Commentationes Mathematicae Universitatis Carolinae

Karel Čuda; Blanka Vojtášková

Monads of indiscernibles

Commentationes Mathematicae Universitatis Carolinae, Vol. 30 (1989), No. 1, 109--119

Persistent URL: http://dml.cz/dmlcz/106711

Monads of indiscernibles

KAREL ČUDA, BLANKA VOJTÁŠKOVÁ

Abstract. In the paper the main properties of monads of indiscernibles in AST are investigated. We prove that the notions of to be a minimal monad and to be a monad of indiscernibles are different even if they have a lot of common properties. The consistency of AST with the existence of Mc Aloon's function is presented.

Keywords: Alternative set theory (AST), indiscernibles, minimal monad, infinite natural numbers, endomorphic universe, interpretability in AST

Classification: Primary 03E70, Secondary 54J05,04A20

Monads in the equivalence $\underset{\{c\}}{\overset{\circ}{=}}$, i.e.classes of decomposition of V acc^^rding to $\underset{\{c\}}{\overset{\circ}{=}}$, correspond with ultrafilters on the ring of $Sd_{\{c\}}$ classes. In this understanding minimal monads correspond to minimal (selective) ultrafilters and monads of indiscernibles to Ramsey's ultrafilters. In classical set theory the notions " to be minimal" and "to be Ramsey's" are for ultrafilters equivalent (see [CN]).

In this paper we shall prove, that in the case of monads the situation is different. We show namely that there is a minimal monad which is not a monad of indiscernibles (see §2). For proving this we use substantially Mc Aloon's function, i.e. the function whose existence was demonstrated in [Mc Al]. In the last paragraph of this paper the consistency of AST with the existence of Mc Aloon's function is given.

For the reader's convenience we remind from [S–Ve],[Č–Vo],[Č–K] several important facts and notions used further.

A class $I \subseteq N$ is called a class of indiscernibles for the language $L_{\{c\}}$ iff for each formula $\varphi(z_1, \dots, z_k) \in SFL_{\{c\}}$ and every two sequences $t_1 < t_2 < \cdots < t_k$ and $u_1 < u_2 < \cdots < u_k$ of elements from I we have $\varphi(t_1, \dots, t_k) \equiv \varphi(u_1, \dots, u_k)$.

Obviously, $I \subseteq N$ is a class of indiscernibles for $L_{\{c\}}$ iff for each $n \in FN$ the class $P_n(I) = \{x; x \subseteq I \ \& \ x \widehat{\approx} n\}$ is a part of a monad in $\underset{\{c\}}{\doteq}$

In [S–Ve] it is constructed a class of indiscernibles for the language L, which is a proper π–class and which is an intersection of countably many Sd classes—notation *Ind*. This class is a monad in $\doteq$ (see [Č–K]). For the language $L_{\{c\}}$ are all these considerations similar, we use notation $Ind_{\{c\}}$.

A monad μ (in $\underset{\{c\}}{\doteq}$) is minimal in the ordering $\underset{\{c\}}{\preceq}$ iff each function $F \in Sd_{\{c\}}$ is either constant or one–one on μ.

Let $x \notin \mathrm{Def}_{\{c\}}$. Then

$$Int_{\{c\}}(x) = \{t; (\forall z_1, z_2 \in \mathrm{Def}_{\{c\}})(z_1 < x < z_2 \equiv z_1 < t < z_2)\}.$$

We denote

$$^{\infty}\mathrm{Def}_{\{c\}}N = (\ \mathrm{Def}_{\{c\}} \cap N) - FN.$$

§1. In [Č–Vo] minimal monads were investigated. One special type of them are monads of indiscernibles.

Theorem 1. *$Ind_{\{c\}}$ is a minimal monad (in $\underset{\{c\}}{\preceq}$).*

PROOF: Let $F \in Sd_{\{c\}}$. We have to prove that $F \restriction Ind_{\{c\}}$ is either constant or a one–one function. Suppose that $F \restriction Ind_{\{c\}}$ is not one–one. Then there are $i_1, i_2 \in Ind_{\{c\}}$ such that $i_1 < i_2$ and $F(i_1) = F(i_2) = a$. The condition $F(z_1) = F(z_2)$ can be, however, described by a formula $\varphi(z_1, z_2) \in SFL_{\{c\}}$. Let now i be an arbitrarily chosen element of $Ind_{\{c\}}$. Then either $i < i_2$ and $\varphi(i, i_2)$ holds or $i > i_1$ and $\varphi(i_1, i)$ is valid. In both the cases we come to $F(i) = a$. Consequently $F \restriction Ind_{\{c\}}$ is a constant function. ■

Remarks. Notice, that in the previous proof the property of homogeneity only for couples of elements of $Ind_{\{c\}}$ have been exploited.

