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Abstract. We construct for any given F_σ–set F in R^3 a linear elliptic system with bounded measurable coefficients and its bounded weak solution in R^3 which is essentially discontinuous on F and essentially continuous on $R^3 \setminus F$.

Keywords: elliptic systems, regularity.

Classification: 35D10, 35J45

1. Introduction. We are interested in linear elliptic systems of the form

$$D_\alpha(A_{ij}^{\alpha\beta}(x)D_\beta u^j) = 0, \qquad i = 1,\dots,M. \tag{1.1}$$

The domain of the functions $A_{ij}^{\alpha\beta}$, u^j is considered to be a nonempty open subset Ω of R^m. The summation convention is used throughout the paper. We suppose that

$$A_{ij}^{\alpha\beta} \in L^\infty(\Omega), \qquad \alpha,\beta = 1,\dots,m; \quad i,j,= 1,\dots,M \tag{1.2}$$

such that there is $\lambda > 0$ for which

$$A_{ij}^{\alpha\beta}(x)\xi_\alpha^i\xi_\beta^j > \lambda|\xi|^2 \qquad \text{for every } \xi \in R^{mM} \text{ and almost every } x \in \Omega. \tag{1.3}$$

By a (weak) solution of the system (1.1) we understand a function

$$u \in W_{\text{loc}}^{1,2}(\Omega, R^M), \qquad u = (u^1,\dots,u^M) \tag{1.4}$$

such that

$$D_\alpha b_i^\alpha = 0, \qquad i = 1,\dots,M \tag{1.5}$$

holds in the sense of distributions on Ω for

$$b_i^\alpha = A_{ij}^{\alpha\beta}D_\beta u^j.$$

According to the classical results of C.B.Morrey, A.Douglis, L.Nirenberg ([**18**], [**4**]) every weak solution of (1.1) is locally Hölder continuous provided $A_{ij}^{\alpha\beta}$ are continuous on Ω. As we can see from the proof of Theorem 3.1 in [**9**], the continuity of coefficients at one point implies the Hölder continuity of the solution in a neighbourhood of this point. On the other hand, the discontinuity of coefficients of (1.1)–(1.3) even at one point can cause the discontinuity of the solution—see the well-known counterexamples of E.De Giorgi, E.Giusti, M.Miranda (see [1], [12], [9]).

Consider now a system

$$D_\alpha(\widetilde{A}_{ij}^{\alpha\beta}(x,u)D_\beta u^j) = 0, \qquad i = 1,\dots,M \tag{1.6}$$

where the coefficients $\widetilde{A}_{ij}^{\alpha\beta}$ are uniformly continuous both in x and u and satisfy conditions analogous to (1.2), (1.3). From the point of view of regularity we can regard (1.6)to be a special case of (1.1)when putting

$$A_{ij}^{\alpha\beta}(x) = \widetilde{A}_{ij}^{\alpha\beta}(x, u(x)).$$

Although we cannot expect in general the everywhere continuity of the solution of (1.6) (see [12]) the following partial regularity result holds (see [9]):

(1.7) "There is an open set $\Omega_0 \subset \Omega$ such that u is locally Hölder continuous on Ω_0 and the $(m-2)$-dimensional Hausdorff measure of $\Omega \setminus \Omega_0$ is zero."

The counterexample of J.Souček (see [19]) gives a solution u of a system (1.1) which is discontinuous on a dense countable set. Hence the partial regularity (1.7) does not hold for solutions of (1.1).

For the solutions of (1.1), analogues of (1.7) are available in terms of generalized continuities only. J.Deny and J.L.Lions [3] proved that every function from $W^{1,2}$ is finely continuous except a set of 2-capacity zero. This result was generalized to $W^{1,p}$ by N.G.Meyers [17]. The relations between capacities and Hausdorff measures (O.Frostman [7], H.Federer and W.P.Ziemer [5], V.G.Mazja and V.P.Havin [15]) show that the exceptional set of a function from $W^{1,p}$ has Hausdorff dimension $m-p$. It was observed by B.Fuglede [8] that fine continuity implies approximate continuity in the sense of A.Denjoy [2]. The size of sets of non-Lebesgue points is estimated in the papers of E.Giusti ([10], [11]), H.Federer and W.P.Ziemer [5], etc.

