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Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 4, 647--656

Persistent URL: http://dml.cz/dmlcz/106680

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
29,4 (1988)

ANOTHER LOOK AT THE LOCALIC TYCHONOFF THEOREM

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Dedicated to Professor M. Katětov on his seventieth birthday

Abstract: A new proof is presented of the familiar result of Johnstone [2] that the coproduct of compact frames is compact by describing the frame coproduct by means of the frame reflection of the corresponding preframe coproduct. Here, a preframe is a partially ordered set which has all finitary meets and updirected joins such that binary meet distributes over the latter, and the frame reflection of a preframe A is shown to be the frame of Scott closed subsets of A. The result of [2] is obtained as a consequence of a number of general facts concerning preframes and the manner in which the frame coproduct can be obtained from the preframe coproduct.

Key words: Frame, frame coproduct, preframe, preframe coproduct, frame reflection of preframes.

Classification: 54D30, 54H99

The usual proofs of the theorem that the coproduct of compact frames is compact employs a description of the frame coproduct as a certain quotient of the free frame on the meet-semilattice coproduct of the given frames and appear to be somewhat lengthy slogs without discernible intermediate stages of independent interest. It is the aim of this note to re-route the proof in a manner that exhibits distinctive steps of separate meaning on the way to the final result. In order to achieve this we introduce a new notion, that of preframe, which is a partially ordered set resembling a frame except that the finitary joins may be missing. Preframes have coproducts as well as reflections to frames, and both processes preserve compactness. The relevance of this for frames is that the frame coproduct can also be described in terms of these processes, and this turns out to work in such a way that its compactness, for given compact frames, becomes an immediate consequence of earlier results.

Although rearranged and with new emphases placed, all details of this proof have essentially appeared in previous arguments. Still, it is felt that the present presentation should help to make an undeniably complex subject matter somewhat more transparent.

This note was written during sabbatical leave spent at the University of Cape Town. Financial assistence from that institution as well as from the Natural Sciences and Engineering Research Council of Canada are gratefully acknowledged.

1. Background. We shall employ the usual lattice theoretic terminology and notation. For general concepts concerning frames we refer to Johnston [3].

Recall that a <u>nucleus</u> on a frame L is a closure operator k on L which preserves binary meet, and that the associated closure system $Fix(k)=\{x\in L \mid k(x)=x\}$ is a frame such that the map $k:L\longrightarrow Fix(k)$ is a frame homomorphism. In the following, we shall encounter nuclei generated by certain data, and as a useful intermediate step we introduce the following notion: a <u>prenucleus</u> on a frame L is a map $k_0:L\longrightarrow L$ such that, for all $x,y\in L$:

$$x\leq k_0(x),\quad \text{if } x\leq y \text{ then } k_0(x)\leq k_0(y),\quad k_0(x)\wedge y\leq k_0(x\wedge y).$$

It follows easily from the first two conditions that $K=Fix(k_0)$ is a closure system, and the associated closure operator is then given by

$$k(x)=\bigwedge t(x\leq t, t\in K).$$

Now we have

Lemma 1. <u>The closure operator</u> k <u>is a nucleus such that the frame homomorphism</u> $k:L\longrightarrow K$ <u>is universal among all frame homomorphisms</u> $h:L\longrightarrow M$ <u>for which</u> $h(x)=h(k_0(x))$ <u>for all</u> $x\in L$.

Proof. As a first step, we have to show that $k(x\wedge y)=k(x)\wedge k(y)$. For this, consider

$$E=\{a\in L \mid x\leq a\leq k(x), a\wedge y\leq k(x\wedge y)\},$$

for any $x,y\in L$. Then $x\in E$ trivially, $a\in E$ implies $k_0(a)\in E$ by the properties of k_0, and for any non-void $X\subseteq E$, $\bigvee X\in E$ by the distribution law of frames. Hence, in particular, $t=\bigvee E\in E$, that is, E has a largest element t. Then $k_0(t)\leq t$, hence $k_0(t)=t$, and since $x\leq t\leq k(x)$ we have $t=k(x)$. This says that $k(x)\wedge y\leq k(x\wedge y)$, and therefore

$$k(x)\wedge k(y)\leq k(x\wedge k(y))\leq k^2(x\wedge y)=k(x\wedge y)$$

as desired.

For the second part of the lemma, since $k(k_0(x))=k(x)$ for all $x\in L$ by the definition of k, we have to show that the given condition on $h:L\longrightarrow M$ implies that $h(k(x))=h(x)$ for all $x \in L$. To this end, let

$$F=\{c \in L \mid x \leq c \leq k(x), h(c)=h(x)\}.$$

Then $x \in F$ trivially, $c \in F$ implies $k_0(c) \in F$ by the hypothesis on h, and for any non-void $X \subseteq F$, $\bigvee X \in F$. As before, it follows that F has a largest element s, and then $k_0(s)=s$, implying that $s=k(x)$. Thus $h(k(x))=h(x)$, as claimed.

Remark. The above k can also be described as the stable transfinite iterate of k_0, and then the lemma can be proved by transfinite induction. Our approach avoids this use of the ordinals.

When considering quotients of a compact frame L, one is often interested in those which are again compact. Since quotients of frames are conveniently described by nuclei, it is useful to have conditions for nuclei that ensure the compactness of their associated quotients. We call a nucleus k on L codense if $k(x)=e$, the unit (=top) of L, implies $x=e$, and finitary if $k(\bigvee D)= \bigvee k[D]$ for any updirected $D \subseteq L$. Note that the latter condition is equivalent to the requirement that Fix(k) be closed under directed joins in L.

