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Smooth functions and zero traces

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Abstract. In the present paper, we prove a possibility of approximation of a function $f \in W^{k,p}(\Omega)$ by smooth functions which vanish on the same part of the boundary as f.

Keywords: Sobolev spaces, density theorems, approximation of boundary values

Classification: 41A30, 46E35

1. Introduction.

In this paper, we consider density of smooth functions in subspaces $V \subset W^{k,p}(\Omega)$ of all functions of the Sobolev space $W^{k,p}(\Omega)$ which vanish on some part of the boundary $\partial\Omega$. It is well known that if V is the space of all functions with zero traces on the whole boundary, then we have $V = W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}$ supposing the boundary to be Lipschitzian (a survey of notations and definitions is written out in the section 2 below; see also [1] or [2]). Under the same assumption we have $W^{k,p}(\Omega) = \overline{C^\infty(\overline{\Omega})}$. One can suppose that for $V = \{u \in W^{k,p}(\Omega); u = 0 \text{ on } \Gamma \subset \partial\Omega\}$, $\mathcal{V} = V \cap C^\infty(\overline{\Omega})$ the density identity $V = \overline{\mathcal{V}}$ holds; some affirmative examples are given in [2] and a more general result of this type is in [3]. In the present paper, we prove a slightly stronger density theorem for a wide class of "zero sets" $\Gamma \subset \partial\Omega$ supposing higher smoothness of the boundary $\partial\Omega$ (depending on k). The main theorem is proved in section 4 as a consequence of the auxiliary Lemma 1. The proof of this lemma — which is essentially a special case of the density theorem — is given in the section 3, while the section 2 contains definitions and notations used in the following.

2. Notations and definitions.

In this section, we briefly summarize notations and concepts used in the following and repeat their main properties needful for our considerations; for details and proofs, see [1] or [2].

By R_M we denote the M–dimensional Euclidean space of points $x = (x_1, \dots, x_M)$; $R_M^+ = \{x \in R_M; x_M > 0\}$ is the "positive halfspace". We shall write usually $M = N$ or $M = N + 1$ and we shall abbreviate for $x \in R_N : x = (x', x_N)$, for $x \in R_{N+1} : x = (x', x_N, x_{N+1})$, where x' stands for $(x_1, \dots, x_{N-1})$.

Having f a real function (with domain of definition $D \subset R_M$) we denote by $\operatorname{supp} f = \overline{\{x \in R_M; f(x) \neq 0\}}$ (the closure with respect to usual Euclidean metric) the support of f. Let $\Omega \subset R_M$ be an open domain, bounded or equal to the whole space R_M, or the halfspace R_M^+. We denote:

$C(\overline{\Omega})$ – the space of all functions f, uniformly continuous on Ω, with compact support

$C^\infty(\overline{\Omega})$ $= \{f \in C(\overline{\Omega}); D^\beta f \in C(\overline{\Omega})$ for all multiindexes $\beta\}$

$(\beta = (\beta_1, \ldots, \beta_M), D^\beta = \dfrac{\partial^{\beta_1+\cdots+\beta_M}}{\partial x_1^{\beta_1} \ldots \partial x_M^{\beta_M}})$

$C_0^\infty(\Omega)$ $= \{f \in C^\infty(\overline{\Omega}); \operatorname{supp} f \subset \Omega\}$

$(C^\infty(\Omega) = \mathcal{E}(\overline{\Omega}), C_0^\infty(\Omega) = \mathcal{D}(\Omega)$ according to [2])

For $p \geq 1, k$ positive integer, we denote by

$L_p(\Omega)$ $= W^{0,p}(\Omega)$– the set of all measurable functions f with finite norm $\|f\|_{0,p;\Omega} = \|f\|_{0,p} = (\int_\Omega |f|^p \, dx)^{1/p}$

$W^{k,p}(\Omega) = \{f \in L_p(\Omega); D^\beta f \in L_p(\Omega)$ (in the sense of distributions) for $|\beta| = \beta_1 + \cdots + \beta_M \leq k$, with the norm $\|f\|_{k,p;\Omega} = \|f\|_{k,p} = (\sum_{|\beta| \leq k} \|D^\beta f\|_{0,p}^p)^{1/p}\}$

$W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}$ (the closure in the space $W^{k,p}(\Omega)$).