This all can be said on a function u from $W^{1,p}$ without using the fact that u solves any equation.

For the solutions of (1.1) we have a $W^{1,p}$-estimate for some $p > 2$ due to N.G.Meyers [16] which implies that the Hausdorff dimension of the exceptional set is less than $m-2$.

The advantage of the system (1.6) consists in E.Giusti's "Main Lemma of Partial Regularity" (see [9]), which states that the solution of (1.6) is Hölder continuous on a neighbourhood of every its Lebesque point.

This leads to the topological interpretation of the proof of (1.7) at least concerning bounded functions, for which the notions of Lebesque points and approximate continuity points coincide. A fine approach to partial regularity is established by J.Frehse [6].

This paper is devoted to the continuity of solutions of (1.1) in the usual sense (i.e. with respect to the Euclidean topology). We construct a system of the type (1.1), (1.2), (1.3) and its bounded weak solution on R^3 whose set of points of essential discontinuity is a given set of the type F_σ. In particular, the solution can be everywhere essentially discontinuous. Thus, the above mentioned partial regularity results using generalized continuities are for general system (1.1) in some sense best possible.

In what follows we shall consider the case $M = m \geq 3$.

2. Souček's method. Let u be a solution of (1.1). Denote

$$a_i^\alpha = D_\alpha u^i, \qquad \text{(potential field)}, \tag{2.1}$$

$$b_i^\alpha = A_{ij}^{\alpha\beta} D_\beta u^j, \qquad \text{(divergence–free field)}. \tag{2.2}$$

Then $a, b \in L^2_{\text{loc}}(\Omega, R^m \times R^m)$. Ellipticity and boundedness conditions on $A_{ij}^{\alpha\beta}$ yields the existence of positive constants λ, μ such that

$$\langle b, a\rangle \geq \lambda \langle a, a\rangle \qquad \text{a.e. in } \Omega, \tag{2.3}$$

$$\langle b, b\rangle \leq \mu^2 \langle a, a\rangle \qquad \text{a.e. in } \Omega, \tag{2.4}$$

where $\langle a, b\rangle$ means the scalar product in $R^m \times R^m$.

Converting this observation we obtain the result due to J.Souček **[19]** which is very useful in the construction of counterexamples.

Theorem 1. *Let u be a given function of $W^{1,2}_{\text{loc}}(\Omega, R^m)$, a its potential field (2.1). Let $b \in L^2_{\text{loc}}(\Omega, R^m \times R^m)$ be a divergence-free field (i.e. $D_\alpha b_i^\alpha = 0 \quad (i = 1, \dots, m)$ on Ω in the sense of distributions). Assume that there are positive constants λ, μ such that (2.3), (2.4) hold.*

Then u is weak solution of a system (1.1), whose coefficients $A_{ij}^{\alpha\beta}$ satisfy the estimate

$$\lambda_0 |\xi|^2 \leq A_{ij}^{\alpha\beta}(x) \xi_\alpha^i \xi_\beta^j \leq \lambda_1 |\xi|^2, \tag{2.5}$$

for all $\xi \in R^m \times R^m$ and almost all $x \in \Omega$, where

$$\frac{\lambda_0}{\lambda_1} = \frac{\frac{\mu}{\lambda} - \sqrt{\frac{\mu^2}{\lambda^2} - 1}}{\frac{\mu}{\lambda} + \sqrt{\frac{\mu^2}{\lambda^2} - 1}}. \tag{2.6}$$

Proof: For $\Theta \in (0, \lambda)$ put

$$A_{ij}^{\alpha\beta} = \Theta \delta_{\alpha\beta}\delta_{ij} + \frac{(b_i^\alpha - \Theta a_i^\alpha)(b_j^\beta - \Theta a_j^\beta)}{\langle b - \Theta a, a\rangle}.$$

