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The homogenous Dirichlet problem for non–elliptic partial differential equations with strong nonlinearities

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Abstract. The existence of weak solutions to certain non–elliptic semilinear partial differential is shown. The use of anisotropic Sobolev spaces makes it possible to apply methods developed in elliptic theory. The generating functions for the Nemytskiĭ operators are not required to be of polynomial growth.

Keywords: Anisotropic Sobolev Spaces, Semilinear Partial Differential Equations, Dirichlet Problem, Strong Nonlinearities

Classification: 35J60, 35J70, 46E35

1.1. In this paper we study, for example, the following differential equations

$$u_{yy} - u_{xxxx} - (g(u_x))_x = f, \tag{1.1}$$

$$\partial_{xxxx}(u_{xxxx} + u_{xxyy} + u_{yyyy}) + g(u) = f \tag{1.2}$$

or

$$u_{xxyy} + g(u) = f. \tag{1.3}$$

Here $g : \mathbf{R} \to \mathbf{R}$ is a function that satisfies $g(t)t \geq 0$ for all $t \in \mathbf{R}$. We will show the existence of weak solutions to a generalized Dirichlet problem. A point in this paper is to show that methods used for elliptic equations with similar nonlinearities apply also to these non–elliptic equations. This is accomplished by chosing an appropriate anisotropic Sobolev space. For linear equations the generalized Dirichlet problem has been studied by Doppel and Jacob [**7**], Jacob [**13**]. For semilinear equations with a different type of nonlinearity an existence theorem using the mountain pass lemma was given in Warnecke [**31**], [**32**]. Equations of type (1.1) are known as Boussinesq equations. The nonlinearities discussed in this paper are not of the same kind that appear in the original Boussinesq equation (cp. Boussinesq [**3**], [**4**] or Zabusky [**34**]). In the original equation one has $g(t) = t^2$, i.e. an even function. This is the type of nonlinearity treated in Warnecke [**31**],[**32**]. In this paper we treat nonlinearities that are generated by odd functions like $g(t) = t^{2p+1}, p \in N$ (cp. Moser [**21**]). In Moser [**21**] equations like (1.1) are called hyperbolic. As will be seen in the analysis below these equations are more elliptic than hyperbolic in any sense of these terms. In order to avoid confusion one should be careful to note the sign in front of the term u_{xxxx}. Kalantarov and Ladyzhenskaya [**15**] are occasionally cited in connection with Boussinesq type equations, but they treated equations like $u_{tt} + u_{xxxx} + g(u_x)_x = 0$. This a semilinear version of the hyperbolic beam equation and has the opposite sign for the fourth order term. The linearized version of equation (1.2) was studied by Herler [**10**]. It describes the deflection of anisotropic shells. The linearized version of equation (1.3) was introduced by Dynkin [**8**] for stochastic processes (cp. Doppel and Jacob [**7**]). Note that this is an example of a non–hypoelliptic operator (for a definition see Hörmander [**12**]) that can be treated using "elliptic" methods.

1.2. We will study semilinear differential equations of the form

$$(1.4) \qquad A(x,D)u(x) + N(u)(x) = f(x), x \in G,$$

in a domain $G \subset \mathbf{R}^N$ (G may be unbounded). Here $A(\cdot, D)$ is a linear differential operator in divergence form (see Section 4.1) and $N(\cdot)$ is a Nemytskiĭ or superposition operator (see Section 4.4). The generating functions for the Nemytskiĭ operators will not be required to be of polynomial growth. The solutions to (1.4) will be sought in a Sobolev space $H_0^A(G)$ that corresponds to the operator $A(\cdot, D)$ (for definitions see Sections 2.5 and 4.1).For certain domains one can define trace operators on the boundary. Thereby the solutions in $H_0^A(G)$ can be shown to satisfy certain homogenous Dirichlet boundary data in a generalized sense (see also Rákosník [**24**] [**25**], Warnecke [**31**]).

If $A(\cdot, D)$ is an elliptic operator of order $2m$ then the Nemytskiĭ operator $N(\cdot)$ may include derivatives up to the order $m-1$. This property will have to be generalized for the non–elliptic operators to be studied here (cp. Chapter 4). The linear differential operator $A(\cdot, D)$ will be an operator of order $2m$ that does not contain all the derivatives of this order relevant for ellipticity. The main questions to be addressed in this paper will be: 1.) Can the Hilbert space methods developed for semilinear elliptic problems be modified for applications involving anisotropic Sobolev spaces? Which derivatives may appear in the Nemytskiĭ operators $N(\cdot)$?

1.3. The generalized Dirichlet problem for non–hypoelliptic linear partial differential equations was studied by Nikol'skiĭ [**23**] as well as Louhivaara and Simader [**19**], [**20**]. The latter papers inspired Doppel and Jacob [**7**], Jacob [**13**], Schomburg [**26**] to study very general classes of linear operators $A(\cdot, D)$ that have an associated bilinear form $A[\cdot, \cdot]$ which satisfies a generalized Gårding's inequality on an anisotropic Sobolev space, see also Jacob and Schomburg [**14**]. It was shown that for linear boundary value problems the well known Hilbert space methods used in elliptic theory (cp. Friedman [**9**] or Showalter [**27**]) may be applied to certain non–hypoelliptic boundary value problems.

1.4. The treatment of equation (1.4) in the case of elliptic operators (including quasilinear elliptic operators) goes back to numerous papers of Brézis, Browder, Hess and others (for references see for example Brézis and Browder [**5**], Browder [**6**], Hess [**11**], Landes [**17**], Lehtonen [**18**], Simader [**28**] and Webb [**33**]). The usual techniques include a priori estimates and cut off functions for the type of nonlinearities considered here. The unbounded nonlinearity is approximated by bounded terms with compact support. This reduced problem is solved using Schauder's fixed point theorem. Then the a priori estimate and Vitali's theorem are used to obtain the solution to the original problem. The main ingredient in the a priori estimate is that the Nemytskiĭ operators $N(\cdot)$ are mainly generated by functions that satisfy $g(t)t \geq 0$. In this paper it is shown that these methods may be modified to work in the setting of anisotropic Sobolev spaces.

1.5. As mentioned above we would like to obtain a large set of derivatives that may be used in the nonlinear operator $N(\cdot)$. This issue hinges on the existence of compact embeddings for anisotropic Sobolev spaces. To the operator $A(\cdot, D)$ one has to find a set of multiindices $K(A)$ such that for bounded domains the embedding $H_0^A(G) \to H_0^{K(A)}(G)$ is compact. A sufficient condition for $K(A)$ will be given in Section 3.5. This requires the use of very general Sobolev inequalities as given in Theorem 3.4 (see also Tafel [**29**]). Results of this type for L^p spaces are due to Besov, I'lin and Nikolskiĭ [**2**]. For compact embeddings see also Rákosník [**24**], [**25**] and Warnecke [**31**], [**32**].

I would like to take the oportunity to thank Prof.K.Doppel for suggesting this research and introducing me to anisotropic Sobolev space. Also I thank Ralf Kieser for his help in proof reading the manuscript.

2 Anisotropic Sobolev Spaces.

2.1. First all it will be necessary to introduce some notation. Let $\mathbf{N}_0^N$ denote the set of **multiindices** $\alpha = (\alpha_1, \dots, \alpha_N)$ with $\alpha_j \in \mathbf{N}_0$ for $j = 1, \dots, N$, $N \in \mathbf{N}(\mathbf{N}_0 = \{0\} \cup \mathbf{N})$. Analogously $\mathbf{N}^N$ denotes the set of multiindices with positive components. We introduce the **order of** α : $|\alpha| := \alpha_1 + \dots + \alpha_N$ and set $\partial^\alpha := \partial_1^{\alpha_1} \dots \partial_N^{\alpha_N}$, where $\partial_k := \frac{\partial}{\partial x_k}$ for $1 \leq k \leq N$. Further we define for $x \in \mathbf{R}^N$ the exponents $x^\alpha := x_1^{\alpha_1} \dots x_N^{\alpha_N}$.

