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Solvability and multiplicity results for variational inequalities

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Abstract. We study the solvability and the multiplicity of solutions of variational inequalities of the following type

$$u \in K : \qquad \langle \lambda u - F(u,\lambda), v - u\rangle \geq 0 \qquad \forall v \in K,$$

where K is a closed convex cone in a real Hilbert space H and $F : H \times R \to H$ is a completely continuous, asymptotically linear map.

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This paper is concerned with inequalities of the following form

$$(1) \qquad u \in K : \qquad \langle \lambda u - Au - g(u,\lambda) - f, u - u\rangle \geq 0 \qquad \forall v \in K,$$

where

$$(A)\begin{cases} H \text{ is a real separable Hilbert space with the scalar product } \langle\cdot,\cdot\rangle, \\ K \text{ is a closed convex cone in } H \text{ with its vertex at zero,} \\ \quad K \neq \emptyset, K \neq H, K \neq \{0\}, \\ A : H \to H \text{ is a completely continuous linear operator,} \\ g : H \times R \to H \text{ is a (nonlinear) completely continuous map,} \\ f \in H \text{ is a right–hand side,} \\ \lambda \in R^+ := (0,+\infty). \end{cases}$$

Using the projection $P_K : H \overset{\text{onto}}{\to} K$ we reformulate the inequality (1) as a nonlinear equation and then we study the solvability of this equation (for sublinear g) using the Leray–Schauder degree.

We prove various multiplicity, existence and non–existence results for the solutions of the inequality

$$(2) \qquad u \in K \qquad \langle \lambda u - Au - f, v - u\rangle \geq 0 \qquad \forall v \in K$$

and as consequence of our considerations we get also the existence of nontrivial solutions of the inequality

$$(3) \qquad u \in K \qquad \langle \lambda u - F(u), v - u\rangle \geq 0 \qquad \forall v \in K$$

where $F : H \to H$ is a completely continuous map, $F(0) = 0$ and $F'(0), F'(\infty)$ fulfil some additional assumptions (in particular $F'(0) \neq F'(\infty)$).

Our assertions imply also some existence results for bifurcation points of variational inequalities; these results are close to the results of Miersemann [7], [8], [9] and Kučera [4], [5], [6]. Moreover, our bifurcations are global (in the sense of Rabinowitz [20]).

Our method is the same as in [11], nevertheless many of our results are new. The reformulation of the problem (1) is just sketched, all details can be found in [11].

Let us mention that another degree-theoretic approach to variational inequalities was used by Szulkin [17],[18], [19] and that our degree $d(\lambda)$ is very close to the degree investigated by Švarc [14], [15], [16] in problems involving operators with jumping nonlinearities (in fact, these two degrees coincide for some special cones in R^n).

In the whole paper we will assume (A).

1. Preliminaries.

We will denote by $\sigma_K(A)$ the set of all (real) eigenvalues of the inequality

$$(4) \qquad u \in K \qquad \langle \lambda u - Au, v - u\rangle \geq 0 \qquad \forall v \in K$$

i.e. the set of all $\lambda \in R$ such that the inequality (4) has a nontrivial solution.

Further denote by $\sigma(A)$ the spectrum of the operator A and put

$$\sigma_K^+(A) := \sigma_K(A) \cap R^+, \quad \sigma^+(A) := \sigma(A) \cap R^+, \quad \text{where } R^+ := (0, \infty).$$

Note that the set $\sigma_K^+(A)$ is closed in R^+ and that the set $\sigma_K(A)$ is bounded by $\pm\|A\|$. In general, the set $\sigma_K(A)$ may contain an open interval even for $H = R^3$ and it may also consist of only one point even for $\dim(H) = +\infty$, A symmetric (see [10], [11]).

Let A^* be the adjoint operator to A. We will denote

$$\begin{aligned}
E(\lambda) &:= \operatorname{Ker}(\lambda I - A),\\
E^*(\lambda) &:= \operatorname{Ker}(\lambda I - A^*),\\
E_K(\lambda) &:= \{u \in K; \quad \langle \lambda u - Au, v - u\rangle \geq 0 \quad \forall v \in K\},\\
E_K^*(\lambda) &:= \{u \in K; \quad \langle \lambda u - A^*u, v - u\rangle \geq 0 \quad \forall v \in K\}.
\end{aligned}$$

Moreover, for $\lambda_0 \in R^+$ we put

$$\begin{aligned}
\lambda_0^+ &:= \inf\{\lambda \in \sigma_K(A); \lambda > \lambda_0\},\\
\lambda_0^- &:= \sup(\{0\} \cup \{\lambda \in \sigma_K(A); \lambda < \lambda_0\}),\\
\beta(\lambda_0) &:= \sum_{\lambda > \lambda_0} \dim\left(\bigcup_{p=1}^{\infty} \operatorname{Ker}(\lambda I - A)^p\right),\\
\gamma(\lambda_0) &:= \sum_{\lambda \geq \lambda_0} \dim\left(\bigcup_{p=1}^{\infty} \operatorname{Ker}(\lambda I - A)^p\right).
\end{aligned}$$

If $\{\lambda_n\}$ is a decreasing sequence of real numbers, $\lambda_n \to \lambda_0, \lambda_n > \lambda_0$, then we shall write $\lambda_n \downarrow \lambda_0$; analogously $\lambda_n \uparrow \lambda_0$. Finally, we put

$$\begin{aligned}
B_R(u_0) &:= \{u \in H; \|u - u_0\| < R\}, \quad B_R := B_R(0),\\
S_1 &:= \{u \in H; \|u\| = 1\},\\
P_K &:= \text{ the projection of } H \text{ onto } K,\\
\partial M &:= \text{ the boundary of } M,\\
\overline{M} &:= \text{ the closure of } M,\\
M^0 &:= \text{ the interior of } M,\\
K^a &:= \{u \in K; (\exists D \subset H, \overline{D} = H)(\forall w \in D)(\exists \varepsilon > 0) \quad u \pm \varepsilon w \in K\},\\
K^A &:= \{u \in K; (\forall w \in \cup_{\lambda \in R} E(\lambda))(\exists \varepsilon > 0) \quad u + \varepsilon w \in K\}.
\end{aligned}$$

Obviously, $K^0 \subset K^a$. If, moreover, A is symmetric, then $K^0 \subset K^A \subset K^a$.

Example 1. Let $\Omega := (0,\pi)^2 \subset R^2, H := W_0^{1,2}(\Omega)$ (the Sobolev space), $\langle u, v\rangle := \int_\Omega \nabla u \cdot \nabla v \, dx, \langle Au, v\rangle := \int_\Omega uv \, dx$, $K := \{u \in H; u \geq 0 \text{ on } M\}$, where $M \subset \Omega$ is a closed set of positive capacity. Then one can easily prove $K^0 = \emptyset$, nevertheless $K^A \neq \emptyset$ (e.g. if $u \geq \varepsilon > 0$ on M, then $u \in K^A$).

Lemma 1. *Let $E^*(\lambda) \cap K^a \neq \emptyset$. Then $E_K(\lambda) = E(\lambda) \cap K$.*

PROOF : Obviously $E(\lambda) \cap K \subset E_K(\lambda)$. We shall prove the converse inclusion. Let $u \in E_k(\lambda)$ and choose $u^* \in E^*(\lambda) \cap K^a$. By the definition of K^a there exists $D \subset H, \overline{D} = H$, such that $(\forall w \in D)(\exists \varepsilon > 0) u^* \pm \varepsilon w \in K$. Putting $v = u + u^* \pm \varepsilon w$ in (4) we obtain

$$0 \leq \langle \lambda u - Au, u^* \pm \varepsilon w\rangle = \langle u, \lambda u^* - A^* u^*\rangle + \langle \lambda u - Au, \pm \varepsilon w\rangle = \pm \varepsilon \langle \lambda u - Au, w\rangle,$$

hence $\lambda u - Au \in D^\perp = \{0\}, \quad u \in E(\lambda)$. ∎

Lemma 2. *Let K be such that it is not a subspace of H (i.e.* $\text{span}(K) \neq K$*). Then there exists $0 \neq u_0 \in K$ such that $\langle u, u_0\rangle \geq 0 \quad \forall u \in K$.*

PROOF : Choose $v_0 \in \text{span}(K) - K$. Then $\{v_0\}$ and K are disjoint closed convex sets, $\{v_0\}$ is compact, and according to Hahn–Banach theorem there exists $0 \neq u_1 \in \overline{\text{span}(K)}$ such that $\langle u, u_1\rangle \geq 0 \quad \forall u \in K$. Put $u_0 := P_K u_1$. Since K is a cone with its vertex at zero, we get using the characterization of the projection P_K

$$\langle u_1 - P_K u_1, P_K u_1\rangle = 0$$

and

$$\langle u_1 - P_K u_1, u\rangle \leq 0 \quad \text{for any } u \in K,$$

which implies $\langle u_0, u\rangle = \langle P_K u_1, u\rangle \geq \langle u_1, u\rangle \geq 0$ for any $u \in K$. Since $\langle u_0, \tilde{u}\rangle \geq \langle u_1, \tilde{u}\rangle > 0$ for suitable $\tilde{u} \in K$, we have $u_0 \neq 0$. ∎

2.Reformulation of the problem and bifurcations.

The problem (1) is equivalent to the equation

$$T(u) = 0, \tag{5}$$

where $T : H \to H, T(u) := u - \frac{1}{\lambda}P_K(Au + g(u,\lambda) + f)$ (see [**11**]).

We shall often write $T(\lambda, f, g)$ or $T(\lambda, f, g, A, K)$ instead of T to indicate the dependence of T on the corresponding parameters (while the other parameters are fixed).