For all $\xi \in R^m \times R^m$ and almost all $x \in \Omega$ we have

$$A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j = \Theta |\xi|^2 + \frac{\langle b - \Theta a, \xi\rangle^2}{\langle b - \Theta a, a\rangle} \geq \Theta |\xi|^2$$

and

$$A_{ij}^{\alpha\beta} \xi_\alpha^i \xi_\beta^j \leq |\xi|^2 (\Theta + \frac{|b - \Theta a|^2}{\langle b - \Theta a, a\rangle}) \leq \frac{\mu^2 - \Theta\lambda}{\lambda - \Theta} |\xi|^2.$$

Choosing $\Theta = \frac{\mu^2 - \mu\sqrt{\mu^2 - \lambda^2}}{\lambda}$ we obtain (2.5), (2.6). ∎

Remark. As it is easy to calculate, the above choice of Θ keeps the ratio λ_0/λ_1 in (2.5) maximal. It will help us to prove that the counterexample constructed in this article has λ_0/λ_1 arbitrarily near to the Koshelev's condition number $K(m)$ which guarantees for $\lambda_0/\lambda_1 > K(m)$ the regularity (see Section 7).

3. Construction of the counterexample. Consider a sequence $\{z_p\}$ of dictinct points of R^m, a sequence $\{w_p\}$ of constant vectors from R^m and a sequence $\{G_p\}$ of positive functions from $C^2(R_+)(p = 1,2\dots)$. Denote

$$\begin{aligned} r_p &= r_p(x) = |x - z_p|, & n_p &= n_p(x) = \frac{x - z_p}{|x - z_p|} \\ f_p &= f_p(x) = \frac{G_p(r_p(x))}{r_p(x)}, & g_p &= g_p(x) = G_p'(r_p(x)). \end{aligned} \tag{3.0}$$

Assume that the objects described above have the following properties:

$$|w_p| < 2 \qquad \text{for all } p \in N, \tag{3.1}$$

$$\begin{aligned} &\text{there exists } \tau \in (0, 0.01) \text{ such that for every} \\ &p \in N \qquad 0 \le -g_p \le \tau f_p \qquad \text{on } R^m \setminus \{z_p\}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\text{for every } R > 0 \text{ we have} \\ &\sum_{p=1}^{\infty} \|f_p\|_{L^2(B_R(0))} < +\infty. \end{aligned} \tag{3.3}$$

Put

$$u_p = u_p(x) = r_p f_p(n_p - w_p), \tag{3.6$_0$}$$

$$(a_i^\alpha)_p = D_\alpha u_p^i = f_p(\delta_{\alpha i} - n_p^\alpha n_p^i) + g_p(n_p^\alpha n_p^i - n_p^\alpha w_p^i), \tag{3.4$_0$}$$

$$(b_i^\alpha)_p = f_p((m-2)\delta_{\alpha i} + n_p^\alpha n_p^i) + g_p(\delta_{\alpha i} - n_p^\alpha n_p^i), \tag{3.5$_0$}$$

$$a = \sum_{p=1}^{\infty} a_p, \tag{3.4}$$

$$b = \sum_{p=1}^{\infty} b_p, \tag{3.5}$$

$$u = \sum_{p=1}^{\infty} u_p. \tag{3.6}$$

Theorem 2. *Let $\Omega = R^m$ and u be defined by (3.6). Then there is a system (1.1)-(1.3) such that u is its weak solution. For any $\tau \in (0, 0.01)$, the system can be constructed in such a way that*

$$\frac{\mu^2}{\lambda^2} \le 1 + \frac{m-1}{(m-2)^2} + 150\tau, \tag{3.7}$$

where μ and λ are the constants from (2.3), (2.4).

PROOF: It will be proved in the next section that one can obtain for each positive τ the functions f_p, g_p given by (3.0) for which (3.2) takes place. We can check that the sum (3.6) converges strongly in $W^{2,1}_{\rm loc}(R^m)$ and the sums (3.4), (3.5) in $L^2_{\rm loc}(R^m)$. It is easy to calculate that $D_\alpha b^\alpha_i = 0$, $D_\alpha u^i = a^\alpha_i, i = 1, \quad \ldots, m$, in the sense of distributions.