In Theorem 1 we have proved that $Ind_{\{c\}}$ is a minimal monad. But this monad of indiscernibles was not more closely specified. Hence this statement is true for each monad of indiscernibles.

In [S–Ve] the existence of a monad of indiscernibles was proved. We show now that—similarly to minimal monads (see [Č–Vo])—there are uncountably many monads of indiscernibles. For proving this we shall use the following assertion from [S–Ve]:

Theorem A. *Let $F \in Sd_{\{c\}}$ be a function with $\mathrm{dom}(F) = P_n(S)$ and $\mathrm{rng}(F) \subseteq \{0,1\}$. Hence $S \in Sd_{\{c\}}$. If S is a proper class then there is a proper class $R \subseteq S$ such that $R \in Sd_{\{c\}}$ and $\mathrm{card}(F''P_n(R)) = 1$ (i.e. there is R which is homogeneous for the partition corresponding to F).*

Theorem 2. *For each countable system of monads $\{\mu_i; i \in FN\}$ in $\underset{\{c\}}{\overset{\circ}{=}}$ there is a monad of indiscernibles (for the language $L_{\{c\}}$) which is a proper class disjoint with all μ_i, $i \in FN$.*

PROOF: Let us enumerate all proper $Sd_{\{c\}}$ classes, denote them $\{X_i; i \in FN\}$, and all functions of $P_n(V)$ into $\{0,1\}$ (for every $n \in FN$)—notation $\{F_i; i \in FN\}$. We shall construct a non–increasing sequence $\{Y_i; i \in FN\}$ of proper $Sd_{\{c\}}$ classes with the following properties: for each $k \leq i$

(1) $Y_{i+1} \subseteq X_k$ or $Y_{i+1} \subseteq V - X_k$;
(2) $Y_{i+1} \cap \mu_k = \emptyset$;
(3) $\mathrm{card}\, F_k'' P_{n_k}(Y_k) = 1$, where n_k denotes the arity of F_k (i.e. Y_k is homogeneous for the partition corresponding to F_k).

Without loss of generality we can suppose that $X_i \subseteq N$ for every $i \in FN$ (since there is a definable isomorphism between V and N—see [V]).

Take, firstly, the class X_1 and put $Y_1 = X_1$. If $Y_1 \cap \mu_1 = \emptyset$, put $\overline{Y}_1 = Y_1$. In the opposite case divide Y_1 into two disjoint proper classes (e.g. even and odd elements—in natural ordering). Then, however, μ_1 is a part of one and only one of them. The subclass of Y_1 (in this division) which is disjoint with μ_1 denote $\overline{Y}_1$. Construct further $F_1 \upharpoonright P_{n_1}(\overline{Y}_1)$, where F_1 is the first element in our enumeration, and apply Theorem A. Then there is a proper class $Y \subseteq \overline{Y}_1$ which is homogeneous for the partition given by $F_1 \upharpoonright P_{n_1}(\overline{Y}_1)$. Construct now $Y \cap X_2$ and $Y \cap (V - X_2)$; from these two classes consider the one which is a proper class—denote it Y_2.

The class Y_{i+1} will be constructed by induction in such a way: assume that Y_i is formed and repeat the above mentioned procedure with "starting points" Y_i, μ_i, F_i. Thus we obtain the required non-increasing sequence $\{Y_i; i \in FN\}$ with properties (1)-(3). Then, however, $\bigcap\{Y_i; i \in FN\}$ is a monad, which is a proper $Sd_{\{c\}}$ class, disjoint with countably many considered monads μ_i and, moreover, homogeneous for all the partitions given by F_i.