We say that a bounded domain Ω is of the type C^k (or $C^{k,1}$) and we write $\Omega \in C^k$ (or $\Omega \in C^{k,1}$) if there exists a finite number of Cartesian co-ordinate systems $x = (x_{1,j}, \ldots, x_{M,j}), j = 1, \ldots, r$, such that the boundary $\partial\Omega$ of Ω is covered by graphs (in these systems) of functions a_j, continuous together with all derivatives up to the order k in an open neighbourhood of the origin of j–th system, (with k–th derivatives being Lipschitzian) and such that these graphs divide locally R_M onto the interior and exterior of Ω. For $\Omega \in C^{0,1}$ or $\Omega = R_M^+$ we denote by Tf the trace of f (on $\partial\Omega$). The "mapping of trace" T is uniquely defined as a continuous mapping from $W^{1,p}(\Omega)$ into $L_p(\Omega)$. It is possible to characterize $W_0^{k,p}(\Omega)$ via traces, namely: $u \in W_0^{k,p}(\Omega)$ iff $u \in W^{k,p}(\Omega)$ and $TD^\beta u = 0$ on $\partial\Omega$ for $|\beta| \leq k-1$. (Hence, supposing $\Omega \in C^{0,1}$ we have $\{u \in W^{k,p}(\Omega); TD^\beta u = 0, |\beta| \leq k-1\} = \overline{C_0^\infty(\infty)}$.) For the $W^{k,p}(\Omega)$, we have $W^{k,p}(\Omega) = \overline{C^\infty(\overline{\Omega})}$ supposing $\Omega \in C^0$ or $\Omega = R_M$ or $\Omega = R_M^+$; moreover, $W^{k,p}(\Omega) = \overline{C_0^\infty(R_M)}$ in the sense of restrictions. The following assertion holds: let $\Omega_1 \subset R_M, \Omega_2 \subset R_M$ be two bounded domains and let $\varphi : \overline{\Omega_1} \to \overline{\Omega_2}$ be a Lipschitzian mapping with Lipschitzian inverse φ_1. Then the mapping $\Phi : u \to v : v(x) = u(\varphi(x))$ is an isomorphism between $W^{1,p}(\Omega_1)$ and $W^{1,p}(\Omega_2)$. Moreover, if φ has Lipschitzian derivatives up to the order $k-1$ as well as φ_1, then the mapping Φ is an isomorphism between $W^{k,p}(\Omega_1)$ and $W^{k,p}(\Omega_2)$. In the following, extension theorems will be helpfull, too:

1. Let $f \in W_0^{k,p}(\Omega), \Omega \subset R_M$. Then $f \in W_0^{k,p}(R_M)$ if we define $f(x) = 0$ for $x \notin \Omega$.

2. Let $f \in W^{k,p}(R_M^+)$. Then $f \in W^{k,p}(R_M)$ if we define $f(x_1,\dots,-x_M) = c_1 f(x_1,\dots,x_M) + c_2 f(x_1,\dots,2x_M) + \dots + c_k f(x_1,\dots,kx_M)$ for $x_M > 0$ with convenient choice of c_i (method of Nikolski).

3. An auxiliary lemma.

Lemma 1. *Let G and $\widetilde{G}$ be two $(N+1)$-dimensional parallelepipeds defined as follows: $G = (0,1)^N \times (0,1), \widetilde{G} = (0,1)^N \times (-1,1)$. Let $G_0 \subset \overline{G}_0 \subset (0,1)^N$ be N-dimensional domain of class C^0 and let us denote by Γ the set $\Gamma = G_0 \times \{0\} \subset \widetilde{G}$. Let, moreover, $P \subset \overline{P} \subset \widetilde{G}$ be an open set. Let $v \in W^{k,p}(G)$ (k positive integer) be such a function that* $\operatorname{supp} v \subset P$, *and $v = 0$ on Γ in the sense of traces. Then there exists a sequence $\{w_n\} \subset C_0^\infty(P)$ such that $\lim_{n\to\infty} \|w_n - v\|_{k,p;G} = 0$ and* $\overline{\Gamma} \cap \operatorname{supp} w_n = \emptyset$.

Proof : According to the assumption $G_0 \in C^0$, there exist $\alpha > 0, \delta > 0$ and r Cartesian systems $(x_{i,j})_{i=1}^N (j = 1,\dots,r)$ and r functions a_j, continuous on $\Delta = (-\delta,\delta)^{N-1}$ such that

(i) $x = (x'_j, x_{N,j}) \in G_0$ for $x'_j \in \Delta$,

$a_j(x'_j) < x_{N,j} < a_j(x'_j) + \alpha$

(ii) $x \in G'_0 = R_N - \overline{G}_0$ for $a_j(x'_j) - \alpha < x_{N,j} < a_j(x'_j)$

(iii) for any $x \in \partial G_0$ there exists j and x'_j such that $x = (x'_j, a_j(x'_j))$.