Fix now $p, q \in N$ and denote $\Theta_{pq} = \langle n_p, n_q\rangle$. We have

$$\begin{aligned}
\langle a_p, a_q\rangle &= f_p f_q(m-2+\Theta^2_{pq})+\\
&+ f_p g_p(1-\Theta^2_{pq} - \langle n_q, w_q\rangle + \Theta_{pq}\langle n_p, w_q\rangle)+\\
&+ f_q g_p(1-\Theta^2_{pq} - \langle n_p, w_p\rangle + \Theta_{pq}\langle n_q, w_p\rangle)+\\
&+ g_p g_q(\Theta^2_{pq} - \Theta_{pq}[\langle w_q, n_p\rangle + \langle w_p, n_q\rangle - \langle w_p, w_q\rangle]),\\
\langle b_p, b_q\rangle &= f_p f_q(m^3 - 4m^2 + 6m - 4 + \Theta^2_{pq})+\\
&+ (f_p g_q + f_q g_p)(m^2 - 3m + 3 - \Theta^2_{pq})+\\
&+ g_p g_q(m-2+\Theta^2_{pq}),\\
\langle b_p, a_q\rangle &= f_p f_q(m^2 - 3m + 3 - \Theta^2_{pq})+\\
&f_p g_q(m-2+\Theta^2_{pq} - (m-2)\langle n_q, w_q\rangle - \Theta_{pq}\langle n_p, w_q\rangle)+\\
&f_q g_p(m-2+\Theta^2_{pq})+\\
&g_p g_q(1-\Theta^2_{pq} - \langle n_q, w_q\rangle + \Theta_{pq}\langle n_p, w_q\rangle).
\end{aligned}$$

Hence, taking into account that $\tau \in (0, 0.01)$, we obtain

$$\begin{aligned}
&\langle b_p, a_q\rangle \geq f_p f_q(m^2 - 3m + 3 - \Theta^2_{pq})(1-4\tau),\\
&f_p f_q(m-2+\Theta^2_{pq})(1-11\tau) \leq \langle a_p, a_q\rangle \leq\\
&\leq f_p f_q(m-2+\Theta^2_{pq})(1+9\tau),\\
&\langle b_p, b_q\rangle \leq f_p f_q(m^3 - 4m^2 + 6m - 4 + \Theta^2_{pq})(1+\tau).
\end{aligned} \tag{3.8}$$

From (3.8) it follows that

$$\langle b, a\rangle = \sum_{p,q}\langle b_p, a_q\rangle \geq \sum_{p,q}\lambda_{pq}\langle a_p, a_q\rangle = \lambda\langle a, a\rangle, \qquad \text{where} \tag{3.9}$$

$$\lambda_{pq} = \frac{(m^2 - 3m + 3 - \Theta^2_{pq})(1-4\tau)}{(m-2+\Theta^2_{pq})(1+9\tau)}, \tag{3.10}$$

$$\lambda = \frac{\sum_{p,q}\lambda_{pq}\langle a_p, a_q\rangle}{\langle a, a\rangle}, \tag{3.11}$$

$$\langle b, b\rangle = \sum_{p,q}\langle b_p, b_q\rangle \leq \sum_{p,q}\mu^2_{pq}\langle a_p, a_q\rangle = \mu^2\langle a, a\rangle, \qquad \text{where} \tag{3.12}$$

$$\mu^2_{pq} = \frac{(m^3 - 4m^2 + 6m - 4 + \Theta^2_{pq})(1+\tau)}{(m-2+\Theta^2_{pq})(1-11\tau)}, \tag{3.13}$$

$$\mu^2 = \frac{\sum_{p,q} \mu_{pq}^2 \langle a_p, a_q\rangle}{\langle a, a\rangle}. \tag{3.14}$$

Now we estimate from (3.10), (3.13)