What remains to verify is that $\bigcap\{Y_i; i \in FN\}$ is a monad of indiscernibles for the language $L_{\{c\}}$. This proof (based on the above proved homogeneity) can be, essentially, find in [S–Ve]. ■

Theorem 3. *There are uncountably many monads of indiscernibles (for the language $L_{\{c\}}$) which are proper classes.*

PROOF: If there is only a countable number of monads of indiscernibles (for $L_{\{c\}}$) then we can construct—using Theorem 2—next one, which will be different from all preceding monads of indiscernibles. ■

In the previous theorem we have proved that there are uncountably many monads of indiscernibles. Now we show that even under each infinite definable number we can construct uncountably many monads of indiscernibles.

Theorem 4. *Let $\alpha \in^{\infty} \mathrm{Def}_{\{c\}} N$. Then for each countable system of monads $\{\mu_i; i \in FN\}$ in $\overset{\circ}{\underset{\{c\}}{=}}$ such that $\mu_i < \alpha$ for every $i \in FN$ there exists a monad ν of indiscernibles (for the language $L_{\{c\}}$) such that $\nu < \alpha$ and ν is disjoint with all μ_i from the given system.*

At first we prove an auxiliary assertion, close to Theorem A, which says that for each infinite $x \in \mathrm{Def}_{\{c\}}$ and for each partition of $P_n(x)$, definable from c, into two sets there exists an infinite set $y \subseteq x$, $y \in \mathrm{Def}_{\{c\}}$, which is homogeneous for this partition.

Lemma 1. $(\forall x \in \mathrm{Def}_{\{c\}} - \mathrm{Fin})(\forall z \subseteq P_n(x) \cap \mathrm{Def}_{\{c\}})(\exists y \in \mathrm{Def}_{\{c\}} - \mathrm{Fin})$ *such that*

$$[y \subseteq x \;\; \& \;\; (P_n(y) \subseteq z \vee P_n(y) \subseteq P_n(x) - z]. \tag{4}$$

PROOF: If follows from Ramsey's theorem that there is an infinite set y fulfilling (4). Hence it remains to prove that there is such an element y, definable from c.

Let us examine the set $t = \{y; y \subseteq x \;\; \& \;\; y \text{ is homogeneous for the given partition}\}$. Then $t \in \mathrm{Def}_{\{c\}}$ and includes some infinite elements. Put

$$\beta = \max\{|y|; y \in t\};$$

note that $\beta \in \mathrm{Def}_{\{c\}}$.

Let further t_1 contains just the elements of t whose cardinality is β. Denote a the largest element of t_1 (in natural ordering); then a is the required element (obviously $a \in \mathrm{Def}_{\{c\}} - \mathrm{Fin}$). ∎

Proof of Theorem 4. It is enough to modify slightly the proof of Theorem 3 using now Lemma 1 instead of Theorem A. The role of proper $Sd_{\{c\}}$ classes play now infinite sets from $Sd_{\{c\}}$ and F_i are characteristic functions of z.

Theorem 5. *Under each $\alpha \in^\infty \mathrm{Def}\, N$ there are uncountably many monads of indiscernibles (for the language $L_{\{c\}}$).*

PROOF: Proof is analogous to the proof of Theorem 3. ∎

§2. From the definition of monads of indiscernibles it follows immediately:

Theorem 6. *Let μ be a monad of indiscernibles for the language $L, e \in \mu$. Then $\mu \cap e$ and $\mu \cap (N - e)$ are monads of indiscernibles for the language $L_{\{e\}}$.*

Remark. There is a question whether the assumption $e \in \mu$, in the previous theorem, is essential. In other words, if we take an arbitrary element f, we are interested if $\mu \cap f$ and $\mu \cap (N - f)$ are monads of indiscernibles for the language $L_{\{f\}}$. The answer is negative as for a convenient f a more stronger assertion then $f \cap \mu$ is not a monad in $\underset{\{f\}}{\overset{\circ}{=}}$ is valid; this is the content of the following theorem from [Č–Vo]:

Theorem B. *Let μ be a monad. There are c, x such that $\mu \cap \mathrm{Int}_{\{c\}}(x)$ is not a monad in $\underset{\{c\}}{\overset{\circ}{=}}$.*

We shall tend to finding a connection between Theorems B and 6. At first we prove one theorem and an auxiliary lemma.

Remark. In the following theorem we show that if a function $F \in \mathrm{Sd}_{\{c\}}$ on $\mathrm{Ind}_{\{c\}}$ "goes down" then its values lie either over all indiscernibles smaller than its arguments or fall under a definable element (and therefore under $\mathrm{Ind}_{\{c\}}$).