Without loss of generality we can suppose α such small that $U_j \subset (0,1)^N$ where $U_j = \{x \in R_N; x'_j \in \Delta, a_j(x'_j) - \alpha < x_{N,j} < a_j(x'_j) + \alpha\} (j = 1,\dots,r)$. Let $U_0 \subset \overline{U}_0 \subset G_0, U_{r+1} \subset \overline{U}_{r+1} \subset \overline{(0,1)^N} - \overline{G}_0$ be such domains that $\bigcup_{j=0}^{r+1} U_j = (0,1)^N$. The domains $V_j, V_j = U_j \times (-1,1)$ cover P, and hence there exists a partition of unity:$\varphi_j \in C_0^\infty(V_j)$ for $j = 0,\dots,r+1, 0 \le \varphi_j(x) \le 1$ for $x \in V_j$ and $\sum_{j=0}^{r+1} \varphi_j(x) = 1$ for $x \in P$. Thus we have $v = \sum_{j=0}^{r+1} v_j$ where $v_j = v \cdot \varphi_j$. It is now sufficient to find sequences $w_{j,n} \in C_0^\infty(P), w_{j,n} \to v_j$ and $\operatorname{supp} w_{j,n} \cap \overline{\Gamma} = \emptyset$; the functions $w_n = w_{0,n} + \dots + w_{r+1,n}$ satisfy the assertion of our lemma. In the following, we construct such sequences for arbitrary $j = 0,1,\dots,r,r+1$.

a. Let $i = 0$. Obviously $v_0 \in W_0^{k,p}(G)$ and we can extend it by zero on the whole R_{N+1}. Hence we can approximate v_0 by a function $v_{0,t} : v_{0,t}(x', x_N, x_{N+1}) = v_0(x', x_N, x_{N+1} - t)$. Then $\lim_{t\to 0+} \|v_{0,t} - v_0\|_{k,p;R_{N+1}} = 0$ in virtue of L_p-mean continuity theorem and for t small enough we have $\operatorname{supp} v_{0,t} \subset P \cap G$. Now it is sufficient to write $w_{0,n} = \omega_n * v_{0,t}$ with $t = \frac{1}{n}$, where ω_n is a sequence of mollifiers with radii tending to zero and by $*$ we denote a convolution.

b. Let $i = r+1$. Defining $v_{r+1} = 0$ on $R_{N+1}^+ - G$ and then extending it by the method of Nikolski we obtain $v_{r+1} \in W_0^{k,p}(\widetilde{G}), \operatorname{supp} v_{r+1} \subset V_{r+1}$ and, moreover, $\operatorname{supp} v_{r+1} \cap \overline{\Gamma} = \emptyset$. Now, again the functions $w_{r+1,n} = \omega_n * v_{r+1}$ satisfy our requirements.

c. Now, let $j = 1, 2, \ldots, r$. Then defining $v_j = 0$ outside G we obtain $v_j \in W^{k,p}(R^+_{N+1})$. Hence, writing $v_{j,s}(x'_j, x_{N,j}, x_{N+1}) = v_j(x'_j, x_{N,j} + s, x_{N+1})(0 < s < \alpha)$, we have $\lim_{s \to 0+} \|v_{j,s} - v_j\|_{k,p,R^+_{N+1}} = 0$. Moreover, supposing a small enough we obtain $\operatorname{supp} v_{j,s} \subset V_j \cap P$.

• Now, let us define $U^1 = \{x \in U_j; x_{N,j} > a_j(x'_j) - s\}, U^2 = \{x \in U_j; x_{N,j} < a_j(x'_j) - \frac{s}{2}\}, V^i = U^i \times (-1,1)(i = 1,2)$. Then there exist two functions $\Psi_i \in C_0^\infty(V^i), 0 \leq \Psi_i(x) \leq 1$, and $\Psi_1(x) + \Psi_2(x) = 1$ if $x \in \operatorname{supp} v_{j,s}$; let us define $v^i_s = v_{j,s} \cdot \Psi_i, i = 1, 2$.

It is obvious that $v^1_s \in W_0^{k,p}(G)$ and we can approximate v^1_s by functions $w^1_{s,m} \in C_0^\infty(P), \operatorname{supp} w^1_{s,m} \cap \overline{\Gamma} = \emptyset$, using the procedure of the point a.