(3.15)
$$\begin{aligned}\frac{\mu_{pq}^2}{\lambda_{pq}\lambda_{rs}} &= \frac{(m^3-4m^2+6m-4+\Theta_{pq}^2)(m-2+\Theta_{rs}^2)(1+\tau)(1+9\tau)^2}{(m^2-3m+3-\Theta_{pq}^2)(m^2-3m+3-\Theta_{rs}^2)(1-4\tau)^2(1-11\tau)} \leq \\ &\leq (\frac{m^3-4m^2+6m-3)(m-1)(1+\tau)(1+9\tau)^2}{(m^2-3m+2)^2(1-4\tau)^2(1-11\tau)} \leq \\ &\leq (1+\frac{m-1}{(m-2)^2})(1+50\tau) \leq 1+\frac{m-1}{(m-2)^2}+150\tau.\end{aligned}$$

From (3.11), (3.14) and (3.15) we get finally

$$\begin{aligned}\frac{\mu^2}{\lambda^2} &= \frac{\sum_{p,q}\mu_{pq}^2\langle a_p, a_q\rangle \sum_{r,s}\langle a_r, a_s\rangle}{\sum_{p,q}\lambda_{pq}\langle a_p, a_q\rangle \sum_{r,s}\lambda_{rs}\langle a_r, a_s\rangle} = \\ &= \frac{\sum_{p,q,r,s} \mu_{pq}^2\langle a_p, a_q\rangle\langle a_r, a_s\rangle}{\sum_{p,q,r,s} \lambda_{pq}\lambda_{rs}\langle a_p, a_q\rangle\langle a_r, a_s\rangle} \leq 1+\frac{m-1}{(m-2)^2}+150\tau.\end{aligned}$$

So the system (1.1)–(1.3) with the solution u given by (3.6_0), (3.6) can be constructed as in Theorem 1 with the divergence–free field b given by (3.5_0), (3.5). As we have proved, it has the property (3.7). ∎

4. The auxiliary functions. The condition (3.2) is satisfied, if the function $G = G_p$ satisfies the differential inequality

$$0 \leq -G'(r) \leq \frac{\tau G(r)}{r}.$$

This inequality is satisfied e.g. if G is defined by the formula

$$G(r) = \kappa(1+\omega r)^{-\tau}, \qquad \text{where}$$
$$\kappa \in (0,1), \quad \omega \in (1,\infty) \text{ and } \tau \in (0, 0.01).$$

For each $R > 0$ we have (defining f_p as in (3.0))

$$\|f_p\|_{L^2(B_R(0))} \leq (\int_0^R G_p^2(r) r^{m-3} dr)^{1/2}.$$

Putting

$$G_p(r) = \kappa_p(1+\omega_p r)^{-\tau},$$

we have

$$\|f_p\|_{L^2(B_R(0))} \leq \kappa_p\omega_p^{-\tau}[\frac{R^{m-2(1+\tau)}}{m-2(1+\tau)}]^{1/2}.$$

Hence if the sequence $\{\omega_p\}$ tends to infinity rapidly enough, the condition (3.3) is satisfied.

5. Boundedness. In this section we show that if $p_k \le \kappa$ we can make the solution u bounded by 2κ. We proceed as in Section 3 specifying the choice of G_p and w_p by recurrent formulae.

Let $p \in N$. Denote

$$s_p = \sum_{q=1}^{p-1} u_q$$

(so in the first step $s_1 = 0$). Find $\delta_p > 0$ such that

$$|s_p(x) - s_p(z_p)| < 2^{-p}\kappa_p \qquad \text{for all } x \in B_{\delta_p}(z_p). \tag{5.1}$$

Now let $\omega_p \in (1, \infty)$ be so great that

$$\omega_p^{-\tau} < 2^{-p} \tag{5.2}$$

and

$$(1 + \omega_p \delta_p)^{-\tau} < 2^{-p}. \tag{5.3}$$

Put

$$w_p = \frac{1}{\kappa_p} s_p(z_p), \tag{5.4}$$

$$G_p(r) = \kappa_p (1 + \omega_p r)^{-\tau}.$$

We define g_p, f_p, u_p etc. as in Section 3.