For minimal monads we proved in [Č–Vo] a weaker form of this assertion. Moreover, we know that there is a minimal monad fulfilling Theorem 7(see also [Č–Vo]) and that this monad is not a monad of indiscernibles (it will be shown here—Theorem 8). The existence of a minimal monad for which Theorem 7 does not hold we are not able to demonstrate.

Theorem 7. *Let $F \in \mathrm{Sd}$ be such a function that $F(t) < t$ for one (and hence each) $t \in \mathrm{Ind}_{\{c\}}$. Then just one of the following properties is true:*

(i) $(\forall t \in \mathrm{Ind}_{\{c\}})\{x; F(t) < x < t\} \cap \mathrm{Ind}_{\{c\}} = \emptyset$

(ii) $(\exists d \in \mathrm{Def}_{\{c\}})(\forall t \in \mathrm{Ind}_{\{c\}})F(t) < d < t$.

PROOF: Let us suppose $\neg(i)$. Then there is $u \in \mathrm{Ind}_{\{c\}}$ for which $F(t) < u < t$ is valid for a certain $t \in \mathrm{Ind}_{\{c\}}$. Let $\varphi(u,t)$ be the formula $F(t) < u$; obviously $\varphi(u,t) \in L_{\{c\}}$ and $\varphi(u,t)$ is true for our couple $\langle u,t\rangle$. Then,however, $\varphi(v,t)$ is valid for every $v \in \mathrm{Ind}_{\{c\}}$ such that $v < t$ (see the definition of Ind). This implies

$F(t) < \mathrm{Ind}_{\{c\}}$. From Theorem 1 of [Č–Vo] it follows that there is $d \in \mathrm{Def}_{\{c\}}$ such that $F(t) < d$. Since $t \in \mathrm{Ind}_{\{c\}}$, we obtain $d < t$. Hence (ii) is fulfilled for t. Because $d \in \mathrm{Def}_{\{c\}}, t \in \mathrm{Ind}_{\{c\}}$ and (ii) is true for t, (ii) is valid for all elements of $\mathrm{Ind}_{\{c\}}$. ■

The reader can realize that also here we have used the property of homogeneity only for couples of elements of $\mathrm{Ind}_{\{c\}}$.

Lemma 2. *Let μ be a monad of indiscernibles (for the language L), $e \in \mu$ and $e <^{\infty} \mathrm{Def}\, N$. Let further $x \,\dot{\in}\, N - FN$ is such an element that there is $f \in \mu$ for which $x \leq f < e$ is valid. Then $\mu \cap e = \mu \cap \mathrm{Int}_{\{e\}}(x)$.*

PROOF: For proving this it is sufficient to realize the following facts: The definition of Int implies the inequality $FN < \mathrm{Int}_{\{e\}}(x) <^{\infty} \mathrm{Def}\, N$. From Theorem 7 we know that for each $a \in \mu$ such that $a < e$ we have $a \in \mathrm{Int}_{\{e\}}(x)$ (for this remember that $y \in \mathrm{Def}_{\{c\}} \equiv (\exists F \in \mathrm{Sd}_{\{c\}})y = F(c)$). Moreover $\mu \cap e$ is a monad of indiscernibles for the language $L_{\{e\}}$ (see Theorem 6). ■

The next theorem demonstrates that minimal monads and monads of indiscernibles are really different notions. We shall namely prove that it is consistent with AST that there are not any monads of indiscernibles under $^{\infty}\mathrm{Def}\, N$ and we already know (see [Č–Vo]) that there are minimal monads here.

For proving Theorem 8 we shall use substantially a result from [**Mc Al**] and the technique of endomorphic universes with standard extension (e.u.s.). If someone would like to modify this assertion for nonstandard models of PA, it is enough to substitute the technique of e.u.s. by an application of a convenient ultrapower.

Definition. We say that a function f is c–Mc Aloon's function (denotation c–Mc Al function) iff $f \in \mathrm{Def}_{\{c\}}$ and $f''FN$ is coincial with $^{\infty}\mathrm{Def}_{\{c\}}\, N$. In the cases that $c \in \mathrm{Def}$ or c is clear from the context we shall omit c from our notation.