2.2. Take $x, y \in \mathbf{R}^N$ then we define a partial ordering by $x \leq y$ iff $x_j \leq y_j$ for all $j = 1, \dots, N$, and resp. $x < y$, iff $x_j < y_j$ for all $j = 1, \dots, N$. Also for $j \in \{1, \dots, N\}$ we define the special multiindices $\varepsilon_j := (\delta_{1j}, \dots, \delta_{Nj})$. Here

$$\delta_{kj} := \begin{cases} 1 & \text{for } k = j \\ 0 & \text{for } k \neq j \end{cases} \qquad \text{denotes the Kronecker symbol.}$$

2.3. Let $M \subset \mathbf{R}^N$ be a finite set, $M = \{a^{(1)}, \dots, a^{(k)}\}, k \in \mathbf{N}, a^{(j)} \in \mathbf{R}^N$ for $j = 1, \dots, k$. Then the **convex hull** $ch\,M$ of M is given by

$$ch\,M := \{x \in \mathbf{R}^N |\ \text{There exist } t_1, \dots, t_k \in [0,1] \text{ such that } \sum_{j=1}^k t_j = 1 \text{ and } x = \sum_{j=1}^k t_j a^{(j)}\}. \tag{2.1}$$

Take $A \neq \emptyset$ to be a finite set of multiindices, i.e. $A \subset \mathbf{N}_0^N, A = \{\alpha^{(1)}, \dots, \alpha^{(k)}\}, k \in$

N. We introduce the following notations:

$\overline{A}$ $:= \{\alpha | \alpha \in \mathbf{N}_0^N$ and there exists a $\sigma \in A$ such that $\alpha \leq \sigma\}$,
$|A|$ $:= \max_{\alpha \in A} |\alpha|$ **(the order of** A**)** ,
$konv\, A := ch\, A \cap \mathbf{N}_0^N$,
$\#A$ – number of elements in A.

2.4. In this paper G will always be a **domain** in $\mathbf{R}^N$, i.e. an open connected subset. We write $G' \subset\subset G$ for a bounded subset G' of G with $\overline{G}' \subset G$. Further we will only consider function spaces of real valued functions. For spaces not introduced in this paper we use the notation of Adams [**1**]. Let $\mathbf{B}_n := \{x \in \mathbf{R}^n | \quad |x| < n\}$ for $n \in \mathbf{N}$ and set

$$\chi_G(x) := \begin{cases} 1 & \text{for } x \in G \\ 0 & \text{for } x \notin G \end{cases}$$ to be the **characteristic function** of a set.

2.5. By $\|\cdot\|_0 = \|\cdot\|_{0,G}$ we denote the norm on $L^2(G)$. For a finite set of multiindices $A \subset \mathbf{N}_0^N$ and an arbitrary function $\varphi \in C_0^\infty(G)$ we define the norm

$$|\varphi|_{A,G} := \{\sum_{\alpha \in A} \|\partial^\alpha \varphi\|_{0,G}^2 + \|\varphi\|_{0,G}^2\}^{\frac{1}{2}} \tag{2.2}$$

One has $\|\varphi\|_{A,G} < \infty$ for all $\varphi \in C_0^\infty(G)$. Obviously $\|\cdot\|_{A,G}$ is a norm. If no ambiguities concerning the domain G occur we will just write $\|\cdot\|_A$.

Now we may introduce the following linear space:
$H_0^A(G)$–the completion of $C_0^\infty(G)$ with respect to the norm $\|\cdot\|_{A,G}$.

If the domain G is unbounded will assume that $A = \overline{A}$ throughout the paper. This condition will allow us to use the Leibniz rule for the differentiation of products. On bounded domains the above restriction is not necessary due to the Poincaré inequality (3.4). It implies that the norms defined by A and $\overline{A}$ are equivalent.

2.6. For $\varphi, \psi \in C_0^\infty(G)$ let us introduce the bilinear form

$$\langle \varphi, \psi \rangle_{A,G} = \sum_{\alpha \in A} \int_G \partial^\alpha \varphi \partial^\alpha \psi \, dx + \int_G \varphi \psi \, dx. \tag{2.3}$$

Since $\langle \varphi, \varphi \rangle_{A,G} = \|\varphi\|_{A,G}^2$ this form is positive define. The Cauchy–Schwarz inequality gives

$$|\langle \varphi, \psi \rangle_{A,G} \leq \|\varphi\|_{A,G} \|\psi\|_{A,G}, \tag{2.4}$$

i.e. $\langle \cdot, \cdot \rangle_{A,G}$ is a continuous scalar product on the space $C_0^\infty(G)$. It follows that

Lemma 2.1. *The space $H_0^A(G)$ is a separable Hilbert space with the scalar product $\langle \cdot, \cdot \rangle_{A,G}$.*

PROOF : See Warnecke [**32**]. ∎

Comment 2.2. If $A = \{\alpha \in \mathbf{N}_0^N \,|\, |\alpha| \le m\}$ for some $m \in \mathbf{N}_0$ one obtains the classical Sobolev space $H_0^{m,2}(G)$ with the norm $\|w\|_{m,2}^2 = \sum_{|\alpha|\le m} \|\partial^\alpha w\|_0^2$ for all $w \in H_0^{m,2}(G)$ (cp. Adams [**1**], Chapter III).

2.7. The elements $v \in H_0^A(G)$ have weak derivatives $\partial^\alpha v, \alpha \in A$, in $L^2(G)$. For let $(\psi_n)_{n\in\mathbf{N}_0} \subset C_0^\infty(G)$ be a sequence such that $\psi_n \to v$ with respect to $\|\cdot\|_A$. Then the sequence $(\partial^\alpha\psi_n)_{n\in\mathbf{N}_0}$ converges in $L^2(G)$ to an element $v_\alpha \in L^2(G)$ for every $\alpha \in A$. Due to the uniqueness of weak derivatives one has $v_\alpha = \partial^\alpha v$ (cp. Adams [**1**], Section 1.57).

3 Embeddings of anisotropic Sobolev Spaces.

3.1. In this chapter we have collected some inequalities that are used to obtain embedding theorems for anisotropic Sobolev spaces. Corollary 3.3 will allow us to deduce continuous embeddings from relations between multiindex sets. We will present the inequalities.

3.2. Let A and A_1 be finite non–empty subsets of $\mathbf{R}^N$ such that $A_1 \subset A$. We define

(3.1)
$$ch(A; A_1) := \{x | x \in \mathbf{R}^N \quad \text{and there exists an element } a \in A_1 \text{ and an } s > 1 \text{ such that } a + s(x-a) \in ch(A)\}$$

(cp. Tafel [**29**], Def. (17.1)).

Theorem 3.1. *(Ehrling's Lemma) Let A_1 and A_2 be two finite subsets of $\mathbf{N}_0^N$ such that $A_1 \neq \emptyset$ and $A := A_1 \cup A_2$. Take $\beta \in \mathbf{N}_0^N$. Then $\beta \in ch(A; A_1)$ iff for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that*

$$\|\partial^\beta\varphi\|_{L^2(\mathbf{R}^N)} \le \varepsilon\|\varphi\|_{A_2,\mathbf{R}^N} + C(\varepsilon)\|\varphi\|_{A_1,\mathbf{R}^N} \tag{3.2}$$

for all $\varphi \in C_0^\infty(\mathbf{R}^N)$. Further $\beta \in konv\, A$ iff a constant $C > 0$ exists such that

$$\|\partial^\beta\varphi\|_{L^2(\mathbf{R}^N)} \le C\|\varphi\|_{A,\mathbf{R}^N} \tag{3.3}$$

for all $\varphi \in C_0^\infty(\mathbf{R}^N)$.

PROOF : This is shown in Tafel [**29**], Theorem (19.2). ∎

3.3. For bounded domains one has the following **Poincaré inequality**:

Theorem 3.2. *Let G be a bounded domain and take $d \in \mathbf{R}^N$ such that $d_j := \sup\{|x_j| \,|x = (x_1,\dots,x_N) \in G\}$, for $j = 1,\dots,N$. Further let $A \subset \mathbf{N}_0^N, A \neq \emptyset$, be a finite multiindex set and $\alpha \in A$ an arbitrary multiindex. Then for every multiindex $\beta \in \mathbf{N}_0^N$ such that $\beta \le \alpha$ the inequality*

$$\|\partial^\beta u\|_0 \le 2^{|\alpha-\beta|} d^{\alpha-\beta} \|\partial^\alpha u\|_0 \tag{3.4}$$

holds for all $u \in H_0^A(G)$.