Lemma 3. *(Apriori estimates). Let $J \subset R^+ - \sigma_K(A)$ be a compact set, $\frac{g(u,\lambda)}{\|u\|} \to 0$ for $\|u\| \to \infty$ (uniformly for $\lambda \in J$). Then*

$$(\forall M > 0)(\exists R > 0) \quad \|f\| \le M, \quad t \in [0,1], \quad \lambda \in J, \quad T(\lambda, f, tg)(u) = 0 \Rightarrow \|u\| < R$$

PROOF : [11, Lemma 2]. ■

As a corollary of Lemma 3 and the homotopy invariance property of the Leray–Schauder degree we get that the degree $\deg(T(\lambda, f, g), 0, B_R)$ is well defined for $\lambda \notin \sigma_K(A)$ and for $R > 0$ sufficiently large and does not depend on f and g. Moreover, if we define

$$d(\lambda) := \deg(T(\lambda, 0, 0), 0, B_r)$$

where $r \in R^+$ is arbitrary, then the function $\lambda \mapsto d(\lambda)$ is locally constant on $R^+ - \sigma_K(A)$.

Remark 1. (i) In [**11**], [**13**] there is given a more general version of Lemma 3; the apriori estimates are proved to be independent on some small perturbations of the cone K. As a consequence of this result we get e.g. the following statement:

Let $K_n (n = 1, 2, \dots)$ be closed convex cones in H with their vertices at zero and let

$$\sup_{u \in \overline{B}_1} \|P_K u - P_{K_n} u\| \to 0 \qquad \text{for } n \to \infty. \tag{6}$$

Let $\lambda \in R^+ - \sigma_K(A)$. Then $\lambda \notin \sigma_{K_n}(A)$ and $d_n(\lambda) = d(\lambda)$ for sufficiently large n, where $d_n(\lambda) := \deg(T, \lambda, 0, 0, A, K_n), 0, B_r)$.

Moreover, carefully reading the proof the proof of [11, Lemma 1] one can see that the condition (6) can be weakened to

$$\sup_{u \in \overline{B}_1} \|P_K Au - P_{K_n} Au\| \to 0 \qquad \text{for } n \to \infty.$$

(ii) Denote by $\chi_K(A)$ the set of all $\mu \in R$ such that the inequality

$$u \in K : \qquad \langle u - \mu Au, v - u \rangle \ge 0 \qquad \forall v \in K$$

has a nontrivial solution. For $\mu \neq \chi_K(A)$ we can define

$$\widetilde{d}(\mu) := \deg(I - \mu P_K A, 0, B_r).$$

Then, obviously, $\mu \in \chi_K(A) \cap R^+ \Leftrightarrow \frac{1}{\mu} \in \sigma_K^+(A)$ and $\widetilde{d}(\mu) = d(\frac{1}{\mu})$ for $\mu \in R^+ - \chi_K(A)$. Moreover, if $\langle Au, u \rangle \ge 0$ for any $u \in H$, then one can easily show $\chi_K(A) \subset R^+$, which implies $\widetilde{d}(\mu) = 1$ for $\mu \le 0$.

Lemma 4. *(Local bifurcations). Let $\lambda_1, \lambda_2 \in R^+ - \sigma_K(A), \lambda_1 > \lambda_2$, $d(\lambda_1) \neq d(\lambda_2)$, $\frac{g(u,\lambda_i)}{\|u\|} \to 0$ for $u \to 0$ $(i = 1,2)$ and let $g(0,\lambda) = 0$ for $\lambda \in (\lambda_2, \lambda_1)$. Then there exists a bifurcation point $\lambda_0 \in (\lambda_2, \lambda_1)$ for the inequality*

$$(7) \qquad u \in K: \qquad \langle \lambda u - Au - g(u,\lambda), v - u\rangle \geq 0 \qquad \forall v \in K,$$

i.e. there exists a sequence (u_n, λ_n) of solutions of (7) such that $u_n \neq 0$ and $(u_n, \lambda_n) \to (0, \lambda_0)$. Particularly, $\lambda_0 \in \sigma_K(A)$.

PROOF : [11, Lemma 3]. ■

Lemma 5. *(Global bifurcation). Let λ_0 be an isolated point of $\sigma_K^+(A)$ with $\lim_{\lambda \to \lambda_0^+} d(\lambda) \neq \lim_{\lambda \to \lambda_0^-} d(\lambda)$. Let $\Omega \subset (H \times R)$ be an open set, $(0, \frac{1}{\lambda_0}) \in \Omega$. Put $\mu_0 := \frac{1}{\lambda_0}$ and suppose $\lim_{\substack{u \to 0 \\ (u,\mu) \in \Omega}} \frac{g(u,\mu)}{\|u\|} = 0$ locally uniformly in μ. Further denote by S the closure (in Ω) of all nontrivial solutions (u, μ) of the inequality*

$$u \in K: \qquad \langle u - \mu Au - g(u,\mu), v - u\rangle \geq 0 \qquad \forall v \in K$$

and let C be the component of S containing the point $(0, \mu_0)$.

Then the set C has at least one of the following properties

(i) *C is not bounded*
(ii) *$\overline{C} \cap \partial\Omega \neq \emptyset$*
(iii) *$C \cap (\{0\} \times R) \neq \{(0, \mu_0)\}$.*

PROOF : is the same as the proof of Rabinowitz's global bifurcation theorem [20], [21] so that we shall just sketch it. We shall use the notation from Remark 1 (ii).

Suppose that C has none of the properties (i)–(iii). Then C is compact and similarly as in [20, Lemma 1.3] we can find an open bounded set $\mathcal{O} \subset \Omega$ such that $C \subset \mathcal{O}, S \cap \partial\mathcal{O} = \emptyset$ and $\mathcal{O} \cap (\overline{B}_\rho \times R) = \overline{B}_\rho \times [\mu_0 - \varepsilon, \mu_0 + \varepsilon]$, where $\varepsilon < \mathrm{dist}(\mu_0, \chi_K(A))$. Moreover, we can choose $\rho > 0$ such that the equation $u = P_K(\mu Au + tg(u,\mu))$ is not solvable for $\mu = \mu_0 \pm \varepsilon, 0 < \|u\| \leq \rho$ and $t \in [0,1]$ (see the proof of [11, Lemma 3]). Put $G := \{(u,\mu); \|u\|^2 + (\mu - \mu_0)^2 < \rho^2 + \varepsilon^2\}$; we may suppose $G \subset \Omega$. Further put

$$H_r^t(u,\mu) := (u - P_K(\mu Au + tg(u,\mu)), t(\|u\|^2 - r^2) + (1-t)(\varepsilon^2 - (\mu - \mu_0)^2)).$$

Using the homotopy invariance property of the Leray–Schauder degree we get (for sufficiently large $R > 0$)

$$0 = \deg(H_R^1, 0, \mathcal{O}) = \deg(H_\rho^1, 0, \mathcal{O}) = \deg(H_\rho^1, 0, G) = \deg(H_\rho^0, 0, G) = \widetilde{d}(\mu_0 - \varepsilon) - \widetilde{d}(\mu_0 + \varepsilon) \neq 0,$$

which is a contradiction. ■

3. Determination of $d(\lambda)$.

The following Theorem 1 is proved in [**11**].

Theorem 1. *(i) If $\lambda > \sup \sigma_K(A), \lambda > 0$, then $d(\lambda) = 1$.*
(ii) Let $\lambda_0 \in \sigma^+(A), \dim E(\lambda_0) = 1, E(\lambda_0) \cap K^0 \neq \emptyset, E^*(\lambda_0) \cap K^0 \neq \emptyset$ and choose $u_0 \in E(\lambda_0) \cap K^0, u_0^* \in E^*(\lambda_0) \cap K^0$. Then $\lambda_0^- < \lambda_0 < \lambda_0^+$ (i.e. λ_0 is an isolated point of $\sigma_K^+(A)$) and moreover,

- (a) *if $\langle u_0, u_0^* \rangle > 0$, then $d(\lambda) = (-1)^{\beta(\lambda_0)}$ for any $\lambda \in (\lambda_0, \lambda_0^+), d(\lambda) = 0$ for any $\lambda \in (\lambda_0^-, \lambda_0)$ and there exists a right-hand side $f \in H$ such that the inequality (2) is not solvable for any λ close to $\lambda_0, \lambda < \lambda_0$;*
- (b) *if $\langle u_0, u_0^* \rangle < 0$, then $d(\lambda) = (-1)^{\gamma(\lambda_0)}$ for any $\lambda \in (\lambda_0^-, \lambda_0), d(\lambda) = 0$ for any $\lambda \in (\lambda_0, \lambda_0^+)$ and there exists a right-hand side $f \in H$ such that the inequality (2) is not solvable for any λ close to $\lambda_0, \lambda > \lambda_0$.*

Remark 2. (i) The assertion $d(\lambda) \neq 0$ for some λ enables us to prove that the corresponding inequality (2) (or (1)) is solvable for any $f \in H$. The assertion $d(\lambda) = 0$ does not guarantee the existence of $f \in H$ such that the inequality (2) is not solvable ([**14**]).
(ii) Using Theorem 1 and Lemma 4 or 5 one can easily prove various assertions about the existence of bifurcations of solutions of the inequality (7) (e.g. [11, Corollary of Theorem 3]). Similar assertions can be proved also using the following Theorems 2,3,4,5,6.
(iii) If K is an intersection of a finite number of halfspaces, then we have more precise information about the structure of the solution set of (2): for $\lambda \notin \sigma_K(A)$ and a generic $f \in H$ the number of solutions of (2) is finite, locally constant and its parity depends only on the parity of $d(\lambda)$ ([11, Theorem 5]).
(iv) If K is a halfspace, $K = \{u \in H; \langle u, u_0 \rangle \geq 0\}, \lambda \in R - \sigma(A)$, then the inequality (2) is (uniquely) solvable for any $f \in H$ iff

$$F(\lambda) := \langle (\lambda I - A)^{-1} u_0, u_0 \rangle > 0$$

and $\lambda \in \sigma_K(A)$ iff $F(\lambda) = 0$. If the operator A is symmetric, then the function F is strictly decreasing on each component of $R - \sigma(A)$ ([11, Lemmas 8,9,10]).

The following four theorems are some analogous to Theorem 1 in the case of multiple eigenvalues and cones with empty interior.