Theorem 8. *Let there exists $f \in \mathrm{Def}$ which is Mc Al function. Then there is a monad of indiscernibles μ such that $\mu <^{\infty} \mathrm{Def}\, N$.*

PROOF: Let A be e.u.s. Then $f \in A$ (since $\mathrm{Def} \subseteq A$). Let us denote

$$F_1 = f \cap [(N - FN) \times FN]$$

and let $g \in A$ be a prolongation of F_1. The assertion will be proved by a contradiction. Let there exists a monad of indiscernibles μ such that $\mu <^{\infty} \mathrm{Def}\, N$. We know that $\mu = \bigcap\{x_n; x_n \in \mathrm{Def}\}$. Let $h \in A$ be such a descending function that $\mu = \bigcap\{h(n); n \in FN\}$. Denote

$$\nu = \mathrm{Ex}(\mu \cap A).$$

Then evidently $\nu \subset \mu$, as for every $\alpha \in \mathrm{Ex}(FN)$ we have $\nu \subset h(\alpha) \subset \mu$.

Since f is Mc Al function,we obtain (thanks to the standard extension) that $f''\,\mathrm{Ex}(FN)$ is coincial with $\mathrm{Ex}(^{\infty}\mathrm{Def}\, N)$. Moreover, obviously,

$$f''\,\mathrm{Ex}(FN) \subseteq \mathrm{Ex}(FN) \cup \mathrm{Ex}(^{\infty}\mathrm{Def}\, N).$$

From the construction of ν we have

(5) $$\mathrm{Ex}(FN) < \nu < \mathrm{Ex}({}^{\infty}\mathrm{Def}\, N).$$

Realize further that

$$g \restriction FN = f \cap [(N - FN) \times FN]$$

and therefore (we suppose that A is an endomorphic universe with standard extension)

(6) $$g \restriction \mathrm{Ex}(FN) = f \cap [(N - \mathrm{Ex}(FN)) \times \mathrm{Ex}(FN)].$$

At the same time

(7) $$g \restriction \mathrm{Ex}(FN) \subseteq \mathrm{Ex}({}^{\infty}\mathrm{Def}\, N) \times \mathrm{Ex}(FN).$$

Notice further that for each fixed $i_2, i_3 \in \mu$ such that $i_2 < i_3$ the following formula holds for each $n \in FN$:

(8) $$(\forall k < n)(f(k) > i_2 \Rightarrow f(k) > i_3).$$

Fix hence i_2, i_3 and investigate all the numbers for which (8) is valid. Obviously, it is true also for some infinitely large natural numbers. This implies, however, that there is $i_1 \in \mu, i_1 < i_2$ such that

(9) $$(\forall \alpha < i_1)(f(\alpha) > i_2 \Rightarrow f(\alpha) > i_3).$$

This formula (it is a set–definable formula) is therefore valid for all triples $\langle i_1, i_2, i_3 \rangle$ of indiscernibles (see the definition of a monad of indiscernibles). Take

$$i_1 \in \mu \cap \mathrm{Ex}(FN)$$

(such i_1 exists since $\mathrm{Ex}(FN)$ is complete and μ is coincial with $N - FN$). Denote

$$\beta = \min g'' i_1.$$

Then $\beta \in \mathrm{Ex}({}^{\infty}\mathrm{Def}\, N)$; for this remember (7) and the fact that $i_1 \in \mathrm{Ex}(FN)$. Further $\beta = f(\alpha)$ for a certain number $\alpha < i_1$ (see formula (6)). Moreover, $\beta <^{\infty} \mathrm{Def}\, N$ since $g''FN \subset g''i_1$ and β is the smallest element of $g''i_1$.

Let us come back to properties of ν; remind for this (5) and the circumstance that $\beta \in \mathrm{Ex}({}^{\infty}\mathrm{Def}\, N)$.

Choose $i_2 \in \nu$; then $i_1 < i_2$ (since $i_1 \in \mathrm{Ex}(FN)$). Furthermore $i_2 < f(\alpha) = \beta \in \mathrm{Ex}({}^{\infty}\mathrm{Def}\, N)$; but $\beta <^{\infty} \mathrm{Def}\, N$ and hence there is $i_3 \in \mu$ such that $i_3 > \beta$. Using now formula (9) for our special α we come to the implication

$$f(\alpha) > i_2 \Rightarrow f(\alpha) > i_3,$$

which is a contradiction since $i_2 < f(\alpha)$, but $i_3 > f(\alpha) = \beta$. This completes the proof. ■

Remarks. The proof can be easily overworked into the parametric version ($f \in \mathrm{Def}_{\{c\}}$). We only have to demand in this case that $c \in A$.