PROOF : See Doppel and Jacob [**7**], Lemma 1. ∎

Corollary 3.3. *Let G be a domain. Take $A, B \subset \mathbb{N}_0^N$ to be two finite multiindex sets such that $B \subset konv\,\overline{A}$. Then the space $H_0^A(G)$ is continuously embedded in the space $H_0^B(G)$. Further our general assumptions on multiindex sets imply $H_0^A(G) \simeq H_0^{konv\,\overline{A}}(G) \simeq H_0^{\overline{A}}(G)$.*

PROOF : Take an arbitrary $\varphi \in C_0^\infty(G)$. Then (3.3) and (3.4), in case G is bounded, otherwise our assumption $A = \overline{A}$ for unbounded domains imply the inequalities

$$\|\varphi\|_B \leq \|\varphi\|_{konv\,\overline{A}} \leq C'\|\varphi\|_{\overline{A}} \leq \widetilde{C}\|\varphi\|_A \leq \widetilde{C}\|\varphi\|_{\overline{A}} \leq \widetilde{C}\|\varphi\|_{konv\,\overline{A}}. \tag{3.5}$$

The constants $C', \widetilde{C} > 0$ have to be chosen appropriately ($C' = \widetilde{C}$ for unbounded domains). Since $C_0^\infty(G)$ is a dense subset of the spaces we are concerned with, the corollary is proved. ∎

We now have a sufficient criterion to tell us when two finite multiindex sets $A, B \subset \mathbb{N}_0^N$ give the same anisotropic Sobolev space, i.e. $H_0^A(G) \simeq H_0^B(G)$. This is true when $\operatorname{konv}\overline{A} = \operatorname{konv}\overline{B}$.

3.4. Our compact embedding theorem will be proved with the help of the compact embedding properties of the usual Sobolev spaces. In fact, we will only need the Rellich lemma, namely the compact embedding of the Sobolev space $H_0^{1,2}(G)$ into $L^2(G)$.

Theorem 3.4. *Let G be a bounded domain. Further take $k \in \mathbb{N}, m \in \mathbb{N}_0$. Then the embedding of the space $H_0^{m+k,2}(G)$ into the space $H_0^{m,2}(G)$ is compact.*

PROOF : See Adams [1], Theorem 6.2. ∎

3.5. We will now define for a given finite multiindex set $A \neq \emptyset$ a multiindex set that gives a compactly embedded space. We set:

$$K(A) := \mathbb{N}_0^N \cap ch(\overline{A}, \{0\}), \tag{3.6}$$

for the notation remember (3.1). Let us consider, for example, the multiindex sets for the equations in Section 1.1. The set $A := \{(4,0),(3,1),(2,2)\}$ contains the multiindices for equation (1.2). Then we have $K(A) = \{(0,0),(1,0),(2,0),(3,0),(0,1),(1,1),(2,1)\}$. In case of equation (1.1) we have $A = \{(2,0),(0,1)\}$ and therefore $K(A) = \{(0,0),(1,0)\}$. Finally, for equation (1.3) the multiindex set is $A = \{(1,1)\}$ giving $K(A) = \{(0,0)\}$.

In order to show that for bounded domains G the anisotropic Sobolev space $H_0^A(G)$ is compactly embedded in the space $H_0^{K(A)}(G)$ we will need the following lemma. Remember the notation $(1) = \{\alpha \in \mathbb{N}_0^N \mid |\alpha| = 1\}$.

Lemma 3.5. *Let G be a bounded domain. Further let A be a finite multiindex set with the property that $(1) \subset konv\,\overline{A}$. Then every sequence $(u_n)_{n\in\mathbb{N}} \subset H_0^A(G)$ that is bounded with respect to the norm $\|\cdot\|_A$ has a subsequence $(u_k)_{k\in\mathbb{N}}$ such that $(\partial^\gamma u_k)_{k\in\mathbb{N}}$ converges in $L^2(G)$ for every $\gamma \in K(A)$.*

PROOF : Take $\gamma \in K(A)$. Then (3.2) and (3.4) imply that for each $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$\|\partial^\gamma u\|_0 \leq \varepsilon\|u\|_A + C(\varepsilon)\|u\|_0 \tag{3.7}$$

for all $u \in H_0^A(G)$ (since $C_0^\infty(G)$ is a dense subset). Since $(1) \subset konv\overline{A}$ Corollary 3.3 implies that

$$\|u\|_{1,2} \leq C' \|u\|_a \tag{3.8}$$

for some constant $C' > 0$ and all $u \in H_0^A(G)$.

Now let $(u_n)_{n\in\mathbf{N}} \subset H_0^A(G)$ be a bounded sequence. We may assume that $\|u_n\|_A \leq 1$ for all $n \in \mathbf{N}$. By (3.8) this sequence is also bounded in the space $H_0^{1,2}(G)$. By the compact embeddings in Theorem 3.4 there exists a subsequence $(u_k)_{k\in\mathbf{N}}$ converging in $L^2(G)$. Using (3.7) for this subsequence we obtain

$$\|\partial^\gamma(u_k - u_{k'})\|_0 \leq 2\varepsilon + C(\varepsilon)\|u_k - u_{k'}\|_0. \tag{3.9}$$

Since ε may be chosen arbitrarily small and $(u_k)_{k\in\mathbf{N}}$ converges in $L^2(G)$ the right hand side of (3.9) can be made arbitrarily small. Therefore, the sequence $(\partial^\gamma u_k)_{k\in\mathbf{N}}$ is a Cauchy sequence in $L^2(G)$. This is true for any $\gamma \in K(A)$. ∎

Corollary 3.6. *Under the assumptions of Lemma 3.5 the embedding of the Sobolev space $H_0^A(G)$ into the Sobolev space $H_0^{K(A)}(G)$ is compact. Therefore, bounded sequences in $H_0^A(G)$ have convergent subsequences in $H_0^{K(A)}$, and weakly convergent sequences in $H_0^A(G)$ converge in $H_0^{K(A)}(G)$.*

Lemma 3.7. *Let $G, G' \subset \mathbf{R}^N$ be domains such that $G' \cap G \neq \emptyset$. Further, take $\psi \in C_0^\infty(G')$ and let A be finite multiindex set. If $u \in H_0^A(G)$ then $\psi u \in H_0^A(G \cap G')$.*

PROOF : The Poincaré inequality for bounded domains, Theorem 3.2, or respectively our assumption $A = \overline{A}$ for unbounded domains imply that the Leibniz rule for derivatives of products may be used in our spaces. This gives $\|\psi u\|_{A,G'} \leq C(\psi)\|u\|_A$. ∎

Lemma 3.8. *Let $G, K \subset \mathbf{R}^N$ be domains such that $K \subset\subset \mathbf{R}^N$ and $G \cap K \neq \emptyset$. Take A to be a finite multiindex set. If $(u_n)_{n\in\mathbf{N}} \subset H_0^A(G)$ is a sequence that is bounded in the norm $\|\cdot\|_A$ then there exists a subsequence $(u_k)_{k\in\mathbf{N}}$ and an element $u \in H_0^A(G)$ such that:*

- (i) *$u_k \rightharpoonup u$ in $H_0^A(G)$ and $\partial^\alpha u_k \rightharpoonup \partial^\alpha u$ in $L^2(G)$ for all $\alpha \in \overline{A}$.*
- (ii) *The restriction of the sequence to the set $K \cap G$ converges $u_{k|G\cap K} \to u_{|G\cap K}$ with respect to the norm $\|\cdot\|_{K(A),G\cap K}$.*
- (iii) *The sequence converges pointwise almost everywhere $\partial^\alpha u_k \to \partial^\alpha u$ in G for all $\alpha \in K(A)$.*

PROOF :

- (i) Follows from the fact that in Hilbert spaces bounded sets are weakly relatively sequentially compact and from Corollary 3.3.
- (ii) Let K' be a domain such that $K \subset\subset K' \subset\subset \mathbf{R}^N$ and take $\varphi \in C_0^\infty(\mathbf{R}^N)$ to be a function with $\varphi = 1$ on $\overline{K}$, $\varphi = 0$ on $\mathbf{R}^N \setminus K'$. According to Lemma 3.7 $\varphi u_k \in H_0^A(G \cap K')$. Since $G \cap K'$ is bounded Corollary 3.6 gives a convergent subsequence in $H_0^{K(A)}(G \cap K')$. Because $\varphi = 1$ on K we obtain (ii).

(iii) By (ii) we have a subsequence $(u_k)_{k\in\mathbf{N}}$ converging with respect to $\|\cdot\|_{K(A),G\cap B_1}$. Then there exists a further subsequence $(u_{k'})_{k'\in\mathbf{N}}$ with $\partial^\alpha u_{k'} \to \partial^\alpha u$ almost everywhere in $G\cap B_1$ for all $\alpha\in K(A)$ (cp. Kolmogorov and Fomin [16] Section 7.2.5). We apply this procedure inductively to the domains $G\cap B_n, n=1,2,3,\ldots$. A diagonal sequence will have all the desired properties.