Theorem 2. *Let $\lambda_0 \in \sigma^+(A), E^*(\lambda_0) \cap K^a \neq \emptyset$.*

(i) Let $(\forall 0 \neq u \in E(\lambda_0) \cap K)(\exists u^* \in E^*(\lambda_0) \cap K) \quad \langle u, u^* \rangle > 0$. Then $\lambda_0^- < \lambda_0, d(\lambda) = 0$ for $\lambda \in (\lambda_0^-, \lambda_0)$ and there exists a right hand side $f \in H$ such that the inequality (2) is not solvable for λ close to $\lambda_0, \lambda < \lambda_0$.

(ii) Let $(\forall 0 \neq u \in E(\lambda_0) \cap K)(\exists u^* \in E^*(\lambda_0) \cap K) \quad \langle u, u^* \rangle < 0$. Then $\lambda_0^+ > \lambda_0, d(\lambda) = 0$ for $\lambda \in (\lambda_0, \lambda_0^+)$ and there exists a right hand side $f \in H$ such that the inequality (2) is not solvable for λ close to $\lambda_0, \lambda > \lambda_0$.

PROOF : We shall prove only the assertion (i), the proof of (ii) is analogous. All assertions will be proved by a contradiction argument.

First suppose there exist $\lambda_n \in \sigma_K^+(A), \lambda_n \uparrow \lambda_0$. Then there exist $u_n \in E_K(\lambda_n) \cap S_1$. Since $\lambda_n \notin \sigma(A)$ for sufficiently large n, we have $u_n \in \partial K$ for $n \geq n_0$ (each solution of an inequality lying in K^0 is simultaneously a solution of the corresponding equation). Using our reformulation of the problem (1) we get

$$u_n = \frac{1}{\lambda_n} P_K A u_n. \tag{8}$$

Without any loss of generality we may suppose $u_n \rightharpoonup u$. Passing to the limit in (8) we obtain

$$u_n \to u = \frac{1}{\lambda_0} P_K A u,$$

since the right hand side in (8) converges strongly. Thus $u \in E_K(\lambda_0) \cap \partial K \cap S_1$ and according to Lemma 1 we get $u \in E(\lambda_0)$. By our assumptions there exists $u^* \in E^*(\lambda_0) \cap K$ such that $\langle u, u^* \rangle > 0$. Putting $v := u_n + u^*$ in the inequality $\langle \lambda_n u_n - A u_n, v - u_n \rangle \geq 0$ we get

$$\begin{aligned} 0 \leq \langle \lambda_n u_n - A u_n, u^* \rangle &= (\lambda_n - \lambda_0)\langle u_n, u^* \rangle + \langle u_n, \lambda_0 u^* - A^* u^* \rangle = \\ &= (\lambda_n - \lambda_0)\langle u_n, u^* \rangle, \end{aligned}$$

hence $\langle u_n, u^* \rangle \leq 0, \langle u, u^* \rangle \leq 0$, which is a contradiction.

Thus we have $\lambda_0^- < \lambda_0$ and it is sufficient to prove that the inequality (2) is not solvable for suitable f and λ close to $\lambda_0 (\lambda < \lambda_0)$. Our assumptions guarantee that $E^*(\lambda_0) \cap K$ is a closed convex cone (with its vertex at zero) and that it is not a subspace of H. According to Lemma 2 there exists $u_0^* \in E^*(\lambda_0) \cap K \cap S_1$ such that

$$\langle u_0^*, u^* \rangle \geq 0 \qquad \text{for any } u^* \in E^*(\lambda_0) \cap K.$$

Suppose that the inequality (2) is solvable for $f := u_0^*$ and $\lambda_n \uparrow \lambda_0$, i.e. there exist $u_n \in K$ such that

$$\langle \lambda_n u_n - A u_n - u_0^*, v - u_n \rangle \geq 0 \qquad \forall v \in K. \tag{9}$$

Putting $v := u_n + u_0^*$ in (9) we obtain (as above)

$$(\lambda_n - \lambda_0)\langle u_n, u_0^* \rangle \geq \|u_0^*\|^2 > 0,$$

which implies $\|u_n\| \to \infty$. We may suppose $\frac{u_n}{\|u_n\|} \rightharpoonup u$; passing to the limit in the equation

$$\frac{u_n}{\|u_n\|} = \frac{1}{\lambda_n} P_K \Big(A \frac{u_n}{\|u_n\|} + \frac{u_0^*}{\|u_n\|}\Big)$$

we get $\frac{u_n}{\|u_n\|} \to u \in E_K(\lambda_0) = E(\lambda_0) \cap K \quad u \in S_1$. According to our assumptions there exists $u^* \in E^*(\lambda_0) \cap K$ such that $\langle u, u^* \rangle > 0$. Putting $v := u_n + u^*$ in (9) we obtain

$$(\lambda_n - \lambda_0)\langle u_n, u^* \rangle \geq \langle u_0^*, u^* \rangle \geq 0,$$

hence $\langle u_n, u^* \rangle \leq 0, \langle u, u^* \rangle \leq 0$, which is a contradiction. ∎

Theorem 3. *Let $\lambda_0 \in \sigma^+(A), E(\lambda_0) \cap K^0 \neq \emptyset, E^*(\lambda_0) \cap K^0 \neq \emptyset$.*

(i) Choose $\varepsilon \in (0, \sqrt{2})$ and $\delta_1, \delta_2 \in (0, \varepsilon)$ such that

$$\delta_1^2 + \delta_2^2 - \frac{1}{2}\delta_1^2\delta_2^2 \leq \varepsilon^2, \qquad \delta_2^2 \leq \varepsilon^2(1 - \frac{\varepsilon^2}{4}) \tag{10}$$

(we can put e.g. $\delta_1 := \delta_2 := \frac{\varepsilon}{\sqrt{2}}$) and suppose there exist $u_0 \in E(\lambda_0) \cap S_1$ and $u_0^* \in E^*(\lambda_0) \cap S_1$ such that

$$S_1 \cap \overline{B_\varepsilon(u_0^*)} \subset K^0, \tag{11}$$

$$\|u_0 - u_0^*\| \leq \delta_1, \tag{12}$$

$$(\forall u \in E(\lambda_0) \cap \partial K \cap S_1)(\exists u^* \in E^*(\lambda_0) \cap K \cap S_1) \quad \|u - u^*\| \leq \delta_2. \tag{13}$$

Then $\lambda_0^+ > \lambda_0$ and $d(\lambda) = (-1)^{\beta(\lambda_0)}$ for any $\lambda \in (\lambda_0, \lambda_0^+)$.

(ii) Let $u_0 \in E(\lambda_0) \cap K^0, \langle u_0, u^* \rangle \leq 0$ for any $u^* \in E^*(\lambda_0) \cap K, \langle u_0, u_0^* \rangle < 0$ for some $u_0^* \in E^*(\lambda_0) \cap K$. Let, moreover,

$$(\forall u \in E(\lambda_0) \cap \partial K \cap S_1)(\exists u^* \in E^*(\lambda_0) \cap K) \quad \langle u, u^* \rangle > 0.$$

Then $\lambda_0^- < \lambda_0$ and $d(\lambda) = (-1)^{\gamma(\lambda_0)}$ for any $\lambda \in (\lambda_0^-, \lambda_0)$.

PROOF : Similarly as in the proof of Theorem 2 we will argue by contradiction.

(i) First suppose that there exist $\lambda_n \in \sigma_K(A)$ such that $\lambda_n \downarrow \lambda_0$ and choose $u_n \in E_K(\lambda_n) \cap S_1$. As in the proof of Theorem 2 we may suppose $u_n \in \partial K$ and $u_n \to u \in E(\lambda_0) \cap \partial K \cap S_1$. By (13) there exists $u^* \in E^*(\lambda_0) \cap K \cap S_1$ such that $\|u - u^*\| \leq \delta_2$. Using (10) and (11) we obtain $\overline{B_{\delta_2}(u_0^*)} \subset K^0$, hence $u - u^* + u_0^* \in K^0, u_n - u^* + u_0^* \in K$ for sufficiently large n. Putting $v := u_n - u^* + u_0^*$ in the inequality

$$\langle \lambda_n u_n - Au_n, v - u_n \rangle \geq 0 \qquad \forall v \in K$$

we get $(\lambda_n - \lambda_0)\langle u_n, u_0^* - u^* \rangle \geq 0$, hence

$$\langle u, u_0^* \rangle \geq \langle u, u^* \rangle = \frac{1}{2}(\|u\|^2 + \|u^*\|^2 - \|u - u^*\|^2) \geq 1 - \frac{1}{2}\delta_2^2,$$

so that $\|u - u_0^*\| \leq \delta_2, u \in S_1 \cap B_\varepsilon(u_0^*) \subset K^0$, which gives us a contradiction. Thus $\lambda_0^+ > \lambda_0$.