The reader may notice that in the previous proof we used—firstly in this paper—the homogeneity for triples.

In [Č–Vo] we investigated the notion of a great distance between infinitely large definable numbers and gave several equivalents of it. When supposing the existence of Mc Al function we can obtain still one interesting necessary and sufficient condition, namely that two infinitely large definable numbers are very far one from the other iff one falls into $\mathrm{Ex}(FN)$ and the other into the convex hull of $\mathrm{Ex}({}^{\infty}\mathrm{Def}\, N)$ for a suitable standard extension.

For proving this assertion we need one theorem from [Č–Vo]:

Theorem C. *There exists a minimal monad μ in the class $X = \{\alpha; FN < \alpha < {}^{\infty}\mathrm{Def}_{\{c\}} N\}$ such that for every $F \in \mathrm{Sd}_{\{c\}}, F : n \to N$, there is $Z \in \mathrm{Sd}_{\{c\}}, Z \supseteq \mu$, such that either (i) or (ii) takes place, where*

(i) $F \restriction Z : FN \to FN$

(ii) $F''Z \cap FN = \emptyset$.

Lemma 3. *Let μ be a monad from Theorem C and f be Mc Al function. Let further $\alpha \in \mu, \beta >>_{FN} \alpha$. Then there is $\alpha_1 < \alpha$ such that $\beta < f(\alpha_1) >>_{FN} \alpha$.*

For the reader's convenience we recall from [Č–Vo]:

$$a <<_{FN} b \equiv (\forall F \in \mathrm{Sd}_0)(F : FN \to FN \Rightarrow F(a) < b)$$

PROOF: Denote

$$\gamma = \max\{f(\alpha_1); \alpha_1 < \alpha \;\;\&\;\; f(\alpha_1) < \beta\}.$$

We shall prove that $\gamma >>_{FN} \alpha$. Suppose the contrary. Then there is a function $g \in \mathrm{Sd}_0$ such that $g : FN \to FN$ and $g(\alpha) \geq \gamma$.

Let us define the function h as follows:

$$h(\xi) = \min(f''\xi - g(\xi)). \tag{10}$$

Then, evidently, $h(\alpha) \geq \beta >>_{FN} \alpha$.

It follows from Theorem C (since $h \in \mathrm{Sd}$ and (i) does not take place) that $h(\alpha) > \eta$ for a certain $\eta \in^{\infty} \mathrm{Def}\, N$. On the other hand, we know (see (10)) that $h(\alpha) <^{\infty} \mathrm{Def}\, N$; for this realize that $g(\alpha) <^{\infty} \mathrm{Def}\, N, g : FN \to FN$, f is Mc Al function and therefore $f''\alpha$ is coincial with ${}^{\infty}\mathrm{Def}\, N$. We come to a contradiction. ∎

Theorem 9. *Let there exists $f \in \mathrm{Sd}$ which is Mc Al function. Then*

$$\alpha <<_{FN} \beta \equiv (\exists \text{e.u.s.} A)(\exists \gamma \in \mathrm{Ex}({}^{\infty}\mathrm{Def}\, N))(\alpha \in \mathrm{Ex}(FN) \;\;\&\;\; \beta > \gamma).$$

PROOF: At first we prove $\Leftarrow$. Notice that this implication remains to be true even if we do not suppose the existence of Mc Al function.

Obviously it suffices to show that $\alpha <<_{FN} \gamma$. Since $\gamma \in \mathrm{Ex}({}^{\infty}\mathrm{Def}\, N)$, we have $\gamma > \mathrm{Ex}(FN)$. But $\alpha \in \mathrm{Ex}(FN)$. Let us apply now one criterion for to be very far from [Č–Vo]:

$$(11) \qquad \alpha <<_{FN} \beta \equiv (\exists \mathrm{e.u.s.}\, A)\alpha \in \mathrm{Ex}(FN) < \beta.$$

Then we obtain $\alpha <<_{FN} \gamma$.