■

4 Operators, Fixed Point Theorem.

4.1. We begin by introducing a class of linear operators. For convenience we will not take the most general definitions. Let $G\subset\mathbf{R}^N$ be a domain and $A\subset\mathbf{N}_0^N$ be a finite multiindex set. Further suppose that A has the property that $\varepsilon_j\in\overline{A}$ for $j=1,\ldots,N$ (i.e. $(1)\subset\overline{A}$). We will consider the following **linear differential operators**

$$A(\cdot,D)=\sum_{\alpha,\beta\in A}(-1)^{|\alpha|}\partial^\alpha a_{\alpha\beta}(\cdot)\partial^\beta. \tag{4.1}$$

We assume that the coefficients $a_{\alpha\beta}:G\to\mathbf{R}$ satisfy $a_{\alpha\beta}\in C_B^m(G)$ for $m=|\alpha|$ (for notation see Adams [1]). We demand that $a_{\alpha\beta}\neq 0$ for at least one pair of multiindices $\alpha,\beta\in A$ with $|\alpha|=|\beta|=|A|$. Therefore $A(\cdot,D)$ will be a differential operator of order $2|A|$. Also we assume $a_{\alpha\beta}=a_{\beta\alpha}$ for all $\alpha,\beta\in A$. To the differential operator $A(\cdot,D)$ and its' multiindex set A we associate the Hilbert space $H_0^A(G)$ with the scalar product $\langle\cdot,\cdot\rangle_A$, defined according to (2.4), cp. Chapter 2.

4.2. Further we define the **bilinear form** associated with the operator $A(\cdot,D)$

$$A[\varphi,\psi]:=\sum_{\alpha,\beta\in A}\int_G a_{\alpha\beta}(x)\partial^\beta\varphi\partial^\alpha\psi\,dx \tag{4.2}$$

for all $\varphi,\psi\in C_0^\infty(G)$. The Cauchy–Schwarz inequality implies the continuity of the bilinear form on the space $H_0^A(G)$, i.e. $A[\varphi,\psi]\le C_2\|\varphi\|_A\|\psi\|_A$ for a suitable constant $C_2>0$. It can therefore be extended to the whole space $H_0^A(G)$.

4.3. We say that $A(\cdot,D)$, respectively $A[\cdot,\cdot]$ satisfies **Gårding's inequality** on the space $H_0^A(G)$ if there exist constants $C_0,C_1\in\mathbf{R},C_0>0$, such that

$$A[u,u]\ge C_0\|u\|_A^2-C_1\|u\|_0^2 \tag{4.3}$$

for all $u\in H_0^A(G)$. Linear operators with this property were, for example, studied by Doppel and Jacob [7], Jacob [13], Jacob and Schomburg [14], Schomburg [26] as well as Herler [10].

4.4. Let $r=\#K(A)$ (cp. Sections 2.3 and 3.5). A function $b:\overline{G}\times\mathbf{R}^r\to\mathbf{R}$ is said to satisfy the **Carathéodory condition** iff

(4.4) (a) $b(x,\cdot)$ is continuous for almost all $x\in\overline{G}$

(b) $b(\cdot,t)$ is measurable for all $t\in\mathbf{R}^r$.

If $\mu : \overline{G} \to \mathbf{R}^r$ is a measurable function then the Carathéodory condition implies that $b(\cdot, \mu(\cdot)) : \overline{G} \to \mathbf{R}$ is measurable (see Vainberg [**30**], §18). We set $\Psi^r(\overline{G}) := \{\mu : \overline{G} \to \mathbf{R}^r \mid \mu \text{ measurable }\}$. Then the function $b(\cdot,\cdot)$ generates the **Nemytskiĭ operator** $\widetilde{B} : \Psi^r(\overline{G}) \to \Psi^1(\overline{G})$ defined by $\mu \mapsto \widetilde{B}(\mu) := b(\cdot, \mu(\cdot))$ for all $\mu \in \Psi^r(\overline{G})$. We will suppose that the multiindices in $K(A)$ are numbered, i.e. $K(A) = \{\alpha_1, \ldots, \alpha_r\}$. For $u \in H_0^A(G)$ we set $\mu(u) := (\partial^{\alpha_1} u, \ldots, \partial^{\alpha_r} u) \in (L^2(G))^r, \alpha_j \in K(A), 1 \le j \le r$. Now we can define the Nemytskiĭ operator $B : H_0^A(G) \to \Psi^1(\overline{G})$ by setting $B(u)(x) := b(x, \mu(u)(x)) = \widetilde{B}(\mu(u))(x)$. For $L^2(G)$ spaces one has the following important property of the Nemytskiĭ operators:

Theorem 4.1. *For all $v \in (L^2(G))^r$ one has $\widetilde{B}(v) \in L^2(G)$ iff there exists a function $a \in L^2(G)$ and a constant $d \ge 0$ such that*

$$|b(x,t)| \le a(x) + d|t| \tag{4.5}$$

for all $(x,t) \in \overline{G} \times \mathbf{R}^r$. In this case the operator $\widetilde{B} : (L^2(G))^r \to L^2(G)$ is automatically continuous and bounded, i.e. it maps bounded sets to bounded sets.

PROOF : This is a special case of Theorem 19.2 in Vainberg [**30**]. ∎

4.5. Let X, Y be Banach spaces. A continuous operator $T : X \to Y$ is said to be **completely continuous** if it maps bounded sets in X to relatively compact sets in Y.

Corollary 4.2. *Let A be a multiindex set with $(1) \subset \overline{A}$. Further suppose that $b : \overline{G} \times \mathbf{R}^r \to \mathbf{R}$ is given as above. If $u \in H_0^A(G)$ and the function $b(\cdot,\cdot)$ satisfies (4.5) then $B : H_0^A(G) \to L^2(G)$ is continuous and bounded. If G is a bounded domain then $B(\cdot)$ is also completely continuous.*

PROOF : Obviously the map $\mu : H_0^A(G) \to (L^2(G))^r$ is continuous and bounded. If G is a bounded domain it is also completely continuous by Corollary 3.6. Since $\widetilde{B} : (L^2(G))^r \to L^2(G)$ is continuous and bounded the operator $B(\cdot)$ has the desired properties. ∎

For completely continuous operators one has the well known

Theorem 4.3. *(Schauder's Fixed Point Theorem)*
Let X be a Banach space and K a closed, bounded, convex, nonempty subset of X. Suppose that the operator $T : K \to K$ is completely continuous. Then T has a fixed point $x \in K$, i.e. $T(x) = x$.

PROOF : See Zeidler [**35**] Theorem 2.1. ∎

5 Bounded Nonlinearities.

5.1. In this chapter we prove an existence theorem for every restricted class of nonlinearities. Namely, the type of nonlinearities one obtains in applying the cut off procedures to be introduced in Section 6.3 (cp. Simader [**28**]).

Lemma 5.1. *Let G be a domain and A a finite nonempty multiindex set. Further suppose we are given for each $\alpha \in K(A)$ a function $b_\alpha : G \times \mathbf{R}^r \to \mathbf{R}, r = \#K(A)$, that satisfies the Carathéodory condition (4.4) as well as the inequality (4.5) with $d = 0$ and a common function $a \in L^2(G)$. We require that the function a has a compact support if G is unbounded. By $B_\alpha(\cdot)$ we denote the respective Nemytskiĭ operators $B_\alpha : H_0^A(G) \to L^2(G)$. Let $f \in L^2(G)$ be an arbitrary but fixed function. Then for all $u, \varphi \in H_0^A(G)$ the equation*

$$\langle N(u), \varphi\rangle_A :\doteq \langle f, \varphi\rangle_0 - \sum_{\alpha\in K(A)} \langle B_\alpha(u), \partial^\alpha \varphi\rangle_0 \tag{5.1}$$

defines a completely continuous operator $N : H_0^A(G) \to H_0^A(G)$. For any $u \in H_0^A(G)$ we have

$$\|N(u)\|_A \leq C_3(f, a). \tag{5.2}$$

PROOF : Since the functions $b_\alpha(\cdot,\cdot)$ satisfy the inequalities (4.5) we may apply Corollary 4.2 to give $B_\alpha(u) \in L^2(G)$ for all $u \in H_0^A(G)$. Therefore, using $\|\partial^\alpha\varphi\|_0 \leq \widetilde{C}\|\varphi\|_A$ ($\widetilde{C}$ as in (3.5)) the right hand side of (5.1) defines a continuous linear functional in φ on $H_0^A(G)$ for every fixed $u \in H_0^A(G)$. The Riesz representation theorem gives us an element $w \in H_0^A(G)$ such that