Theorem 3. *Under the above specification, for every $p \in N$ we have*

$$|s_p| < 2\kappa \qquad \textit{a.e. on } R^m.$$

PROOF: By means of induction we prove the following claim:
For each $p \in N$ we have

$$|s_p| \le 2\kappa(1 - 2^{-p}). \tag{5.5}$$

For $p = 1$ we have $s_1 = 0$. Let $p \ge 1$ and (5.5) be satisfied. Choose $x \in B_{\delta_p}(z_p)$. Then from (5.5), (5.1) and (5.4) it follows

$$\begin{aligned} |s_{p+1}(x)| &\le |s_p(x) - s_p(z_p)| + |s_p(z_p) + u_p(x)| \le \\ &\le \kappa_p 2^{-p} + |s_p(z_p)|(1 - \frac{r_p f_p}{\kappa_p}) + |\kappa_p n_p| \frac{r_p f_p}{\kappa_p} \le \\ &\le \kappa_p 2^{-p} + \max(|s_p(z_p)|, \kappa_p) \le \kappa_p 2^{-p} + 2\kappa(1 - 2^{-p}) \le \\ &\le 2\kappa(1 - 2^{-p-1}). \end{aligned}$$

Now choose x outside $B_{\delta_p}(z_p)$. Then from (5.3), (5.5) we obtain

$$\begin{aligned} |s_{p+1}(x)| &\le |s_p(x)| + |u_p(x)| \le \\ &\le \kappa_p 2^{-p} + 2\kappa(1 - 2^{-p}) \le 2\kappa(1 - 2^{-p-1}). \quad \blacksquare \end{aligned}$$

6. Continuity and discontinuity. It is well known that the set of all points of discontinuity of an arbitrary function is a set of type F_σ. We shall prove a curious converse: Every set $F \subset R^m$ of type F_σ is the set of all discontinuity points for a solution of an elliptic system with L^∞–coefficients.

Since a function from $W^{1,2}$ is, in fact, defined up to a set of measure zero, it is more meaningful to work with essential continuity. A measurable function v is said to be essentially continuous at a point $z \in R^m$ if

$$\operatorname*{osc\,ess}_{x \to z} f(x) = 0,$$

where

$$\operatorname*{osc\,ess}_{x \to z} f(x) = \inf\nolimits_{\delta>0} \inf\nolimits_{Z \subset R^m, \operatorname{meas} Z=0} \sup\nolimits_{x,y \in B_\delta(z)\setminus Z} |f(x) - f(y)|.$$

If we replace a function $f \in L^\infty_{\rm loc}$ by its essential limsup (in each coordinate, if a vector–valued function is considered), then we obtain a representative of f which is defined everywhere and which is continuous exactly at the points of essential continuity of f.

Certainly, the results of Section 3 remain valid if we use coupled indices (k,p) instead of single ones. Consider a sequence $\{F_k\}$ of closed sets and denote by F their union. Find distinct points $z_{k,p} (k, p \in N)$ such that for every $k \in N$ the set

$$\{z_{k,p}; p \in N\}$$

is dense in F_k. Further, find compact sets $K_{k,p} (k, p \in N)$ such that each $K_{k,p}$ has Lebesque density at $z_{k,p}$ equal one and does not meet the set

$$\{z_{l,q}; l, q \in N, (l, q) \neq (k, p)\}.$$

For every fixed k we construct $u_{k,p}$, $a_{k,p}$, $b_{k,p}$ etc. in spirit of Section 5 in such a way that

(6.1) $\qquad \kappa_{k,p} = 2^{-k-1},$

(6.2) $\qquad |\sum_{q=1}^{p} u_{k,q}| < 2^{-k} \qquad$ for every $p, k \in N$,

(6.3) $\qquad |u_{k,p}| < 2^{-k-p}$ on $K_{l,q}$ whenever $k, l, p, q \in N, l + q < k + p$,

(6.4) $\qquad |u_{k,p}| < 2^{-k-p}$ outside $B_{2^{-p}}(z_{k,p})$

(it is guaranteed by choice $\delta_{k,p} < \operatorname{dist}(z_{k,p}, \bigcup_{l+q<k+p} K_{l,q}), \quad \delta_{k,p} < 2^{-p}$)

and

(6.5) $$\|u_{k,p}\|_{W^{1,2}(B_R(0))} + \|b_{k,p}\|_{L^2(B_R(0))} \leq \operatorname{const}(R) 2^{-k-p}.$$