For proving $\Rightarrow$ recall (from [Č–Vo]) that there are α_1, β_1 such that $\alpha < \alpha_1 < \beta_1 < \beta$ and $\alpha_1, \beta_1 \in \mu$, where μ is the monad from Theorem C. Moreover, $\beta_1 >>_{FN} \alpha_1$ (μ is minimal). In accordance with Lemma 3 there exists therefore $\overline{\alpha}_1 < \alpha_1$ such that $\beta_1 > f(\overline{\alpha}) >> \alpha_1$. Apply now (11) for α_1 and $f(\overline{\alpha})$, We obtain that there is an endomorphic universe with standard extension A such that $\alpha_1 \in \mathrm{Ex}(FN)$ and $f(\overline{\alpha}) \notin \mathrm{Ex}(FN)$. But $f(\overline{\alpha}) \in \mathrm{Ex}({}^{\infty}\mathrm{Def}\, N)$, since f is Mc Al function. For completing the proof put $\gamma = f(\overline{\alpha})$. ■

§3 In the paper [Mc Al] it is shown how to construct a countable model of PA, in which Mc Al function f is definable without parameter. Furthemore, one can find there an instruction for the construction of such a countable model of PA, in which do not exist definable—without parameter—infinitely large natural numbers, but for a convenient parameter Mc Al function, with this parameter, can be defined. Both these construction are realized by a suitable modification of Henkin's construction of the countable model for countable consistent theory. Since Henkin's construction requires only the possibility to speak about countable parts of natural numbers, one can repeat it in the second order arithmetic. Therefore we may realize it, and hence both the above mentioned constructions, in the alternative set theory (AST).

Theorem 10. *The theory*

$$T_1 \equiv (\ AST + \textit{the existence of Mc Al function } f \in \mathrm{Def})$$

is interpretable in AST.

Theorem 11. *The theory*

$$T_2 \equiv [\ AST\ + \mathrm{Def} = V_{FN} + [(\exists c)(\exists f)(f \in \mathrm{Def}_{\{c\}}\ \ \&\ \ f \textit{ is Mc Al function})]]$$

is interpretable in AST.

Proofs of Theorems 10 and 11:. From the remark which precedes to Theorem 10 it follows that in AST we have for our disposal two countable structures, that are models of PA, and therefore also of ZF_{Fin}. Moreover, one of them—denotation $\mathcal{S}_2$—is elementary equivalent with FN; in $\mathcal{S}_2$ Mc Al function f with parameter is definable. In the other structure—let us denote it $\mathcal{S}_1$—this function is definable without parameter.

Let $\mathcal{U}$ be a nontrivial ultrafilter on FN. Let us construct the ultrapowers of $\mathcal{S}_1, \mathcal{S}_2$ with FN as the index class. We assert that from these ultrapowers we can obtain the required interpretations of T_1 and T_2 in AST.

Since the respective interpretations are described, under weaker assumptions, in [S] and because ultrapower belongs to usual mathematical constructions, we shall present here only a brief description of these interpretations. We shall take stronger assumptions than in [S] (the axiom of choice) which should make the reader's understanding easier.

Further we shall work with $\mathcal{S}_1$ and $\mathcal{S}_2$. Since these structures are models of ZF_{Fin}, we shall use for the relation memberskip the symbol ε.

Let us denote

$$V^{**} = \{f; f : FN \to \mathcal{S}_2\}$$

and let

$$f \widehat{=} g \equiv \{t; f(t) = g(t)\} \in \mathcal{U}.$$

Construct $V^* = V^{**}/_{\widehat{=}}$ and take (we suppose the axiom of choice) the representatives of factor classes in such a manner that from classes of almost constant functions (constant on an element of $\mathcal{U}$) we choose as the representative the respective constant function.

Put further

$$f \widehat{\in} g \equiv \{t; f(t) \varepsilon g(t)\} \in \mathcal{U}.$$

Classes are now all subclasses of V^* and sets are such classes X that there is $f \in V^*$ such that $X = \{g; g \in V^* \ \& \ g \widehat{\in} f\}$.

The scheme of existence of classes (i.e. Morse's scheme) in the interpretation we obtain from the same scheme in the theory (when using the initial structure and the ultrafilter $\mathcal{U}$ as the parameters). The axioms for sets we have in the interpretations immediately since they are true, thanks to Lös's theorem, in the structures.

Realize further that

$$FN^* = \{f; (\exists n \in FN) \operatorname{rng}(f) = \{n\}\}.$$

This at once implies that there is a one–one mapping between FN^* and FN; hence the property of countability is absolute. For verifying the axiom of two cardinalities it suffices to prove that there is at least one uncountable class in V^*. Notice, because of this, that V^* is uncountable since it contains all constant functions. It implies, however, that we can embed V into V^* and thus an uncountable class exists (in V the axiom of two cardinalities holds).