$$\langle w, \varphi\rangle_A = \langle f, \varphi\rangle_0 - \sum_{\alpha\in K(A)} \langle B_\alpha(u), \partial^\alpha \varphi\rangle_0 \tag{5.3}$$

for all $\varphi \in H_0^A(G)$. Now define the operator $N : H_0^A(G) \to H_0^A(G)$ by setting $N(u) := w$ as in (5.1). We will now show that the operator $N(\cdot)$ is completely continuous. Let us set $\varphi = N(u) - N(v), u, v \in H_0^A(G)$, in (5.1) then we obtain

$$\begin{aligned}
\|N(u) - N(v)\|_A^2 &= \langle N(u), N(u) - N(v)\rangle_A - \langle N(v), N(u) - N(v)\rangle_A \\
&= \sum_{\alpha\in K(A)} \langle B_\alpha(u), \partial^\alpha(N(u)) - N(v))\rangle_0 \\
&- \sum_{\alpha\in K(A)} \langle B_\alpha(v), \partial^\alpha(N(u)) - N(v))\rangle_0 \\
&\leq \left(\dot{\sum_{\alpha\in K(A)}} \|B_\alpha(u) - B_\alpha(v)\|_0\right) \|N(u) - N(v)\|_{K(A)}
\end{aligned}$$

i.e..

$$\|N(u) - N(v)\|_A \leq \sum_{\alpha\in K(A)} \|B_\alpha(u) - B_\alpha(v)\|_0. \tag{5.4}$$

Alternatively one also obtains

$$\|N(u) - N(v)\|_A^2 \leq \left(\sum_{\alpha\in K(A)} \|B_\alpha(u)\|_0 + \|B_\alpha(v)\|_0\right) \|N(u) - N(v)\|_{K(A)}. \tag{5.5}$$

From (5.4) we obtain the continuity of the operator $N(\cdot)$ since the operators $B_\alpha : H_0^A(G) \to L^2(G)$ are continuous due to Corollary 4.2.

We will now show (5.2). From (4.5) with $d = 0$ and $G' := \operatorname{supp} a(\cdot) \subset G$ we have

$$\|B_\alpha(u)\|_0^2 = \int_G |b_\alpha(x, \mu(u)(x))|^2\, dx \le \int_{G'} |a(x)|^2\, dx = \|a\|_0^2. \tag{5.6}$$

Using (5.1), (5.6) and (3.5) with $r = \#K(A)$ we obtain

$$\begin{aligned} \langle N(u), \varphi\rangle_A &\le \|f\|_0\|\varphi\|_0 + \sum_{\alpha\in K(A)} \|B_\alpha(u)\|_{0,G'}\|\partial^\alpha\varphi\|_{0,G'} \\ &\le \|f\|_0\|\varphi\|_0 + r\cdot\|a\|_0\|\varphi\|_{K(A)} \\ &\le C_3(f,a)\|\varphi\|_A \end{aligned} \tag{5.7}$$

for a suitable constant $C_3(f,a) > 0$. Setting $\varphi := N(u)$ this immediately implies

$$\|N(u)\|_A \le C_3(f,a) \tag{5.8}$$

for all $u \in H_0^A(G)$.

Now let us take (5.5) and use (5.6) to get

$$\|N(u) - N(v)\|_A^2 \le 2r\cdot\|a\|_0\|N(u) - N(v)\|_{K(A),G'}. \tag{5.9}$$

Take $(u_n)_{n\in\mathbf{N}} \subset H_0^A(G)$ to be a bounded sequence. Due to (5.8) the sequence $(N(u_n))_{n\in\mathbf{N}}$ is also bounded in $H_0^A(G)$. If G is a bounded domain then Corollary 3.6 gives a subsequence $(N(u_k))_{k\in\mathbf{N}}$ converging with respect to the norm $\|\cdot\|_{K(A),G}$. If G is an unbounded domain then we obtain a subsequence converging with respect to the norm $\|\cdot\|_{K(A),G'}$ by applying Lemma 3.8 (ii). Using (5.9) we may now conclude that this subsequence converges with respect to the norm $\|\cdot\|_A$. Therefore, the operator $N(\cdot)$ is completely continuous. ∎

Theorem 5.2. (Existence theorem for bounded nonlinearities) *Let G be a domain. Take $A[\cdot,\cdot]$ to be a continuous and positive bilinear form, i.e. it satisfies Gårding's inequality (4.3) with $C_1 = 0$. also suppose that for each $\alpha \in K(A)$ we are given a function $b_\alpha : \overline{G}\times\mathbf{R}^r \to \mathbf{R}$ that satisfies the assumptions of Lemma 5.1. Then there exists for each $f \in L^2(G)$ an element $u \in H_0^A(G)$ such that*

$$A[u,\varphi] + \sum_{\alpha\in K(A)} \langle B_\alpha(u), \partial^\alpha\varphi\rangle_0 = \langle f, \varphi\rangle_0 \tag{5.10}$$

for all $\varphi \in H_0^A(G)$.

Proof : Since the functional $A[u,\cdot] : H_0^A(G) \to \mathbf{R}$ is linear and bounded for each $u \in H_0^A(G)$ we may apply the Riesz representation theorem to obtain a bounded linear operator $T : H_0^A(G) \to H_0^A(G)$ such that

$$A[u,\varphi] = \langle Tu, \varphi\rangle_A.$$

We have assumed that $A[u,u] = \langle Tu, u\rangle_A \geq C_0\|u\|_A^2$. Therefore, the operator T is continuously invertible with $\|T^{-1}\| \leq C_0^{-1}$. Taking $N(\cdot)$ defined as in (5.1) we may rewrite (5.10) as follows

$$\langle Tu, \varphi\rangle_A = A[u,\varphi] = \langle N(u), \varphi\rangle_A$$

for all $\varphi \in H_0^A(G)$. We see that (5.10) is equivalent to the equation $Tu = N(u)$ or $u = T^{-1}N(u)$. By Lemma 5.1 the operator $T^{-1}N(\cdot)$ is completely continuous. Due to (5.2) it maps the closed ball

$$K = \{u | u \in H_0^A(G), \|u\|_A \leq C_3(f,a)C_0^{-1}\}$$

into itself. The statement of the theorem now follows by applying the Schauder Fixed Point Theorem 4.3. ∎

6 Existence Theorem for strong Nonlinearities.

6.1. The following conditions for nonlinearities are generalizations of the conditions given by Lehtonen [18] (see also Simader [28]). In the case of elliptic operators the condition N (ii) is due to Browder [6].

Condition (N). Let G be a domain and A a finite multiindex set with $(1) \subset A$. For each $\alpha \in K(A)$ we assume we are given a function $b_\alpha : \overline{G}\times\mathbf{R}^r \to \mathbf{R}, r = \#K(A)$, that satisfies the Carathéodory condition (4.4). Further we will suppose that the following conditions hold:

(i) The functions b_α may be split into two parts: $b_\alpha(x,t) = h_{\alpha 1}(x,t) + h_{\alpha 2}(x,t)$ for $x \in \overline{G}, t \in \mathbf{R}^r$. The functions $h_{\alpha 1}(\cdot,\cdot)$ and $h_{\alpha 2}(\cdot,\cdot)$ shall both satisfy the Carathéodory condition (4.4).

(ii) For the functions $h_{\alpha 1}$ we suppose the existence of a function $k_0 \in L^1(G), k_0 \leq 0$, such that

$$\sum_{\alpha\in K(A)} h_{\alpha 1}(x,t)t_\alpha \geq k_0(x) \tag{6.1}$$

for almost all $x \in G$ and all $t \in \mathbf{R}^r, t = (t_{\alpha 1}, \ldots, t_{\alpha r})$. Further we suppose that for each $\varepsilon > 0$ there exists a function $k_\varepsilon \in L^1_{\mathrm{loc}}(G)$ such that

$$|h_{\alpha 1}(x,t)| \leq \varepsilon \sum_{\beta\in K(A)} h_{\beta 1}(x,t)t_\beta + k_\varepsilon(x) \tag{6.2}$$

for almost all $x \in G$, all $t \in \mathbf{R}^r$ and all $a \in K(A)$.