Put

$$u = \sum_{k,p} u_{k,p}. \tag{6.6}$$

Theorem 4. *The solution of a system of the type (1.1) constructed in this section has the following properties:*

a) *u is bounded,*
b) *u is essentially discontinuous at all points of F,*
c) *u is essentially continuous at all points of $R^m \setminus F$.*

Proof: a) is obvious. b) Consider first a point $z_{k,p}$. By (6.3) the sum (6.6) converges uniformly on $K_{k,p}$. Further, all $u_{l,q}$ $(l,q) \neq (k,p)$, are continuous on $K_{k,p}$. The function $u_{k,p}$ behaves near $z_{k,p}$ like

$$2^{-k-1}\frac{x - z_{k,p}}{|x - z_{k,p}|}.$$

Hence $\operatorname*{osc\,ess}_{x \to z_{k,p}} u(x) \geq 2^{-k}$.

By obvious topological argument we see that $\operatorname*{osc\,ess}_{x \to z} u(x) \geq 2^{-k}$ for all $z \in F_k$.

c) Choose $z \in R^m \setminus F$ and $\varepsilon > 0$. Find $k, p \in N$ such that $2^{-p} + 2^{-k} < \varepsilon$ and $2^{-p+1} < \operatorname{dist}(z, F_1 \cup F_2 \cup \cdots \cup F_k)$. Then by (6.4)

$$|u_{l,q}| < 2^{-l-q} \qquad \text{on } B_{2^{-p}}(z) \text{ for each } l \in \{1, \ldots, k\} \text{ and } q > p$$

and by (6.2)

$$\Big|\sum_{q=1}^{\infty} u_{l,q}\Big| \leq 2^{-l} \qquad \text{on } R^m \text{ for every } l \in N.$$

Hence

$$\Big|\sum_{l>k \text{ or } q>p} u_{l,q}\Big| < 2^{-k} + 2^{-p} < \varepsilon \text{ on } B_{2^{-p}}(z).$$

Since the functions $u_{1,q}$ are continuous on a neighbourhood of z for $l \leq k$ and $q \leq p$, we deduce $\operatorname*{osc\,ess}_{x \to z} u(x) < 2\varepsilon$. ■

Remarks.

1) Given a closed set F, we can by similar method construct a solution u of a system of a type (1.1) which is locally unbounded precisely at the points of F.
2) In a subsequent paper we give an example of a system of 6 equations of a type (1.6) in the dimension $n = 3$ such that the set of discontinuities of a solution is not isolated.

7. Relation to Koshelev's condition number. In [13] A.I.Koshelev proved that if the eigenvalues of a symmetric matrix $A^{ij}_{\alpha\beta}$ are placed in the interval $\langle\lambda_0,\lambda_1\rangle$, where

$$\frac{\lambda_0}{\lambda_1} > K(m) = \frac{\sqrt{1+\frac{(m-2)^2}{(m-1)}}-1}{\sqrt{1+\frac{(m-2)^2}{(m-1)}}+1},$$

then all weak solutions of (1.1) are locally Hölder continuous in $\Omega \subset R^m$.

Observe that in our example

$$\frac{\lambda_0}{\lambda_1} = \frac{\frac{\mu}{\lambda}-\sqrt{(\frac{\mu}{\lambda})^2-1}}{\frac{\mu}{\lambda}+\sqrt{(\frac{\mu}{\lambda})^2-1}}$$

and it is a decreasing function of μ/λ. Thus using (3.7)

$$\frac{\lambda_0}{\lambda_1} > K(m) - \omega\tau$$

where ω is sufficiently large positive constant. For τ small enough, λ_0/λ_1 can be compressed arbitrarily near to $K(m)$. It means that if $\lambda_0/\lambda_1 < K(m)$ we cannot control the smallness of the singular set of a solution u by the distance of λ_0/λ_1 and $K(m)$ as such a solution can develop singularities on arbitrary F_σ–set in R^m.

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