We shall prove now (the axiom of prolongation)*. Let $f_1, f_2, \ldots, f_n$ be a countable class of functions and F a class function such that $F(n) = f_n$, where $f_n \in V^*$. We tend to find a set prolongation of F, i.e. a function g for which

$$g^n(\bullet) \widehat{=} f_n(\bullet),$$

where the index up denotes variables in the sense of the interpretation and $\bullet$ variable in the index class (FN) from the construction of the ultrapower. Put

$$g(1) = \{< f_1(1), 1 >\}$$
$$g(2) = \{< f_1(2), 1 >, < f_2(2), 2 >\}$$
$$g(3) = \{< f_1(3), 1 >, < f_2(3), 2 >, < f_3(3), 3 >\}, \text{etc.}$$

Evidently g is a set function (owing to the axiom of prolongation in the theory) and for each $n < k$ we have

$$g^n(k) = f_n(k).$$

Hence g is the function we have looked for.

The class V^* can be well ordered since the restriction of well ordering of V on V^* is the class (in the sense of the interpretation), which is well ordering and hence it is well ordering also in the sense of interpretation.

Thus we have shown that, when following the above described procedure, we obtain the interpretations of AST. For completing the proof of both theorems it is sufficient to demonstrate the interpretability of Mc Al function definable with and without parameter. We show, for this purpose, that the constant function with the value of Mc Al function is Mc Al function in the interpretation.

Firstly we shall investigate Mc Al function $f \in \text{Def}$. Then the following statement holds:

(S): For each $\alpha \in^\infty \text{Def}\, N$ there is $n \in FN$ such that

$$FN < f(n) < \alpha.$$

We prove that $(\exists! f)\varphi(f)$, where φ is a set formula describing Mc Al function—this assertion is, however, a corollary of Loś's theorem. Furthermore, we verify $(S)^*$.

Let $(\alpha \in^\infty \text{Def}\, N)^*$, then

$$[(\exists!\alpha)\, \varphi\, (\alpha) \;\;\&\;\; \alpha \in N \;\;\&\;\; (\forall n \in FN)\alpha > n]^*.$$

Formula φ is a set formula and therefore, thanks to Loś's theorem, φ defines just one element which fulfills this assertion in S_1. This implies that the elements of $^\infty \text{Def}\, N$ are just the constants from $^\infty \text{Def}\, N$ in S_1. Hence, for proving $(S)^*$, it suffices to use the absoluteness of FN and $^\infty \text{Def}\, N$ and to apply again Loś's theorem for fixed elements of FN and $^\infty \text{Def}\, N$.

In the case that f is in the model defined by a parameter c, we must add c as a logical constant into the language. Then there exists a function $\varphi(f, c)$, which describes Mc Al function. Repeating the above mentioned process we can transfer $^\infty \text{Def}_{\{c\}}\, N$ and realize that the parameter c is such a constant function that $f(n) = c$ for each $n \in FN$. The rest of the proof is quite analogous.

References

[V] Vopěnka P., "Mathematics in the Alternative Set Theory," Teubner–Texte, Leipzig, 1979.

[Č–Vo] Čuda K., Vojtášková B., *Minimal monads*, Comment.Math.Univ.Carolinae **28** (1987), 691-706.

[Č–K] Čuda K., Kussová B., *Monads in basic equivalences*, Comment.Math.Univ.Carolinae **24** (1983), 437-452.

[S–Ve] Sochor A., Vencovská A., *Indiscernibles in the alternative set theory*, Comment.Math. Univ. Carolinae **22** (1981), 785-798.

[S] Sochor A., *Metamathematics of the alternative set theory II*, Comment.Math.Univ.Carolinae **23** (1982), 55-79.

[Mc Al] Mc Aloon K., "Diagonal methods and strong cuts in models of arithmetic,," Logic Colloq 1977, Wroclaw, North Holland Publ.Company, 1978, pp. 171-181.
[CN] Comfort W.W., Negrepontis S., "The Theory of Ultrafilters," Springer–Verlag, 1974.

Matematický ústav UK, Sokolovská 83,180 00 Praha 8, Czechoslovakia

(*Received* July 30,1988)