(iii) For the functions $h_{\alpha 2}$ we suppose that there exist functions $k_1 \in L^2(G), k_2 \in L^1(G), k_2 \leq 0$, and constants $C_4, C_5, C_6 \geq 0, \widetilde{C}\cdot C_5 < C_0$, ($\widetilde{C}$ as in (3.5)) $C_6 \geq C_1$ (C_0, C_1 as in Gårding's inequality (4.3)) such that

$$|h_{\alpha 2}(x,t)| \leq k_1(x) + C_4|t| \tag{6.3}$$

for all $\alpha \in K(A)$ (cp. (4.5)) and

$$\sum_{\alpha\in K(A)} h_{\alpha 2}(x,t)t_\alpha \geq k_2(x) - C_5|t|^2 + C_6|t_0|^2 \tag{6.4}$$

for almost all $x \in G$ and all $t \in \mathbf{R}^r$.

Remark 6.1. The type of nonlinearities discussed in the paper of Simader [28], where $p_\alpha(x)g_\alpha(t_\alpha)$ with $p_\alpha \in L^1_{\mathrm{loc}}(G), p_\alpha \geq 0, g_\alpha \in C^0(\mathbf{R})$ and $g_\alpha(t_\alpha)t_\alpha \geq 0$, satisfy the condition (N) with $h_{\alpha 2}(x,t) = 0, k_0(x) = 0$ and

$$k_\varepsilon(x) = \sum_{\alpha \in K(A)} 2\left\{\max_{t_\alpha \in [\frac{-1}{\varepsilon},\frac{1}{\varepsilon}]} |g_\alpha(t_\alpha)|\right\} p_\alpha(x).$$

This is true since

$$|p_\alpha(x)g_\alpha(t_\alpha| \leq \varepsilon p_\alpha(x)g_\alpha(t_\alpha)t_\alpha$$

for $\varepsilon|t_\alpha| \geq 1$. For $\varepsilon|t_\alpha| \leq 1$ one has

$$\begin{aligned}|p_\alpha(x)g_\alpha(t_\alpha)| &\leq \varepsilon p_\alpha(x)g_\alpha(t_\alpha)t_\alpha + p_\alpha(x)|g_\alpha(t_\alpha)(1-\varepsilon t_\alpha)| \\ &\leq \varepsilon p_\alpha(x)g_\alpha(t_\alpha)t_\alpha + 2p_\alpha(x)|g_\alpha(t_\alpha)|.\end{aligned}$$

Examples. We will restrict our attention to nonlinearities consisting of odd powers. Let us take the linear operator $A_1 u = u_{yy} - u_{xxxx}$. In Section 3.5 we had seen that $K(A_1) = \{(0,0),(1,0)\}$. Therefore our conditions include equations of the following kind

$$u_{xxxx} - u_{yy} + u^{2p+1} + (u_x^{2q+1})_x = f,$$

with $p, q \in \mathbf{N}$. Taking $A_2 u = \partial_{xxxx}(u_{xxxx} + u_{xxyy} + u_{yyyy})$ we may construct nonlinear terms that include the following derivatives $u, u_x, u_{xx}, u_{xxx}, u_y, u_{xy}$ and u_{xxy}. For example we could take

$$\partial_{xxxx}(u_{xxxx} + u_{xxyy} + u_{xxxx}) + (u_{xy}^{2p+1})_{xy} + (u_{xxy}^{2q+1})_{xxy} = f$$

for $p, q \in \mathbf{N}$. Finally looking at $A_3 u = u_{xxyy}$ we may only take functions of u, i.e.

$$u_{xxyy} + u^{2p+1} = f$$

with $p \in \mathbf{N}$.

6.2. We will now introduce the **cut off operators** $A_n(\cdot)$. Let $f : \mathbf{R}^m \to \mathbf{R}, m \in \mathbf{N}$, be an arbitrary function then we set

$$sgn f(x) := \begin{cases} f(x)/|f(x)| & \text{for } f(x) \neq 0 \\ 0 & \text{for } f(x) = 0. \end{cases}$$

The cut of operators A_n may now be defined as follows

$$A_n f(x) := \begin{cases} f(x) & \text{for } |f(x)| \leq n \\ n\, sgn f(x) & \text{for } |f(x)| > n. \end{cases}$$

If $f \in L^2(G)$ then $A_n f \in L^2(G)$. For $n > 0$ and $\alpha \in K(A)$ we define

$$h_{\alpha j,n}(x,t) := \chi_{B_n}(x) A_n h_{\alpha j}(x,t) \quad j = 1,2. \tag{6.5}$$

From now on we will denote the respective Nemytskiĭ operators using capital letters, e.g. $H_{\alpha 1,n}(u)$ for $h_{\alpha 1,n}(\cdot, \mu(u)(\cdot))$ etc.

By $i : \mathbf{R} \to \mathbf{R}$ we denote the identity map, i.e. the map defined by $i(t) = t$. As above we set $i_n(x,t) : \chi_{B_n}(x) \cdot A_n i(t)$ or

$$I_n(u)(x) := i_n(x, \mu(u)(x)). \tag{6.6}$$

Then $i_n(\cdot,\cdot)$ satisfies the assumptions for $b_0(\cdot,\cdot)$ in Lemma 5.1.

Remark 6.2. Take $(x,t) \in \overline{G} \times \mathbf{R}^r$. If $h_{\alpha 1}(x,t)t_\alpha \geq 0$ then $h_{\alpha 1,n}(x,t)t_\alpha \geq 0$. Therefore, in (6.1) one can replace the terms $h_{\alpha 1}(\cdot,\cdot)$ by $h_{\alpha 1,n}(\cdot,\cdot)$. In on the other hand $h_{\alpha 1}(x,t)t_\alpha < 0$ then we have $h_{\alpha 1,n}(x,t)t_\alpha \geq h_{\alpha 1}(x,t)t_\alpha$. This implies that in (6.1) all functions $h_{\alpha 1}(\cdot,\cdot)$ may be replaced by the respective cut off function $h_{\alpha 1,n}(\cdot,\cdot), n \in \mathbf{N}$. For the functions $h_{\alpha 2}$ and inequality (6.4) the same is true by analogy.

The conditions N (ii) do not guarantee that $H_{\alpha 1}(u) \in L^2(G)$ for $u \in H_0^A(G)$. We will therefore need the following technical lemma that will be applied in the proof of Theorem 6.5. The condition (6.7) will be verified there.

Lemma 6.3. *Let $(u_k)_{k\in\mathbf{N}}$ a sequence in $H_0^A(G)$ that converges weakly to $u \in H_0^A(G)$ such that for all $\alpha \in K(A)$ one has $\partial^\alpha u_k \to \partial^\alpha u$ pointwise almost everywhere in G. Suppose that the functions $h_{\alpha 1}(\cdot,\cdot)$ satisfy condition (N(i), (ii)). Further assume that*

$$(6.7) \qquad \sup_{k\in\mathbf{N}} \left| \int_G \sum_{\alpha\in K(A)} H_{\alpha 1,n}(u_k)\partial^\alpha u_k \, dx \right| \leq M < \infty$$

for a constant $M \in \mathbf{R}$. Then one obtains $H_{\alpha 1}(u) \in L^1_{\mathrm{loc}}(G)$ and $\sum_{\alpha\in K(A)} H_{\alpha 1}(u)\partial^\alpha u \in L^1(G)$. For any function $\varphi \in C_0^\infty(G)$ and every $\alpha \in K(A)$ one has

$$(6.8) \qquad \int_G H_{\alpha 1,k}(u_k)\partial^\alpha \varphi \, dx \to \int_G H_{\alpha 1}(u)\partial^\alpha \varphi \, dx$$

for $k \to \infty$. Furthermore one also obtains $\langle I_k(u_k), \varphi\rangle_0 \to \langle u, \varphi\rangle_0$ for $k \to \infty$.

