_{j_1,\dots,j_q}(j^1_0 g_s) = 0$. The induced G^1_m-equivariancy means that for every $A = (a^i_j) \in G^1_m$, $A^{-1} = (\tilde{a}^i_j)$, it holds

$$a^{i_1}_{l_1} \cdots a^{i_p}_{l_p} \gamma^{l_1\dots l_p}_{r_1\dots r_q}(g_{ij}, 0)\tilde{a}^{r_1}_{j_1} \cdots \tilde{a}^{r_q}_{j_q} = \gamma^{i_1\dots i_p}_{j_1\dots j_q}(g_{rs}\tilde{a}^r_i \tilde{a}^s_j, 0). \tag{1}$$

Now, let us fix some indices $i_1, \dots, i_p$, $j_1, \dots, j_q$ and let

$$n(i) = \sum_{k=1}^{p} \delta^i_{i_k} + \sum_{k=1}^{q} \delta^i_{j_k} .$$

Since $p - q$ is odd, there is some index k with $n(k)$ odd. We choose such k and consider the transformation $a^i_j = (-1)^{\delta^k_i} \delta^i_j$, i.e. we change just the orientation of the k-th axis. Such transformations belong to the isotropy group of the canonical metric g_s, so that (1) implies

$$(-1)^{n(k)} \gamma^{i_1\dots i_p}_{j_1\dots j_q}(j^1_0 g_s) = \gamma^{i_1\dots i_p}_{j_1\dots j_q}(j^1_0 g_s).$$

Since $n(k)$ is odd, the proposition follows. ∎

Corollary 1. *The Levi-Cività connection is the only first order natural connection.*

PROOF : Since the connections are sections of a natural affine bundle with the associated natural vector bundle $T^{(1,2)}$, the difference of every two natural connections is a natural tensor and therefore the zero operator. ∎

Let us remark, that we have used our knowledge of the Levi-Cività connection. The advantage of some other methods (e.g. the method of differential equations) is that we get all natural operators directly during some calculations.

At the end of this note, we discuss how the situation might change if we restrict the naturality condition to orientation preserving morphisms. That is to say, for each oriented Riemannian manifold (M, g) we have to define a linear connection

$\Gamma_{(M,g)}$ and for every local isometry $f : (M,g) \to (M',g')$ which preserves the orientations we require, $f^*\Gamma_{(M',g')} = \Gamma_{(M,g)}$. In the setting of natural bundles, we have first to change their domain category to the category of m-dimensional oriented manifolds and orientation preserving local diffeomorphisms. The whole general theory of natural bundles and operators does not essentially change and we only have to repeat the above considerations, but with the equivariancy restricted to the subgroup $G^2_{m+} \subset G^2_m$ of the jets of orientation preserving local diffeomorphisms.

We remark that using the method of differential equations, one always works with the connected component of the unit of the group in question. That is why, solving the corresponding equations (see [**2**]) we are not able to get directly our Corollary 1, we could get only the following results.

Proposition 2. *Let $\Gamma : R_s \to T^{(1,2)}$ be a first order operator on pseudometrics of signature s on oriented m-dimensional manifolds, natural with respect to all orientation preserving morphisms. If $n \geq 4$ or $n = 2$, then Γ is the zero operator. For $n = 3$ there is an one parameter family of such operators generated by the vector product. If $n = 1$, then there is an one parameter family with the coordinate expression $\Gamma(g) = c\sqrt{|g|}$, $c \in \mathbf{R}$.*

PROOF : Let $\gamma^i_{jk}(j^1_0 g_s)$ be the G^2_{m+}-equivariant map corresponding to Γ. According to Lemma 1, we may restrict ourselves to the values on the canonical metric g_s with signature s on $\mathbf{R}^m$.

If $n = 1$, then we deal with a function γ defined on positive or negative real numbers and the naturality condition gives

$$\gamma(g) = c\sqrt{|g|} \qquad c \in \mathbf{R}.$$

This clearly is a G^2_{1+}-equivariant map, so that it gives rise to a natural operator for every $c \in \mathbf{R}$.

Now assume $n \geq 2$ and let us denote

$$c^i_{jk} = \gamma^i_{jk}(j^1_0 g_s).$$

We shall consider the linear transformations $f_{p,q} : \mathbf{R}^m \to \mathbf{R}^m$,$1 \leq p,g \leq m$, $p \neq q$ determined by the matrix

$$a^i_j = (-1)^{(\delta^i_p + \delta^j_q)}\delta^i_j \ ,$$

i.e. we change the orientation on the p-th and q-th axes. These maps are orientation preserving and belong to the isotropy group of the canonical pseudometric g_s. Let us discuss the naturality condition for these transformations:

$$c^i_{jk} = (-1)^{(\delta^i_p + \delta^i_q + \delta^j_p + \delta^j_q + \delta^k_p + \delta^k_q)} c^i_{jk} \ . \tag{2}$$

For every choice of p,q we get a condition on the constants c^i_{jk}

(i) Assume $i = j = k = p$. Then (2) implies $c^i_{jk} = 0$.
(ii) If all the indices i, j, k, p are different, i.e $m \geq 4$, and if we take $q = i$, then we get $c^i_{jk} = 0$.

(iii) Let $i \neq j$, $j = k$, $i = p$, $j = q$. Then $c^i_{jk} = 0$.
(iv) Similarly, $i = j$, $j \neq k$, $i = p$, $k = q$, or the relations $i = k$, $j \neq k$, $i = p$, $j = q$, imply $c^i_{jk} = 0$.

So if $m = 2$ or $m \geq 4$, we have proved that Γ is the zero operator. But if $m = 3$, then c^i_{jk} with all indices different might be non-zero.

Let us assume first the signature is 0 and consider the linear transformations with the following matrices

$$(3) \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The naturality with respect to these transformations gives

$$c^1_{23} = -c^1_{32} = c^2_{31} = -c^2_{13} = c^3_{12} = -c^3_{21}$$

In this way we have got just the components of the $SO(3)$-invariant tensor corresponding to the vector prod'ict in the Euclidean space $\mathbf{R}^3$ and the conclusion in dimension 3 follows from Lemma 1.

If the signature is one we proceed analogously using transformations depending on a real parameter t

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}$$

As before, the naturality condition with respect to these transformations, together with the formula $(\cosh t)^2 - (\sinh t)^2 = 1$, imply the equalities

$$c^1_{23} = -c^1_{32} = -c^2_{31} = c^2_{13} = -c^3_{12} = c^3_{21}.$$

Since the transformations (4) are generators of the isotropy group of the canonical metric, there is a uniquely determined G^1_m-equivariant map transforming metrics of signature 1 into $(1,2)$-tensors, and we can apply Lemma 1.

If g is a metric of signature s, then the metric $-g$ has signature $3 - s$, so that the remaining two cases follow and the Proposition is proved. ∎

Corollary 2. *The Levi-Cività connection is the only first order natural connection with respect to orientation preserving local diffeomorphisms in dimensions different from one and three. All natural connections on three-dimensional pseudo-Riemannian manifolds of signature s form an one parameter family* $\Gamma = \Gamma_{\text{Levi-Cività}} + cV$, $c \in \mathbf{R}$, *where* V *is the zero order operator induced by the vector product. In dimension one, there also is such one parameter family but then* V *means the zero order operator induced by the scalar product.*

The simple method used in our proofs is powerful also in other situations, cf. [3], where similar technique appears independantly on the present paper.

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Mathematical Institute of the ČSAV, branch Brno, Mendlovo nám. 1, CS-662 82 Brno

(*Received* March 9,1989)