Now let us consider the inequality

$$u \in K : \qquad \langle \lambda u - Au - (\lambda - \lambda_0)u_0, v - u \rangle \geq 0 \qquad \forall v \in K. \tag{14}$$

This inequality has for $\lambda > \lambda_0$ the solution $u := u_0 \in K^0$, which is its unique solution in K^0 for $\lambda \notin \sigma(A)$ (since each solution of the inequality lying in K^0 is also a solution of the corresponding equation). Thus for $\rho > 0$ small, $R > 0$ large, λ close to $\lambda_0 (\lambda > \lambda_0)$ and $T := T(\lambda, (\lambda - \lambda_0)u_0, 0)$ we get

$$d(\lambda) = \deg(T, 0, B_R) = \deg(T, 0, B_\rho(u_0)) + \deg(T, 0, B_R - \overline{B_\rho(u_0)}) =$$
$$= (-1)^{\beta(\lambda_0)} + \deg(T, 0, B_R - \overline{B_\rho(u_0)}),$$

since $T(u) = u - \frac{1}{\lambda}P_K(Au + (\lambda - \lambda_0)u_0) = u - \frac{1}{\lambda}(Au + (\lambda - \lambda_0)u_0)$ for $u \in B_\rho(u_0)$. We shall prove $\deg(T, 0, B_R - \overline{B_\rho(u_0)}) = 0$. To prove this it is sufficient to show that the inequality (14) does not have solution in ∂K for λ sufficiently close to $\lambda_0 (\lambda > \lambda_0)$. Suppose the contrary, i.e. there exist $\lambda_n \downarrow \lambda_0$ and $u_n \in \partial K$ such that

(15) $$\langle \lambda_n u_n - Au_n - (\lambda_n - \lambda_0)u_0, v - u_n \rangle \geq 0 \qquad \forall v \in K.$$

Choosing $v := u_n + u_0^*$ we get $(\lambda_n - \lambda_0)\langle u_n - u_0, u_0^* \rangle \geq 0$, so that

(16) $$\langle u_n, u_0^* \rangle \geq \langle u_0, u_0^* \rangle \geq 1 - \frac{1}{2}\delta_1^2.$$

Hence $\|u_n\| \geq c > 0$ and we may suppose $\frac{u_n}{\|u_n\|} \to u$. Passing to the limit in the equation

$$\frac{u_n}{\|u_n\|} = \frac{1}{\lambda_n} P_K(A\frac{u_n}{\|u_n\|} + (\lambda_n - \lambda_0)\frac{u_0}{\|u_n\|})$$

we get $\frac{u_n}{\|u_n\|} \to u \in E_K(\lambda_0) \cap \partial K \cap S_1$. Further $\frac{u_n}{\|u_n\|} \in \partial K \cap S_1$, thus $\|\frac{u_n}{\|u_n\|} - u_0^*\| \geq \varepsilon$, which implies $\langle \frac{u_n}{\|u_n\|}, u_0^* \rangle \leq 1 - \frac{1}{2}\varepsilon^2$. The last inequality and (16) imply

(17) $$\|u_n\| \geq \frac{2 - \delta_1^2}{2 - \varepsilon^2}.$$

By (13) there exists $u^* \in E^*(\lambda_0) \cap K \cap S_1$ such that

(18) $$\|u - u^*\| \leq \delta_2,$$

thus $u_0^* + u - u^* \in K^0$. Choosing $v := u_n + u_0^* - u^* \in K$ in (15) and dividing this inequality by $\|u_n\|$ we obtain

$$(\lambda_n - \lambda_0)\langle \frac{u_n}{\|u_n\|} - \frac{u_0}{\|u_n\|}, u_0^* - u^* \rangle \geq 0.$$

Using the last inequality together with (16), (17) and (18) we get

$$\langle u, u_0^* \rangle \geq \langle u, u^* \rangle + \limsup_{n \to \infty} \frac{1}{\|u_n\|}((\langle u_0, u_0^* \rangle - \langle u_0, u^* \rangle \geq$$
$$\geq 1 - \frac{1}{2}\delta_2^2 + \frac{2 - \varepsilon^2}{2 - \delta_1^2}(1 - \frac{1}{2}\delta_1^2 - 1) \geq 1 - \frac{1}{2}\varepsilon^2,$$

so that $u \in S_1 \cap \overline{B_\varepsilon(u_0^*)} \subset K^0$, which gives us a contradiction.

(ii) The proof of $\lambda_0^- < \lambda_0$ is the same as that in Theorem 2. Similarly as in the proof of (i) it is now sufficient to prove that the inequality (14) does not have solution in ∂K for λ close to $\lambda_0, \lambda < \lambda_0$. Suppose the contrary, i.e. there exist $u_n \in \partial K$ and $\lambda_n \uparrow \lambda_0$ such that (15) is valid. Choosing $v := u_n + u^*, u^* \in E^*(\lambda_0)$, we get

(19) $$\langle u_n, u^* \rangle \leq \langle u_0, u^* \rangle \leq 0,$$

which implies (putting $u^* := u_0^*$) $\|u_n\| \geq c > 0$. As in the proof of (i) we get now

$$\frac{u_n}{\|u_n\|} \to u \in E_K(\lambda_0) \cap \partial K \cap S_1.$$

By (19) we have $\langle u, u^* \rangle \leq 0$ for any $u^* \in E_K^*(\lambda_0)$, which gives us a contradiction with our assumptions. ∎

Remark 3. If $E(\lambda_0)\cap K^0 \neq \emptyset \neq E^*(\lambda_0)\cap K^0$ and $\dim E(\lambda_0) = 1$, then Theorem 1 enables us to compute the degree $d(\lambda)$ in a neighbourhood of λ_0 in a generic case (if $\langle u_0, u_0^*\rangle \neq 0$). Unfortunately, if $\dim E(\lambda_0) > 1$, then Theorems 2 and 3 do not give us such general answer. The following Theorem 4 guarantees that under additional assumption (20) we are able to compute $d(\lambda)$ for $\lambda > \lambda_0$ again in a generic case (cf. Remark 4).

Theorem 4. *Let $\lambda_0 \in \sigma^+(A), \dim E(\lambda_0) \geq 2, E^*(\lambda_0)\cap K^a \neq \emptyset$ and let moreover,*

$$(20)\qquad (\forall u \in E(\lambda_0)\cap \partial K \cap S_1)(\exists u^* \in E^*(\lambda_0)\cap K) \qquad \langle u, u^*\rangle < 0.$$

Choose $u_0^* \in E^*(\lambda_0)\cap K \cap S_1$ such that $\langle u_0^*, u^*\rangle \geq 0$ for any $u^* \in E^*(\lambda_0)\cap K$ (see Lemma 2) and denote $M := E(\lambda_0)\cap S_1 \cap (E^*(\lambda_0)^\perp \oplus \{cu_0^*; c \geq 0\})$. Then $\lambda_0^+ > \lambda_0$ and for any $\lambda \in (\lambda_0, \lambda_0^+)$ we have

$$\text{(i)}\qquad d(\lambda) = (-1)^{\beta(\lambda_0)} \qquad \textit{if } M \subset K^0,$$

$$\text{(ii)}\qquad d(\lambda) = 0 \qquad \textit{if } M \cap K = \emptyset.$$

Remark 4. Let $\{u_i\}_{i=1}^m, \{u_i^*\}_{i=1}^m$ be orthonormal basis of $E(\lambda_0), E^*(\lambda_0)$, respectively, and let $\det(\langle u_i, u_j^*\rangle) \neq 0$. Then the set M in Theorem 4 consists of exactly one point (see [**13**]).

PROOF of Theorem 4: The proof of $\lambda_0^+ > \lambda_0$ is the same as that in Theorem 2. We shall show that for λ close to $\lambda_0 (\lambda > \lambda_0)$ the inequality

$$(21)\qquad u \in \partial K: \qquad \langle \lambda u - Au - u_0^*, v - u\rangle \geq 0 \qquad \forall v \in K$$

is not solvable and, moreover,

$$(\alpha)\qquad R(\lambda, A)u_0^* \in K^0 \qquad \text{if} \quad M \subset K^0,$$

$$(\beta)\qquad R(\lambda, A)u_0^* \notin K \qquad \text{if} \quad M \cap K = \emptyset,$$

where $R(\lambda, A) := (\lambda I - A)^{-1}$. Using these facts one can prove the assertions of Theorem 4 similarly as in the proofs of Theorems 2 and 3.

First suppose that there exist $u_n \in \partial K$ and $\lambda_n \downarrow \lambda_0$ such that

$$(22)\qquad \langle \lambda_n u_n - Au_n - u_0^*, v - u_n\rangle \geq 0 \qquad \forall v \in K.$$

Putting $v := u_n + u_0^*$ we get $\langle u_n, u_0^*\rangle \geq \frac{1}{\lambda_n - \lambda_0}\|u_0^*\| \to +\infty$, thus $\|u_n\| \to \infty$. Passing to the limit in the equation

$$\frac{u_n}{\|u_n\|} = \frac{1}{\lambda_n} P_K\left(A\frac{u_n}{\|u_n\|} + \frac{u_0^*}{\|u_n\|}\right)$$

we get $\frac{u_n}{\|u_n\|} \to u \in E(\lambda_0)\cap\partial K\cap S_1$. Choosing $v := u_n + u^*, u^* \in E^*(\lambda_0)\cap K$, we get from (22) $\langle u_n, u^*\rangle \geq \frac{1}{\lambda_n - \lambda_0}\langle u_0^*, u^*\rangle \geq 0$, hence $\langle u, u^*\rangle \geq 0$ for any $u^* \in E^*(\lambda_0)\cap K$, which gives us a contradiction.

It remains to prove the assertions (α) and (β). To prove them it is sufficient to show that for any sequence $\{\lambda_n\}$, where $\lambda_n \downarrow \lambda_0$, there exists a subsequence (which we will denote again by $\{\lambda_n\}$) such that

$$\frac{R(\lambda_n, A)u_0^*}{\|R(\lambda_n, A)u_0^*\|} \to u \in M.$$

Thus let $\lambda_n \downarrow \lambda_0$. Let us write $R(\lambda_n, A)u_0^* = u_n + w_n$, where $u_n \in E(\lambda_0), w_n \in E(\lambda_0)^\perp$. Then we have

$$u_0^* = (\lambda_n - \lambda_0)u_n + (\lambda_n I - A)w_n. \tag{23}$$

Further, for a suitable $c > 0$ and for n sufficiently large we have

$$\|(\lambda_n I - A)w_n\| \geq c\|w_n\|, \tag{24}$$

since $w_n \in E(\lambda_0)^\perp$. Multiplying the equation (23) by u_0^* we get

$$1 = (\lambda_n - \lambda_0)(\langle u_n, u_0^*\rangle + \langle w_n, u_0^*\rangle). \tag{25}$$

Suppose $(\lambda_n I - A)w_n \to u_0^*$. Then by (23) we get $(\lambda_n - \lambda_0)u_n \to 0$ and (24) implies that the sequence $\{w_n\}$ is bounded, which gives us a contradiction with (25). Thus we may assume

$$\|(\lambda_n I - A)w_n - u_0^*\| \geq \varepsilon > 0. \tag{26}$$

Using (26) and (23) we obtain $\|u_n\| \to \infty$, by (23), (24) and (26) there exists $\delta > 0$ such that

$$(\lambda_n - \lambda_0)\|u_n\| = \|(\lambda_n I - A)w_n - u_0^*\| \geq \delta \cdot \max(\|w_n\|, 1),$$

hence $\frac{w_n}{\|u_n\|} \to 0$,

$$\lim_{n\to\infty} \frac{R(\lambda_n, A)u_0^*}{\|R(\lambda_n, A)u_0^*\|} = \lim_{n\to\infty} \frac{u_n + w_n}{\|u_n + w_n\|} = \lim_{n\to\infty} \frac{u_n}{\|u_n\|} = u \in E(\lambda_0) \cap S_1$$

(for a suitable subsequence of $\{u_n\}$). Moreover, by (25) we get $\lim_{n\to\infty} \langle \frac{u_n}{\|u_n\|}, u_0^*\rangle \geq 0$, thus $\langle u, u_0^*\rangle \geq 0$. Finally, for $u^* \in E^*(\lambda_0), u^* \perp u_0^*$, we have $\langle R(\lambda_n, A)u_0^*, u^*\rangle = \langle u_0^*, R(\lambda_n, A^*)u^*\rangle = \frac{1}{\lambda_n - \lambda_0}\langle u_0^*, u^*\rangle = 0$, thus $\langle u, u^*\rangle = 0$.