PROOF : Obviously the functions $h_{\alpha 1,k}(\cdot,\cdot)$ will also satisfy the Carathéodory condition (4.4). Therefore the functions $h_{\alpha 1,k}(\cdot,\mu(u_k)(\cdot))$ and $h_{\alpha 1}(\cdot,\mu(u)(\cdot))$ are measurable (cp. Vainberg [30], Theorem 18.3). Take $x \in G$ with $h_{\alpha 1}(x,\cdot)$ continuous and $\partial^\alpha u_k(x) \to \partial^\alpha u(x)$ for all $\alpha \in K(A)$. Then the functions $h_{\alpha 1,k}(x,\cdot)$ are continuous for all $k \in \mathbf{N}$. Now it follows that for any $k \in \mathbf{N}$: $h_{\alpha 1,k}(x,\mu(u_n)(x)) \to h_{\alpha 1,k}(x,\mu(u)(x))$ for $n \to \infty$ and $h_{\alpha 1,k}(x,\mu(u)(x)) \to h_{\alpha 1}(x,\mu(u)(x))$ for $k \to \infty$. This implies for the diagonal sequence $n = k$ that $h_{\alpha 1,k}(x,\mu(u_k)(x)) \to h_{\alpha 1}(x,\mu(u)(x))$ for almost all $x \in G$. Due to inequality (6.1) we have $((\sum_{\alpha\in K(A)} h_{\alpha 1,k}(x,\mu(u_k)(x))\partial^\alpha u_k(x)) - k_0(x) \geq 0)$. Therefore our assumption (6.7) and Fatou's Theorem imply that $\sum_{\alpha\in K(A)} h_{\alpha 1}(x,\mu(u))\partial^\alpha u = \sum_{\alpha\in K(A)} H_{\alpha 1}(u)\partial^\alpha u \in L^1(G)$ and we also obtain $\int_G \sum_{\alpha\in K(A)} H_{\alpha 1}(u)\partial^\alpha u \, dx \leq M$.

Now let $G'' \subset\subset G' \subset\subset G$ be otherwise arbitrarily chosen domains. Then from inequality (6.2) we get

(6.9)
$$\begin{aligned}\int_{G''} |H_{\alpha 1,k}(u_k)(x)|\,dx &= \int_{G''} |h_{\alpha 1,k}(x,\mu(u_k)(x))|\,dx\\ &\le \varepsilon\left[\int_{G''}\left(\sum_{\beta\in K(A)} h_{\beta 1,k}(x,\mu(u_k)(x))\partial^\beta u_k(x)\right) - k_0(x)\,dx\right]\\ &+\int_{G''} k_0(x)\,dx + \int_{G''} k_\varepsilon(x)\,dx\\ &\le \varepsilon M + 2\varepsilon\|k_0\|_{L^1(G)} + \|k_\varepsilon\|_{L^1(G'')}\end{aligned}$$

for any $\varepsilon > 0$.

This implies that the sequence $(H_{\alpha 1,k}(u_k))_{k\in N}$ which converges pointwise almost everywhere has uniformly absolute continuous integrals on G' (cp. Natanson [22], Chapter VI.3). For, take any $\lambda > 0$ then there exists a $\delta(\lambda) > 0$ such that $\nu(G'') < \delta(\lambda)$ (here ν denotes the N-dimensional Lebesgue measure) implies that $\int_{G''}|H_{\alpha 1,k}(u_k)|\,dx < \lambda$. (For a given λ one may choose ε so small that $\varepsilon M + 2\varepsilon\|k_0\|_{L^1(G)} < \lambda/2$. Now one can take $\delta(\lambda) > 0$ small enough so that the absolute continuity of the Lebesgue integral implies $\|k_\varepsilon\|_{L^1(G'')} < \lambda/2$ for all $G'' \subset\subset G'$ with $\nu(G'') < \delta(\lambda)$.)

Since G' was chosen arbitrarily we now obtain $H_{\alpha 1}(u) \in L^1_{\mathrm{loc}}(G)$ by applying Vitali's Theorem (for $N = 1$ see Natanson [22], Theorem VI.3.2). Analogously as in (6.9) one can show for any arbitrary but fixed function $\varphi \in C_0^\infty(G)$, setting $G'' = \operatorname{supp}\varphi$ as well as $\max\limits_{\alpha\in K(A)}\max\limits_{x\in G''}|\partial^\alpha\varphi(x)| = C(\varphi)$,

$$\int_{G''} |H_{\alpha 1,k}(u_k)\partial^\alpha\varphi|\,dx \le C(\varphi)[\varepsilon M + 2\varepsilon\|k_0\|_{L^1(G)} + \|k_\varepsilon\|_{L^1(G'')}].$$

Now we may again apply Vitali's Theorem in order to obtain the convergence in (6.8). Finally let us set $h_{\alpha 1}(\cdot,\cdot) = 0$ for $\alpha \neq 0$ and set $h_{01,k}(x,t) = i_k(x,t_0)$. Then due to Remarks 6.1 and the condition N (ii) is fulfilled. Therefore we may apply what we have just shown to this case and obtain $\langle I_k(u_k),\varphi\rangle_0 \to \langle u,\varphi\rangle_0$ for $k\to\infty$. ■

Now we will prove the important a priori inequality.

Lemma 6.4. *(A priori Inequality) Let $A[\cdot,\cdot]$ be a continuous bilinear form defined on the space $H_0^A(G)$ that satisfies Gårding's inequality (4.3). Suppose that for each $\alpha \in K(A)$ we are given a function $b_\alpha(\cdot,\cdot)$ such that the condition (N) is fulfilled. Take $f \in L^2(G)$, $u \in H_0^A(G)$ and suppose that for some $n \in \mathbf{N}$*

(6.10)
$$\begin{aligned}&A[u,\varphi] + C_1\langle u,\varphi\rangle_0 - C_1\langle I_n(u),\varphi\rangle_0\\ &\quad+\sum_{\alpha\in K(A)}\langle H_{\alpha 1,n}(u) + H_{\alpha 2,n}(u),\partial^\alpha\varphi\rangle_0 = \langle f,\varphi\rangle_0\end{aligned}$$

is valid for all $\varphi \in H_0^A(G)$. *Then one has*

$$\|u\|_A \le C(k_0, k_2, f), \tag{6.11}$$

indepently of $n \in \mathbf{N}$.

PROOF : By using (4.3) and (6.10) for $u \in H_0^A(G)$ we obtain the estimate

(6.12)

$$C_0\|u\|_A^2 \le A[u,u] + C_1\|u\|_0^2 = -\sum_{\alpha\in K(A)} \langle H_{\alpha 1,n}(u) + H_{\alpha 2,n}(u), \partial^\alpha u\rangle_0 + \langle f, u\rangle_0 + C_1\langle I_n(u), u\rangle_0.$$

Since $0 \le I_n(u)\cdot u \le |u|^2$ this implies

$$C_0\|u\|_A^2 \le -\Big(\sum_{\alpha\in K(A)} \langle H_{\alpha 1,n}(u), \partial^\alpha u\rangle_0 + \langle H_{\alpha 2,n}(u), \partial^\alpha u\rangle_0\Big) + \langle f, u\rangle_0 + C_1\|u\|_0^2.$$

Using the modified inequalities (6.1) and (6.4) (cp. Remark 6.2), the fact that $C_1 - C_6 \le 0$ due to condition N (ii) and taking $\widetilde{C}$ as in (3.5) we obtain from (6.12) the estimates

(6.13)

$$\begin{aligned} C_0\|u\|_A^2 &\le \int_G -k_0(x)\,dx + \int_G -k_2(x)\,dx \\ &\quad + C_5\sum_{\alpha\in K(A)}\int_G |\partial^\alpha u|^2\,dx + (C_1 - C_6)\|u\|_0^2 + \|f\|_0\|u\|_0 \\ &\le \|k_0\|_{L^1(G)} + \|k_2\|_{L^1(G)} + C_5\|u\|_{K(A)}^2 + \|f\|_0\|u\|_0 \\ &\le C(k_0,k_2) + \widetilde{C}C_5\|u\|_A^2 + \|f\|_0\|u\|_A. \end{aligned}$$

Since due to condition N (iii) we have $C_0 - \widetilde{C}C_5 > 0$, we may chose an $\varepsilon > 0$ such that $\frac{\varepsilon}{2} < c_0 - \widetilde{C}C_5$. Now using $2ab \le \varepsilon a^2 + \frac{1}{\varepsilon}b^2$ we obtain

$$\begin{aligned} (C_0 - \widetilde{C}C_5)\|u\|_A^2 &\le C(k_0,k_2) + \|f\|_0\|u\|_A \\ &\le C(k_0,k_2) + \frac{1}{2\varepsilon}\|f\|_0^2 + \frac{\varepsilon}{2}\|u\|_A^2. \end{aligned}$$

Since $(C_0 - \widetilde{C}C_5) - \frac{\varepsilon}{2} > 0$ we have shown that $\|u\|_A^2$ is bounded, i.e. that (6.11) holds. ∎

Theorem 6.5. *(Existence Theorem for strong Nonlinearities) Let* $G \subset \mathbf{R}^N$ *be a domain. Take* $A[\cdot,\cdot]$ *to be a continuous bilinear form that satisfies Gårding's inequality (4.3). Suppose we are given functions* $b_\alpha(\cdot,\cdot), \alpha \in K(A)$ *that satisfy the* *condition (N). Then for each function $f \in L^2(G)$ there exists a solution $u \in H_0^A(G)$ satisfying $B_\alpha(u) = b_\alpha(\cdot,\mu(u))(\cdot) \in L^1_{\mathrm{loc}}(G)$ as well as $\sum\limits_{\alpha\in K(A)} b_\alpha(\cdot,\mu(u)(\cdot))\partial^\alpha u \in L^1(G)$ such that the functional equation*

$$A[u,\varphi] + \sum_{\alpha\in K(A)} \int_G B_\alpha(u)\partial^\alpha\varphi\,dx = \langle f,\varphi\rangle_0 \tag{6.14}$$

holds for all $\varphi \in C_0^\infty(G)$.