On the other hand, we have $\operatorname{supp} v^2_s \subset V^2$ and according to the point b we can construct a sequence $w^2_{s,m} \in C_0^\infty(P \cap V^2), w^2_{s,m} \to v^2_s$. Now, to obtain the desired sequence $w_{j,n}$, we chose, for arbitrary n, s small enough and then m great enough, and then we write $w_{j,n} = w^1_{s,m} + w^2_{s,m}$. ■

4. The main density theorem.

Let $\Omega \subset C^{0,1}$ be an $(N+1)$–dimensional domain and let $\Gamma \subset \partial\Omega$ be a relatively open set, (i.e. Γ is open in the metric space $\partial\Omega$). According to the smoothness of $\partial\Omega$, there exist $\delta > 0, \alpha > 0$ and r cartesian systems $(x_{i,j})_{i=1}^{N+1}$ $(j = 1, \ldots, r)$, analogous to that ones in the proof of Lemma 1, with functions A_j (which correspond to functions a_j from this proof) being Lipschitzian. Now we say that Γ has $C^{0,*}$ property if for arbitrary $j = 1, \ldots, r$ the projection $G_{0,j}$ of $\Gamma \cap W_j$ to Δ in the direction of $x'_{N+1,j}$ axis ($W_j = \{x \in R_{N+1}; (x'_j, x_{N,j}) \in \Delta, A_j(x'_j, x_{N,j}) - \alpha < x_{N+1} < A_j(x'_j, x_{n,j}) + \alpha\}$) is the domain of class C^0. (In this sense, the property $C^{0,*}$ depends not only on Ω and Γ but on the covering of $\partial\Omega$, too. Hence, more precisely, we say Γ to have $C^{0,*}$ property if there exists a covering described above).

Theorem 1. *Let $\Omega \in C^{k-1,1}$ (k positive integer) be $(N+1)$–dimensional domain and let (for convenient covering of $\partial\Omega$) $\Gamma \subset \partial\Omega$ has $C^{0,*}$ property. Let a function $u \in W^{k,p}(\Omega)(p \geq 1)$ be equal to zero on Γ in the sense of traces. Then there exists a sequence $v_n, v_n \in C_0^\infty(R_{N+1}), \operatorname{supp} v_n \cap \overline{\Gamma} = \emptyset, v_n \to u$ in the space $W^{k,p}(\Omega)$.*

PROOF : First, we add to the system W_j an open set $W_0, W_0 \subset \overline{W_0} \subset \Omega$ to form a covering of $\overline{\Omega}$. It is easy to see that we are able to construct, for arbitrary $j = 1, \ldots, r$, an open parallelepiped $\Delta' \subset \overline{\Delta'} \subset \Delta$ (not obviously parallel to the interval $(-\delta, \delta)^N = \Delta$) such that the sets $W'_j = \{(x'_j, X_{N,j}); A_j(x'_j, x_{N,j}) - \alpha < x_{N+1,j} < A_j(x'_j, x_{N,j}) + \alpha\}$ have the same covering properties as the W_j, and, moreover, the open set $G_{0,j} \cap \Delta'$ is of the type C^0. So we can construct a partition of unity Φ_i with respect to the covering W'_j of $\overline{\Omega}$. The functions $u_j = u \cdot \Phi_i$ have their support contained in $W'_j(j = 1, \ldots, r)$, or in $\Omega(j = 0)$, respectively, and so, if we construct sequences $v_{j,n}$ of smooth functions, $v_{j,n} \to u_j, \operatorname{supp} v_{j,n} \subset W'_j \cdot \operatorname{supp} v_{j,n} \cap (\overline{\Gamma} \cap W'_j) = \emptyset \operatorname{supp}, v_{0,n} \subset \Omega$, we can write $v_n = \sum_j v_{j,n}$. To this end, we map the set $W_j(j = 1, \cdots, r)$ onto the parallelepiped $\widetilde{G} = (0,1)^N \times (-1,1)$ defining $\Phi(x) = (Y(x'_j, x'_{N,j}), y_{N+1}(x))$, with $Y : y_i = \frac{1}{2\delta} x_{i,j} + 1(i = 1, \ldots, N)$ and $y_{N+1} = \frac{1}{\alpha}(x_{N+1,j} - A_j(x'_j, x_{N,j}))$. This mapping has Lipschitzian derivatives up to the order $k-1$ (as well as the functions A_j), and hence it is sufficient to apply Lemma 1 on the function $v(y) = u_j(\Phi_{-1}(y))$, with $P = \Phi(W_j'), G_0 = Y(G_{0,j} \cap \Delta')$, and then to use the isomorphism property of the mapping Φ (see Section 2). The existence of a sequence $v_{0,n}$ is obvious, and hence the proof is finished. ∎

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