The properties of u proved above imply $u \in M$. ■

Theorem 5. *Let A be symmetric, $\lambda_0 \in \sigma^+(A), E(\lambda_0) \cap K^A \neq \emptyset$. Then $\lambda_0^- < \lambda_0 < \lambda_0^+$ and*

$$d(\lambda) = 0 \quad \textit{for} \quad \lambda \in (\lambda_0^-, \lambda_0),$$
$$d(\lambda) = -(1)^{\beta(\lambda_0)} \quad \textit{for} \quad \lambda \in (\lambda_0, \lambda_0^+).$$

PROOF : The assertion $\lambda_0^- < \lambda_0$ and $d(\lambda) = 0$ for $\lambda \in (\lambda_0^-, \lambda_0)$ is guaranteed by Theorem 2(i).

Denote by P_0 the orthogonal projection of H onto the space $\bigoplus_{\lambda \geq \lambda_0} E(\lambda)$ and choose $u_0 \in E(\lambda_0) \cap K^A \cap S_1$. First we will show that

$$(\alpha)\begin{cases} \text{for } \lambda > \lambda_0, \text{ sufficiently close to } \lambda_0, \text{ the inequality} \\ (27) \qquad u \in K: \qquad \langle \lambda u - Au - (\lambda - \lambda_0)u_0, v - u\rangle \geq 0 \qquad \forall v \in K \\ \text{has the unique solution } u := u_0. \end{cases}$$

Suppose the contrary, i.e. there exist $\lambda_n \downarrow \lambda_0$ and $u_n \neq u_0$ such that

$$T(\lambda_n, (\lambda_n - \lambda_0)u_0, 0)(u_n) = 0.$$

Choosing $v := u_n + u_0$ in the corresponding inequality we get $(\lambda_n - \lambda_0)\langle u_n - u_0, u_0\rangle \geq 0$, hence $\langle u_n, u_0\rangle \geq \|u_0\|^2 = 1, \|u_n\| \geq 1$. Put $\widetilde{u}_n := \frac{u_n}{\|u_n\|}$. We have

$$\widetilde{u}_n = \frac{1}{\lambda_n} P_K(A\widetilde{u}_n + \frac{\lambda_n - \lambda_0}{\|u_n\|} u_0) \tag{28}$$

and passing to the limit we get $\widetilde{u}_n \to u \in E(\lambda_0) \cap K \cap S_1$. Since $u_0 \in K^A$, we have $(u_0 + P_0(\widetilde{u}_n - u)) \in K$ for sufficiently large n. Choosing $v := u_0 + P_0(\widetilde{u}_n - u) = u_0 - u + P_0\widetilde{u}_n$ in the inequality corresponding to (28) we get

$$\begin{aligned} (29) \qquad 0 &\leq \langle \lambda_n \widetilde{u}_n - A\widetilde{u}_n - \frac{\lambda_n - \lambda_0}{\|u_n\|} u_0, u_0 - u + (P_o - I)\widetilde{u}_n\rangle = \\ &= (\lambda_n - \lambda_0)\langle \widetilde{u}_n - \frac{u_0}{\|u_n\|}, u_0 - u\rangle + \langle \lambda_n \widetilde{u}_n - A\widetilde{u}_n, (P_0 - I)\widetilde{u}_n\rangle \leq \\ &\leq (\lambda_n - \lambda_0)\langle \widetilde{u}_n, u_0 - u\rangle \end{aligned}$$

since $\langle -u_0, u_0 - u\rangle \leq 0$ and

$$\langle \lambda_n \widetilde{u}_n - A\widetilde{u}_n, (P_0 - I)\widetilde{u}_n\rangle = - \sum_{\lambda_{(s)} < \lambda_0} (\lambda_n - \lambda_{(s)})(c_s^n)^2 \leq 0, \tag{30}$$

where $\lambda_{(s)}$ are eigenvalues of the operator A and c_s^n are corresponding Fourier coefficients of $\widetilde{u}_n$. The inequality (29) implies $\langle \widetilde{u}_n, u_0 - u\rangle \geq 0$ and passing to the limit we obtain $\langle u, u_0\rangle \geq \|u\|^2 = 1$, hence $u = u_0$. By (29) we get now $\langle \lambda_n \widetilde{u}_n - A\widetilde{u}_n, (P_0 - I)\widetilde{u}_n\rangle = 0$, which implies (together with (30)) $\widetilde{u}_n = P_0\widetilde{u}_n$. Since $u_0 \in K^A$, we obtain further

$$A\widetilde{u}_n = \lambda_0 u_0 + A(\widetilde{u}_n - u_0) = \lambda_0 u_0 + P_0 A(\widetilde{u}_n - u_0) \in K$$

for sufficiently large n, thus (28) implies

$$\widetilde{u}_n = \frac{1}{\lambda_n}(A\widetilde{u}_n + \frac{\lambda_n - \lambda_0}{\|u_n\|} u_0),$$

i.e. $\lambda_n \widetilde{u}_n - A\widetilde{u}_n = \frac{\lambda_n - \lambda_0}{\|u_n\|} u_0$. Since $(\lambda_n I - A)$ is an isomorphism for n sufficiently large, we have $\widetilde{u}_n = \frac{u_0}{\|u_n\|}$. Since $\widetilde{u}_n, u_0 \in S_1$, we get $\|u_n\| = 1$, hence $u_n = \widetilde{u}_n = u_0$, which is a contradiction.

Thus we have proved the assertion (α). In the same way as in the proof of (α) one can show $\lambda_0^+ > \lambda_0$; this proof is left to the reader. In what follows we shall prove

$$(\beta)\begin{cases} \text{if } \lambda > \lambda_0, \lambda < \inf\{\lambda \in \sigma(A); \lambda > \lambda_0\} \text{ and } \eta > 0 \text{ is} \\ \text{sufficiently small, then} \\ \deg(T(\lambda, (\lambda - \lambda_0)u_0, 0), 0, B_\eta(u_0)) = (-1)^{\beta(\lambda_0)}. \end{cases}$$

Obviously, the assertions (α) and (β) imply $d(\lambda) = (-1)^{\beta(\lambda_0)}$ for $\lambda \in (\lambda_0, \lambda_0^+)$, which we are to prove.

So let λ fulfil the inequalities in (β). Put $f := (\lambda - \lambda_0)u_0$ and define the following homotopy

$$H(t,u) := u - \frac{t}{\lambda} P_K(Au + f) - \frac{1-t}{\lambda}(Au + f), \qquad t \in [0,1].$$

Obviously, $H(1,\cdot) = T(\lambda, f, 0)$ and $\deg(H(0,\cdot), 0, B_\eta(u_0)) = (-1)^{\beta(\lambda_0)}$. Thus it is sufficient to prove $H(t,u) \neq 0$ for $t \in [0,1]$ and $u \in \partial B_\eta(u_0)$, where $\eta > 0$ is sufficiently small. Suppose the contrary, i.e. there exist $u_n \neq u_0, u_n \to u_0$, and $t_n \in [0,1]$ such that $H(t_n, u_n) = 0$. Using the equality $H(t_n, u_n) - H(t_n, u_0) = 0$ we get

$$(31)\quad u_n - u_0 = \frac{t_n}{\lambda}(P_K(Au_n + f) - (Au_0 + f)) + \frac{1 - t_n}{\lambda}((Au_n + f) - (Au_0 + f)).$$

We shall show

$$(32)\qquad P_K(Au_n + f) - (Au_n + f) = o(\|u_n - u_0\|) \qquad (\text{for } n \to \infty),$$

which (together with (31)) gives us

$$w_n = \frac{1}{\lambda} A w_n + o(1) \qquad (\text{where } w_n := \frac{u_n - u_0}{\|u_n - u_0\|})$$

and passing to the limit in this equation we get $w_n \to w \in E(\lambda) \cap S_1$, which contradicts $\lambda \notin \sigma(A)$. To prove (32) let us choose $\delta > 0$ and write $u_n - u_0 = \sum_p t_p^n u_{(p)}$, where $u_{(p)}$ are eigenfunctions of A forming an orthonormal basis in $H, u_{(p)} \in E(\lambda_{(p)})$, where $|\lambda_{(1)}| \geq |\lambda_{(2)}| \geq \ldots$. Since $u_0 \in K^A$, there exist $\varepsilon_p > 0$ such that $u_0 \pm \varepsilon_p u_{(p)} \in K$. Choose p_0 such that $|\lambda_{(p_0)}| < \delta$. Further choose $\tau > 0$ such that $|\lambda_{(p)}|\tau < \frac{\lambda}{p_0}\varepsilon_p$ for any $p = 1, 2, \ldots, p_0$ and suppose $\|u_n - u_0\| < \tau$. Then we have

$$Au_n + f = \lambda u_0 + A(u_n - u_0) = \underbrace{\frac{\lambda}{p_0} \cdot \sum_{p=1}^{p_0} (u_0 + \frac{\lambda_{(p)} t_p^n p_0}{\lambda} u_{(p)})}_{z_1^n} + \underbrace{\sum_{p > p_0} \lambda_{(p)} t_p^n u_{(p)}}_{z_2^n}.$$

Since $|\frac{\lambda_{(p)}t_p^n p_0}{\lambda}| \leq \frac{|\lambda_{(p)}|p_0}{\lambda}\|u_n - u_0\| \leq \frac{|\lambda_{(p)}|p_0}{\lambda}\tau < \varepsilon_p$ for $p \leq p_0$, we have $z_1^n \in K$, hence

$$\|(Au_n + f) - P_K(Au_n + f)\| \leq \|(Au_n + f) - z_1^n\| = \|z_2^n\| =$$
$$= \sqrt{\sum_{p>p_0}(\lambda_{(p)}t_p^n)^2} \leq |\lambda_{(p_0)}| \cdot \|u_n - u_0\| < \delta\|u_n - u_0\|.$$

Thus we have proved (32) and simultaneously the assertion (β) and the whole Theorem 5. ∎

In the following theorem we describe some other situations in which the degree $d(\lambda)$ can be determined.