PROOF : Take $f \in L^2(G)$ arbitrarily but fixed. Due to $|h_{\alpha j,n}(x,t)| \le n\chi_{B_n}(x)$, $j = 1,2$, and $i_n(x,t) \le n\chi_{B_n}(x)$ these functions fulfill the assumptions of Lemma 5.1. The bilinear form $A[\cdot,\cdot]+C_1\langle\cdot,\cdot\rangle_0$ is positive. Therefore we may apply Theorem 5.2 for each $n \in \mathrm{N}$ to obtain a solution $u_n \in H_0^A(G)$ to the equation

$$\begin{aligned} a[u_n,\varphi] + C_1\langle u_n,\varphi\rangle_0 - C_1\langle I_n(u_n),\varphi\rangle_0 \\ + \sum_{\alpha\in K(A)} \langle H_{\alpha1,n}(u_n) + H_{\alpha2,n}(u_n), \partial^\alpha\varphi\rangle_0 = \langle f,\varphi\rangle_0 \end{aligned} \tag{6.15}$$

for any $\varphi \in C_0^\infty(G)$. Due to Lemma 6.4 the sequence of solutions $(u_n)_{n\in\mathrm{N}}$ is bounded in $H_0^A(G)$ independently of $n \in \mathrm{N}$. Using (6.3) we deduce that

$$\|H_{\alpha2,n}(u_n)\|_0 \le \|k_1\|_0 + C_4\|u_n\|_{K(A)} \tag{6.16}$$

holds, i.e. for every α the sequence $(H_{\alpha2,n}(u_n))_{n\in\mathrm{N}}$ is bounded in $L^2(G)$. Using $A[u_n,u_n] + C_1\|u_n\|_0^2 \ge 0$, $\|I_n(u_n)\|_0 \le \|u_n\|_0$ and (6.15), (6.4) we obtain

$$\begin{aligned} \sum_{\alpha\in K(A)} \langle H_{\alpha1,n}(u_n),\partial^\alpha u_n\rangle_0 &= -A[u_n,u_n] - C_1\|u_n\|_0^2 + C_1\langle I_n(u_n),u_n\rangle_0 \\ &\quad - \sum_{\alpha\in K(A)} \langle H_{\alpha2,n}(u_n),\partial^\alpha u_n\rangle_0 + \langle f,u_n\rangle_0 \\ &\le C_1\|u_n\|_0^2 + \|k_2\|_{L^1(G)} + \widetilde{C}C_5\|u_n\|_A^2 \\ &\quad - C_6\|u_n\|_0^2 + \|f\|_0\|u_n\|_0. \end{aligned}$$

Because of the assumption $C_1 - C_6 \le 0$ this implies that

$$\sum_{\alpha\in K(A)} \langle H_{\alpha1,n}(u_n),\partial^\alpha u_n\rangle_0 \le \|k_2\|_{L^1(G)} + \widetilde{C}C_5\|u_n\|_A^2 + \|f\|_0\|u_n\|_A.$$

On the other hand we may deduce from (6.15) and (6.16) (C_2 as given in Section

4.2)

$$
\begin{aligned}
-\sum_{\alpha\in K(A)} \langle H_{\alpha 1,n}(u_n), \partial^\alpha u_n\rangle_0 &= A[u_n,u_n] + C_1\|u_n\|_0^2 - C_1\langle I_n(u_n), u_n\rangle_0 \\
&\quad + \sum_{\alpha\in K(A)} \langle H_{\alpha 2,n}(u_n), \partial^\alpha u_n\rangle_0 - \langle f, u_n\rangle_0 \\
&\le C_2\|u_n\|_A^2 + \sum_{\alpha\in K(A)} \left(\|k_1\|_0 + C_4\|u_n\|_{K(A)}\right)\|\partial^\alpha u_n\|_0 \\
&\quad + \|f\|_0\|u_n\|_0 \\
&\le C_2\|u_n\|_A^2 + \widetilde{C}\left[\left(\|k_1\|_0 + \widetilde{C}C_4\|u_n\|_A\right)\|u_n\|_A\right. \\
&\quad \left. + \|f\|_0\|u_n\|_A\right].
\end{aligned}
$$

We have now shown that the sequence $(\sum_{\alpha\in K(A)} \langle H_{\alpha 1,n}(u_n), \partial^\alpha u_n\rangle_0)_{n\in N}$ is bounded in $\mathbf{R}$.

Since the sequence $(u_n)_{n\in N}$ is bounded in $H_0^A(G)$ we obtain from Lemma 3.8 a subsequence $(u_k)_{k\in \mathbf{N}}$ that converges weakly to an element $u \in H_0^A(G)$. Further we have $\partial^\alpha u_k \to \partial^\alpha u$ pointwise almost every where in G for all $\alpha \in K(A)$. Now we may apply Lemma 6.3 to $\langle I_k(u_k), \varphi\rangle_0 + \sum_{\alpha\in K(A)} \langle H_{\alpha 1,k}(u_k), \partial^\alpha \varphi\rangle_0$. Due to $i_k(x,t_0)t_0 \ge 0$ and Remark 6.2 the condition N (ii) is satisfied even if $h_{01,k}(\cdot,\cdot)$ is replaced by $h_{01,k}(\cdot,\cdot) + i_k(\cdot,\cdot)$. Therefore we obtain $H_{\alpha 1}(u) \in L^1_{\mathrm{loc}}(G)$, $\sum_{\alpha\in K(A)} H_{\alpha 1}(u)\partial^\alpha u \in L^1(G)$ and we get the convergence

$$
\begin{aligned}
&\langle I_k(u_k), \varphi\rangle_0 + \sum_{\alpha\in K(A)} \langle H_{\alpha 1,k}(u_k), \partial^\alpha \varphi\rangle_0 \\
&\qquad \to \langle u, \varphi\rangle_0 + \sum_{\alpha\in K(A)} \langle H_{\alpha 1}(u), \partial^\alpha \varphi\rangle_0
\end{aligned}
$$

for any $\varphi \in C_0^\infty(G)$. Due to (6.16) the subsequence may be chosen such that $H_{\alpha 2,k}(u_k)$ converges weakly in $L^2(G)$. Since $\partial^\alpha u_k \to \partial^\alpha u$ converges pointwise almost everywhere it follows that the sequence $H_{\alpha 2,k}(u_k) \to H_{\alpha 2}(u)$ converges pointwise almost everywhere. Because weak limits are unique it follows that $H_{\alpha 2,k}(u_k)$ converges weakly to $H_{\alpha 2}(u)$ in $L^2(G)$, i.e. $\langle H_{\alpha 2,k}(u_k), \partial^\alpha\varphi\rangle \to \langle H_\alpha 2(u), \partial^\alpha \varphi\rangle$ for all $\varphi \in C_0^\infty(G)$.

Since the map $A[\cdot,\varphi] : H_0^A(G) \to \mathbf{R}$ defines for any given $\varphi \in C_0^\infty(G)$ a continuous linear functional the weak convergence of the sequence $(u_k)_{k\in\mathbf{N}}$ implies that $A[u_k,\varphi] \to A[u,\varphi]$ for $k \to \infty$. Now take $\varphi \in C_0^\infty(G)$ arbitrarily but fixed. Then we take the limit $k \to \infty$ in (6.15) to obtain

$$
A[u,\varphi] + \sum_{\alpha\in K(A)} \langle H_{\alpha 1}(u) + H_{\alpha 2}(u), \partial^\alpha \varphi\rangle_0 = \langle f, \varphi\rangle_0
$$

for all $\varphi \in C_0^\infty(G)$, i.e. (6.14) holds. ∎

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