Theorem 6. *(i) Let* $\dim H < \infty$, *let* K *be such that it is not a subspace of* H *and let* $0 < \lambda < \inf_{u\in S_1}\langle Au, u\rangle$. *Then* $\lambda \notin \sigma_K(A)$ *and* $d(\lambda) = 0$.

(ii) Let $\lambda_0 > 0, E^*(\lambda_0) \cap K^a \neq \emptyset, E(\lambda_0) \cap K = \{0\}$. *Then* $\lambda \notin \sigma_K(A)$ *and* $d(\lambda) = 0$.

(iii) Let A *be a symmetric and put* $\lambda_0 := \sup_{u\in S_1\cap K}\langle Au, u\rangle$. *Suppose that* $\lambda_0 \in R^+ - \sigma(A)$ *and* $\text{card}(E_K(\lambda_0) \cap S_1) = 1$. *Let* $u_0 \in E_K(\lambda_0) \cap S_1$ *and suppose there exists* $w_0 \neq 0$ *and* $\varepsilon > 0$ *such that*

$$B_\varepsilon(u_0) \cap K = \{u \in B_\varepsilon(u_0); \langle u - u_0, w_0\rangle \geq 0\}. \tag{33}$$

Then $\lambda_0^- < \lambda_0$ *and* $d(\lambda) = 0$ *for* $\lambda \in (\lambda_0^-, \lambda_0)$.

Proof : The assertions (i), (ii) are proved in [11]. Suppose that the assumptions of (iii) are fulfilled. We shall prove (by contradiction) that the inequality (2) does not have solution for λ close to $\lambda_0(\lambda < \lambda_0)$ and for a suitable f; the proof of $\lambda_0^- < \lambda_0$ can be carried out in the same way. Suppose that for $\lambda_n \uparrow \lambda_0$ and $f_n := -\lambda_n u_0 + Au_0$ there exist $u_n \in K$ such that $\langle \lambda_n u_n - Au_n - f_n, v - u_n\rangle \geq 0$ for any $v \in K$, i.e.

$$\langle \lambda_n(u_n + u_0) - A(u_n + u_0), v - u_n\rangle \geq 0 \qquad \forall v \in K. \tag{34}$$

Choosing $v := u_n + u_0$ in (34) we obtain

$$\langle(\lambda_n I - A)u_n, u_0\rangle \geq -\langle(\lambda_n I - A)u_0, u_0\rangle > 0, \tag{35}$$

since $\langle Au_0, u_0\rangle = \lambda_0\|u_0\|^2 = \lambda_0$. Moreover, choosing $v := 2u_n$ and $v := 0$ in (34) we get $\langle(\lambda_n I - A)(u_n + u_0), u_n\rangle = 0$, thus according to (35) we have

$$\langle(\lambda_n I - A)u_n, u_n\rangle = -\langle(\lambda_n I - A)u_0, u_n\rangle < 0, \tag{36}$$

hence $\langle Au_n, u_n\rangle > \lambda_n\|u_n\|^2$, so that $u_n \neq 0$ and

$$\frac{\langle Au_n, u_n\rangle}{\|u_n\|^2} \to \lambda_0 = \sup_{0\neq u\in K}\frac{\langle Au, u\rangle}{\|u\|^2}. \tag{37}$$

We may suppose $\frac{u_n}{\|u_n\|} \rightharpoonup u$. Then $u \in K \cap \overline{B}_1$ and (37) implies $\langle Au, u\rangle = \lambda_0, u \in E_K(\lambda_0) \cap S_1$, since the functional $v \mapsto \langle Av, v\rangle$ attains at $v := u$ its maximum in $K \cap \overline{B}_1$. Hence $u = u_0, \frac{u_n}{\|u_n\|} \to u$ (since $\|\frac{u_n}{\|u_n\|}\| \to \|u\|$). Moreover, we have

$$(38) \qquad \begin{aligned} 0 &= \langle(\lambda_n I - A)(u_n + u_0), u_n\rangle = \\ &= \langle(\lambda_n I - A)u_n, u_n\rangle + (\lambda_n - \lambda_0)\langle u_0, u_n\rangle + \langle(\lambda_0 I - A)u_0, u_n\rangle. \end{aligned}$$

Since $\langle(\lambda_n I - A)u_n, u_n\rangle < 0$ by (36) and $(\lambda_n - \lambda_0)\langle u_0, u_n\rangle < 0$ for sufficiently large n, it is sufficient to prove $\langle(\lambda_0 I - A)u_0, u_n\rangle = 0$ and (38) will yield us a contradiction. According to (33) we have $(\lambda_0 I - A)u_0 = tw_0$ for some $t > 0$ and so it is sufficient to prove $\langle w_0, u_n\rangle = 0$ for large n. Suppose the contrary. Then by (33) we get $u_n \in K^0$, hence u_n is the solution of the equation corresponding to (34), i.e. $u_n = -u_0$. Nevertheless, $-u_0 \notin K$, which gives us a contradiction. ∎

Remark 5. The assertion of Theorem 6(iii) can be proved (for some special problems) also if the condition (33) is not fulfilled ([**13**]).

Example 2. Let $H := W_0^{1,2}(0,\pi), \langle u, v\rangle := \int_0^\pi u'v'\,dx, \langle Au, v\rangle := \int_0^\pi uv\,dx, K := \{u \in H; u(x_1) \le 0, u(x_2) \ge 0\}$, where $x_1 = \frac{2}{5}\pi, x_2 = \frac{2}{3}\pi$. Then u is a solution of the inequality (4) iff

$$(39) \qquad \begin{cases} \lambda u''(x) - u(x) = 0 & \text{in } (0,x_1) \cup (x_1,x_2) \cup (x_2,\pi) \\ u(0) = u(\pi) = 0 \\ u(x_1) \le 0, u(x_2) \ge 0, u'_-(x_1) \le u'_+(x_1), u'_-(x_2) \ge u'_+(x_2) \\ (u'_-(x_1) - u'_+(x_1))u(x_1) = 0, (u'_-(x_2) - u'_+(x_2))u(x_2) = 0 \end{cases}$$

Solving (39) we get $\sigma_K(A) \cap [\frac{1}{16}, +\infty) = \{\frac{4}{9}, \frac{9}{25}, \frac{1}{4}, \frac{4}{25}, \frac{1}{9}, \frac{9}{100}, \frac{1}{16}\}$ and using our results we can derive following facts:

	$d(\lambda)$	follows from
$\lambda > 4/9$	1	Theorem 1(i)
$\lambda \in (9/25, 4/9)$	0	Theorem 6(iii) ($\lambda_0 = 9/25$)
$\lambda \in (1/4, 9/25)$	-1	Theorem 1(ii) } ($\lambda_0 = 1/4$)
$\lambda \in (4/25, 1/4)$	0	Theorem 1(ii) } ($\lambda_0 = 1/4$)
$\lambda \in (1/9, 4/25)$	1	Theorem 1(ii) + Remark 1 } ($\lambda_0 = 1/9$)
$\lambda \in (9/100, 1/9)$	0	Theorem 1(ii) + Remark1 } ($\lambda_0 = 1/9$)
$\lambda \in (1/16, 9/100)$	-1	Theorem 1(ii) ($\lambda_0 = 1/16$)

(Remark 1 can be used e.g. with $K_n := \{u \in H; u(x_1) \le 0, u(x_2 + \frac{1}{n}) \ge 0\}$).

Example 3. Let $H := W_0^{1,2}(\Omega)$, where $\Omega := (0,\pi)^2, \langle u,v\rangle := \int_\Omega \nabla u \cdot \nabla v\, dx$, $\langle Au,v\rangle := \int_\Omega uv\, dx, K := \{u \in H; u \ge 0 \text{ on } M\}$, where $M := (\frac{1}{6}\pi, \frac{1}{3}\pi) \times (0,\pi)$. Using similar arguments as in [11, Example 2] one can easily show $\sigma_K(A) \cap [\frac{1}{5}, +\infty) = \{\frac{1}{5}, \frac{4}{13}, \frac{1}{2}\}$ and using Theorem 5 we get

$$
\begin{aligned}
d(\lambda) &= 1 && \text{for } \lambda > 1/2,\\
d(\lambda) &= 0 && \text{for } \lambda \in (4/13, 1/2),\\
d(\lambda) &= -1 && \text{for } \lambda \in (1/5, 4/13).
\end{aligned}
$$

Remark 6. Theorem 2 and 6 (ii) were used in [13] to get some existence results for eigenvalues of inequalities of reaction–diffusion type; these results imply some destabilizing effect of unilateral conditions for the system of reaction–diffusion equations and generalize in many directions results proved in [1], [2], [12].

4. Multiplicity results.

If $\lambda > \sup_{u\in B_1} \langle Au,u\rangle$, then the operator $\lambda I - A$ is strictly monotone, so that the inequality (2) has a unique solution for any $f \in H$ (e.g. [3]). Nevertheless, for $\lambda < \sup_{u\in B_1} \langle Au,u\rangle$ we may lose the uniqueness.

Theorem 7. *Let $\lambda \in R^+ - (\sigma_K(A) \cup \sigma(A)), d(\lambda) \ne (-1)^{\beta(\lambda)}$ and let $f \in (\lambda I - A)(K^A)$ (if A is symmetric) or $f \in (\lambda I - A)(K^0)$. Then the inequality (2) has at least two solutions. If, moreover, K is an intersection of finitely many halfspaces, then for each $\delta > 0$ there exists $\tilde f \in B_\delta(f)$ such that the inequality (2) with the right-hand side $\tilde f$ has at least $|(-1)^{\beta(\lambda)} - d(\lambda)| + 1$ solutions.*

PROOF : Let $f = (\lambda I - A)u$, where $u \in K^0$ (or $u \in K^A$ and A be symmetric). In both cases we know that u is an isolated solution of the equation

$$
T(\lambda, f, 0)(u) = 0 \tag{40}
$$

and that $\deg(T(\lambda,f,0),0,B_\varepsilon(u)) = (-1)^{\beta(\lambda)}$ for sufficiently small ε (for $u \in K^A$ and A symmetric this fact was proved in the proof of Theorem 5). Thus we have

$$
d(\lambda) = \deg(T(\lambda,f,0),0,B_R) = \deg(T(\lambda,f,0),0,B_R - \overline{B_\varepsilon(u)}) + (-1)^{\beta(\lambda)}.
$$

Since $d(\lambda) \ne (-1)^{\beta(\lambda)}$, the equation (40) has at least one solution in $B_R - \overline{B_\varepsilon(u)}$. If, moreover, K is an intersection of finitely many halfspaces, then [11, Theorem 5] implies that for any $\delta > 0$ we can find $\tilde f \in B_\delta(f)$ such that $\tilde f \in (\lambda I - A)(K^0)$ (or $\tilde f \in (\lambda I - A)(K^A)$) and $\tilde f$ is a regular value of T, i.e. all solutions u_i of (40) are isolated and $\deg(T(\lambda,\tilde f,0),0,B_\varepsilon(u_i)) = \pm 1$ for sufficiently small ε. ∎

Corollary. *Let $\lambda \in R^+ - (\sigma_K(A) \cup \sigma(A)), d(\lambda) = 0, f \in (\lambda I - A)(K^0)$ (or $f \in (\lambda I - A)(K^A)$ and A be symmetric). Then (2) has at least two solutions.*

Example 4. Let A be symmetric, let $\lambda_1 > \lambda_2 \geq 0$ be the two largest eigenvalues of A, let $E(\lambda_1) \cap K = \{0\}, K^A \neq \emptyset$ and let λ_1 have an odd algebraic multiplicity (i.e. $\gamma(\lambda_1)$ is odd). Let $\lambda > \max(\lambda_2, \sup_{u \in K \cap B_1} \langle Au, u\rangle), \lambda < \lambda_1$. Then $d(\lambda) = 1$ by Theorem 1(i) and $(-1)^{\beta(\lambda)} = -1$, so that any $f \in (\lambda I - A)(K^A)$ we have at least two solutions of (2) (or 3 solutions, if K is an intersection of finitely many halfspaces).

Example 5. Let Ω be a bounded domain in $R^n, H := W_0^{1,2}(\Omega), \langle u, v\rangle := \int_\Omega \nabla u \cdot \nabla v \, dx, \langle Au, v\rangle := \int_\Omega uv \, dx, K = K^+ := \{u \in H; u \geq 0\}$. Let λ_1 be the first eigenvalue of A and let $e_1 \in E(\lambda_1) \cap S_1$. We may suppose $e_1 > 0$ in Ω; using similar arguments as in [11, Example 2] one can prove that $\sigma_K^+(A) = \{\lambda_1\}, E_K(\lambda_1) \cap S_1 = \{e_1\}$. Choosing the test function $v := u + e_1$ in the inequality

$$u \in K: \qquad \langle \lambda u - Au - e_1, v - u\rangle \geq 0 \qquad \forall v \in K$$

we get that this inequality is not solvable for $\lambda < \lambda_1$, so that $d(\lambda) = 0$ for $\lambda < \lambda_1$. Further choose $f \in \widetilde{K}^S := \{u \in H; \langle u, v\rangle < 0 \quad \forall v \in K - \{0\}\}$ and $\lambda_0 \in (0, \lambda_1)$. Then $u := 0$ is a trivial solution of (2) (with $\lambda := \lambda_0$) and we can use the idea of Szulkin [17], [19] to prove that the inequality (2) (with $\lambda := \lambda_0$) has at least two solutions:
Choose $\Lambda > \lambda_1$ and first let us prove that the inequality (2) has no solution in $\overline{B_\varepsilon(0)}$ for any $\lambda \in [\lambda_0, \Lambda]$ and $\varepsilon > 0$ sufficiently small. Suppose the contrary, i.e. there exist $0 \neq u_n \to 0$ and $\lambda_n \in [\lambda_0, \Lambda]$ such that $\langle \lambda_n u_n - Au_n - f, v - u_n\rangle \geq 0$ for any $v \in K$. Dividing the equation $\langle \lambda_n u_n - Au_n - f, u_n\rangle = 0$ by $\|u_n\|^2$ and passing to the limit (assuming $\frac{u_n}{\|u_n\|} \to u \in K, \lambda_n \to \lambda > 0$) we get

$$\lambda - \langle Au, u\rangle = \lim_{n\to\infty} \frac{1}{\|u_n\|}\langle f, \frac{u_n}{\|u_n\|}\rangle \leq 0$$

hence $u \neq 0, \langle f, u\rangle = 0$, which gives us a contradiction. Thus we have

$$\begin{aligned} 0 = d(\lambda_0) &= \deg(T(\lambda_0, f, 0), 0, B_\varepsilon) + \deg(T(\lambda_0, f, 0), 0, B_R - \overline{B}_\varepsilon) = \\ &= \deg(T(\Lambda, f, 0), 0, B_\varepsilon) + \deg(T(\lambda_0, f, 0), 0, B_R - \overline{B}_\varepsilon) = \\ &= 1 + \deg(T(\lambda_0, f, 0), 0, B_R - \overline{B}_\varepsilon), \end{aligned}$$

which implies the existence of a solution in $B_R - \overline{B}_\varepsilon$.

Moreover, we have $K^A \neq \emptyset$: if $\{e_k\}_{k=1}^\infty$ are eigenfunctions of A forming an orthonormal basis in H, then e.g. $u := \sum_{k=1}^{\infty} \frac{|e_k|}{k^2} \in K^A$. Hence we can apply Corollary of Theorem 7 to prove a multiplicity result for $f \in (\lambda I - A)(K^A)$. Since $(\lambda I - A)(K^A) \not\subset \widetilde{K}^S$, we get also new right-hand sides with multiple solutions (in comparison to the Szulkin's result).

Theorem 8. *Let A be symmetric, let $\lambda_1 > \lambda_2 \geq 0$ be the two largest eigenvalues of A. Let $\dim E(\lambda_1) = 1, e_1 \in E(\lambda_1) \cap S_1, \overline{B_\delta(e_1)} \subset K^0$ (obviously $\delta < 1$). Put $J := [\lambda_2 + (1-\delta^2)(\lambda_1 - \lambda_2), \lambda_1)$ and choose $\lambda_0 \in J$. Then $\lambda_0 \notin \sigma_K(A)$, the inequality*

$$(41) \qquad u \in K: \qquad \langle \lambda_0 u - Au - e_1, v - u\rangle \geq 0 \qquad \forall v \in K$$

does not have solution (which implies $d(\lambda_0) = 0$) and for any $f \in \widetilde{K} := \{u \in H; \langle u, v\rangle \leq 0 \quad \forall v \in K\}, f \neq 0$, the inequality (2) (with $(\lambda := \lambda_0)$ has exactly two solutions.

PROOF : Let $\{e_i\}_{i=1}^{\infty}$ be eigenvectors of A forming an orthonormal basis in H.

First suppose $\lambda_0 \in \sigma_K(A), u \in E_K(\lambda_0) \cap S_1$. Then $u = \sum_{i=1}^{\infty} c_i e_i$, where $\sum_{i=1}^{\infty} c_i^2 = 1$, and using the equality $\lambda_0 \|u\|^2 = \langle Au, u\rangle$ we obtain $(\lambda_1 - \lambda_0)c_1^2 = \sum_{i\geq 2}(\lambda_0 - \lambda_i)c_i^2 \geq (\lambda_0 - \lambda_2)(1 - c_1^2)$, which implies

$$(42) \qquad c_1^2 \geq \frac{\lambda_0 - \lambda_2}{\lambda_1 - \lambda_2}.$$

Since $\lambda_0 \notin \sigma(A)$, we have $u \in \partial K$. Since $\overline{B_\delta(e_1)} \subset K^0$, we get $\langle u, e_1\rangle^2 = c_1^2 < 1 - \delta^2$. Thus we have

$$\frac{\lambda_0 - \lambda_2}{\lambda_1 - \lambda_2} \leq c_1^2 < 1 - \delta^2,$$

which implies $\lambda_0 \notin J$ and gives us a contradiction.

Now suppose that $u \in K$ is a solution of (41). Choosing $v := u + e_1$ we get $(\lambda_0 - \lambda_1)\langle u, e_1\rangle \geq 1$, hence $\langle u, e_1\rangle < 0$. Moreover, $\langle \lambda_0 u - Au - e_1, u\rangle = 0$, so that $\langle Au, u\rangle > \lambda_0 \|u\|^2$, which implies (as in the derivation of (42))

$$(43) \qquad \langle \frac{u}{\|u\|}, e_1\rangle^2 > \frac{\lambda_0 - \lambda_1}{\lambda_1 - \lambda_2}.$$

Since the unique solution of the equation $\lambda_0 u - Au = e_1$ does not belong to K, we have $u \in \partial K$ and thus

$$(44) \qquad \langle \frac{u}{\|u\|}, e_1\rangle^2 < 1 - \delta^2.$$

The inequalities (43) and (44) imply $\lambda_0 \notin J$, which is a contradiction.

Finally, choose $f \in \widetilde{K} - \{0\}$. If $\lambda \geq \lambda_0$, then $u := 0$ is the unique solution of (2) lying in ∂K: if, on the contrary, there exists a solution $u \in \partial K - \{0\}$ of (2), then $\langle \lambda u - Au - f, u\rangle = 0, \langle Au, u\rangle = \lambda\|u\|^2 - \langle f, u\rangle \geq \lambda\|u\|^2$, which implies (similarly as above) $\lambda \notin J$ and also $\lambda < \lambda_1$, thus it gives us a contradiction. Further choose $\Lambda > \lambda_1$. The equation $\lambda u - Au = f$ is not solvable in $\overline{B_\varepsilon(0)}$ for any $\lambda \in [\lambda_0, \Lambda]$ and $\varepsilon < \frac{\|f\|}{\Lambda + \|A\|}$ and thus $u := 0$ is the unique solution of (2) in $\overline{B_\varepsilon(0)}$ for any $\lambda \in [\lambda_0, \Lambda]$. Hence

$$1 = d(\Lambda) = \deg(T(\Lambda, f, 0), 0, B_\varepsilon(0)) + \deg(T(\lambda_0, f, 0), 0, B_\varepsilon(0)).$$

On the other hand we know

$$0 = d(\lambda_0) = \deg(T(\lambda_0, f, 0), 0, B_\varepsilon) + \deg(T(\lambda_0, f, 0), 0, B_R - \overline{B}_\varepsilon(0)),$$

so that there exists a solution $u^0 \in B_R - \overline{B_\varepsilon(0)}$ of the inequality

$$u \in K: \qquad \langle \lambda_0 u - Au - f, v - u\rangle \geq 0 \qquad \forall v \in K. \tag{45}$$

Since the inequality (45) does not have solution in $\partial K - \{0\}$, we have $u^0 \in K^0$, i.e. u^0 is uniquely determined. Hence (45) has exactly two solutions: 0 and u^0. ■

In the following theorem we shall use this notation: if A_α is a completely continuous linear operator in H, then we put

$$d_\alpha(\lambda) := \deg(T(\lambda, 0, 0, A_\alpha, K), 0, B_r).$$

Theorem 9. *Let $F : H \to H$ be a completely continuous map, let A_0, A_∞ be completely continuous linear operators and let*

$$\lim_{u\to 0} \frac{F(u) - A_0 u}{\|u\|} = 0, \qquad \lim_{\|u\|\to\infty} \frac{F(u) - A_\infty u}{\|u\|} = 0. \tag{46}$$

Let, moreover, $1 \notin \sigma_K(A_0) \cup \sigma_K(A_\infty)$ and $d_0(1) \neq d_\infty(1)$. Then there exists a nontrivial solution of the inequality

$$u \in K: \qquad \langle u - F(u), v - u\rangle \geq 0 \qquad \forall v \in K. \tag{47}$$

PROOF : Putting $g_\infty(u, \lambda) = F(u) - A_\infty u$ we get using Lemma 3

$$d_\infty(1) = \deg(T(1, 0, g_\infty, A_\infty, K), 0, B_R) = \deg(I - P_K F, 0, B_R)$$

for sufficiently large $R > 0$. On the other hand, putting $g_0(u, \lambda) = F(u) - A_0 u$ we get (as in the proof of [11, Lemma 3])

$$d_0(1) = \deg(T(1, 0, g_0, A_0, K), 0, B_\varepsilon) = \deg(I - P_K F, 0, B_\varepsilon)$$

for sufficiently small $\varepsilon > 0$. Hence

$$\deg(I - P_K F, 0, B_R - \overline{B}_\varepsilon) = d_\infty(1) - d_0(1) \neq 0,$$

which implies the existence of a nontrivial solution of (47). ■

Example 6. Let Ω be a bounded regular domain in $R^n, H := W_0^{1,2}(\Omega), \langle u, v\rangle := \int_\Omega \nabla u \cdot \nabla v \, dx, \langle Au, v\rangle := \int_\Omega uv \, dx, \langle F(u), v\rangle := \int_\Omega f(u) v \, dx$, where $f \in C(R, R)$, $f(0) = 0$. Suppose there exist $f'(0)$ and $f'(\infty) := \lim_{|t|\to\infty} \frac{f(t)}{t}$ and put $A_0 := f'(0)A, A_\infty := f'(\infty)A$. Then one can easily verify (46). Suppose that $f'(0)$, $f'(\infty) \notin \chi_K(A)$ and $\tilde{d}(f'(0)) \neq \tilde{d}(f'(\infty))$ (see Remark 1 (ii)). Then Theorem 9 implies the existence of a nontrivial solution of (47).

5. Variational inequalities in R^2.

In this section se shall show how the structure of the solution set of (2) depends on λ in a very special case.

Suppose $H := R^2$, A is symmetric with eigenvalues $\lambda_1 > \lambda_2 > 0, e_i \in E(\lambda_i) \cap S_1, w_i \in S_1 (i = 1,2), 0 < \langle w_2, e_2\rangle < \langle w_1, e_2\rangle, K := \{u \in H; \langle u, w_1\rangle \geq 0, \langle u, w_2\rangle \geq 0\}$ (see Fig.1). Denote

$$K_i := \{u \in K; u \perp w_i, u \neq 0\} \qquad (i = 1,2),$$
$$K_i^\lambda := (\lambda I - A)(K_i) \qquad (i = 1,2),$$
$$K_0^\lambda := (\lambda I - A)(K^0),$$
$$\widetilde{K} := \{c_1 w_1 + c_2 w_2; c_1 \leq 0, \quad c_2 \leq 0\}.$$

An element $u \in H$ is a solution of (2) iff exactly one of the following four conditions is fulfilled:

(C0) $\quad u \in K^0, \quad f = (\lambda I - A)u$

(C1) $\quad u \in K_1, \quad \lambda u - Au - f = tw_1 \quad$ for some $t \geq 0$

(C2) $\quad u \in K_2, \quad \lambda u - Au - f = tw_2 \quad$ for some $t \geq 0$

(C3) $\quad u = 0, \quad f \in \widetilde{K}.$

Thus the right–hand sides f, for which is the inequality (2) solvable, can be described in the following way: $f \in M_0 \cup M_1 \cup M_2 \cup M_3$, where $M_0 := K_0^\lambda, M_i := K_i^\lambda + \{tw_i; t \leq 0\} \quad (i = 1,2), M_3 := \widetilde{K}$. Moreover, the number of solutions of (2) is for $\lambda \in R^+ - \sigma_K(A)$ and generic f (see [11]) given by the number of indices i such that $f \in M_i$.

Denote $\lambda_I^1 > \lambda_I^2$ the eigenvalues of (4) which correspond to the eigenvectors lying in K_1, K_2. Then $\lambda_1 > \lambda_I^1 > \lambda_I^2 > \lambda_2, \lambda_I^i \in \sigma(P_i A_{/\{w_i\}^\perp})$, where $P_i : H \to \{w_i\}^\perp$ is the orthogonal projection. Using Fig.2 – 6 one obtains the following multiplicity results:

	$d(\lambda)$	the number of solutions of (2) for generic f	see
$\lambda > \lambda_1$	1	1	Fig.2
$\lambda \in (\lambda_I^1, \lambda_1)$	1	1,3	Fig.3
$\lambda \in (\lambda_I^2, \lambda_I^1)$	0	0,2	Fig.4
$\lambda \in (\lambda_2, \lambda_I^2)$	-1	1,3	Fig.5
$\lambda \in (0, \lambda_2)$	0	0,2,4 (or 0,4)	Fig.6

Similar discussions can be made also for another inequalities in R^2 (or R^3) and some of the results of these considerations can be used as conjectures for inequalities in a general Hilbert space, e.g. one sees how to choose a right–hand side $f \in H$, for which the inequality (2) "should not" have solution.

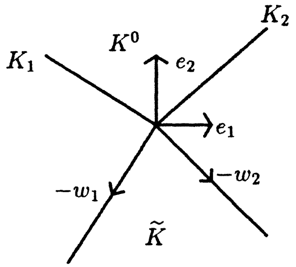


Fig.1

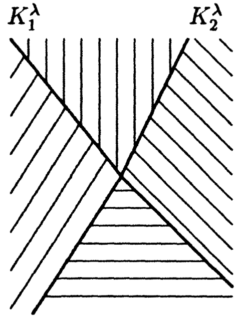


Fig.2 ($\lambda > \lambda_1$)

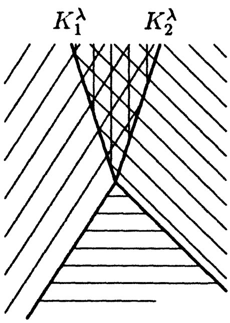


Fig.3 ($\lambda_I^1 < \lambda < \lambda_1$)

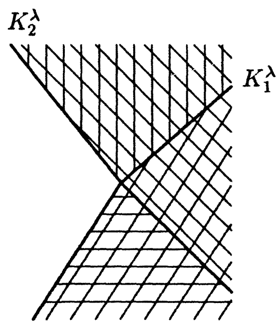


Fig.4 ($\lambda_I^2 < \lambda < \lambda_I^1$)

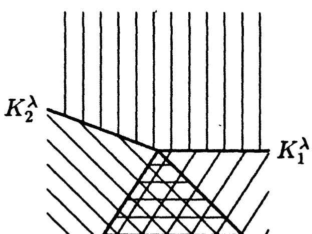


Fig.5 ($\lambda_2 < \lambda < \lambda_I^2$)

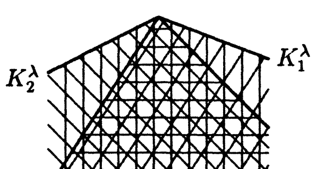
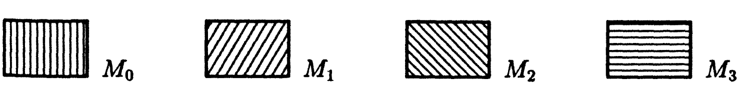


Fig.6 ($\lambda_2 - \varepsilon < \lambda < \lambda_2$)



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