The desired result now is

Lemma 2. For any codense or finitary nucleus k on a compact frame L, the frame Fix(k) is again compact.

Proof. Recall that, for any $X \subseteq \text{Fix}(k)$, its join in Fix(k) is $k(\bigvee X)$ where $\bigvee$ means join in L. Now, if $D \subseteq \text{Fix}(k)$ is updirected then, for codense k, $e=k(\bigvee D)$ implies $e= \bigvee D$ and hence $e \in D$ by the compactness of L, showing that Fix(k) is compact. Similarly, for finitary k, $e=k(\bigvee D)$ implies $e= \bigvee D$ since always $k(\bigvee D)= \bigvee D$, and thus again the desired result $e \in D$.

2. Preframes. A preframe is a partially ordered set A in which all finitary meets and all updirected joins exist, and for any $x \in A$ and updirected $D \subseteq A$

$$x \wedge \bigvee D= \bigvee x \wedge t (t \in D).$$

Note that an updirected set is by definition non-void, and thus a preframe need not have smallest element although it does have a largest one, the meet of the empty set.

Examples of preframes abound: to begin with, every frame is a preframe, every filter in a preframe is a subpreframe, every algebraic lattice is a preframe and so is, more generally, every continuous lattice.

A preframe homomorphism is a map between preframes preserving all finitary meets and all updirected joins. The resulting category will be called

PFrm. The category **Frm** of all frames and their homomorphisms is then a (non-full) subcategory of **PFrm**.

For any preframe A, let $\mathfrak{D}A$ be the frame of all down-sets in A, that is, those $X \subseteq A$ such that $x \leq z$ and $z \in X$ implies $x \in X$. Also, recall that in any partially ordered set with updirected joins, a subset is called *Scott closed* if it is a downset closed under all updirected joins. Particular Scott closed sets are always the principal down-sets $\downarrow a$ consisting of all elements below or equal to a. For any preframe A, $\mathfrak{S}A$ will be the closure system of all Scott closed subsets of A. Then $\mathfrak{S}A \subseteq \mathfrak{D}A$, and σ will be the closure operator on $\mathfrak{D}A$ corresponding to $\mathfrak{S}A$. Also, $\downarrow : A \longrightarrow \mathfrak{S}A$ will be the map taking $a \in A$ to $\downarrow a$.

Now we have

Proposition 1. *For any preframe* A, σ *is a nucleus and hence* $\mathfrak{S}A$ *a frame, and the map* $\downarrow : A \longrightarrow \mathfrak{S}A$ *is the universal preframe homomorphism from* A *to frames*.

Proof. Define $\sigma_0 : \mathfrak{D}A \longrightarrow \mathfrak{D}A$ by $\sigma_0(x) = \{\bigvee D \mid D \subseteq X \text{ updirected}\}$. This is indeed again a down-set: if $a \leq \bigvee D$ for some updirected $D \subseteq C$ then $\{a \wedge t \mid t \in D\} \subseteq X$ and also updirected, with join $a \wedge \bigvee D = a$ by the properties of preframes, and therefore $a \in \sigma_0(X)$. Further, since any $\{x\}$ is updirected with join x, $X \subseteq \sigma_0(X)$, and $\sigma_0(X) = X$ iff $X \in \mathfrak{S}A$. We prove that σ_0 is a prenucleus which will imply that σ is a nucleus by Lemma 1. Of the remaining two conditions to be checked, the preservation of the partial order is obvious, and so we turn to the last one. If $a \in \sigma_0(X) \cap Y$ for some $X, Y \in \mathfrak{D}A$ then $a = \bigvee D$ with some updirected $D \subseteq X$ and $a \in Y$. Since $t \leq a$ for each $t \in D$ the latter implies $t \in Y$, hence $D \subseteq X \cap Y$ and therefore $a \in \sigma_0(X \cap Y)$.

Next we have to show that $\downarrow : A \longrightarrow \mathfrak{S}A$ is a preframe homomorphism. As to meet, $\downarrow e = A$ for the unit e of A so that $\downarrow$ preserves the units, and since $\downarrow(a \wedge b) = (\downarrow a) \cap (\downarrow b)$ for any $a, b \in A$, it also preserves binary meet. This shows all finitary meets are preserved. For join, if $D \subseteq A$ is any updirected set then

$$\bigvee_{t \in D} (\downarrow t) = \sigma\Big[\bigcup_{t \in D} (\downarrow t)\Big] \geq \downarrow(\bigvee D)$$

since $D \subseteq \bigcup (\downarrow t)(t \in D)$, and this implies equality, the reverse inequality being trivial because $t \leq \bigvee D$.

Finally, we have to verify the universality of $\downarrow : A \longrightarrow \mathfrak{S}A$. Let $f : A \longrightarrow L$ be any preframe homomorphism from A into a frame L. Then, as a familiar property of $\mathfrak{D}A$, there exists a unique frame homomorphism $g : \mathfrak{D}A \longrightarrow L$ such that $g \downarrow = f$; in fact, $g(X) = \bigvee f[X]$. In order to see that g factors through

$\sigma : \mathcal{P}A \longrightarrow \mathcal{U}A$, it suffices (Lemma 1) to show $g\sigma_0 = g$, that is

$$\bigvee f[\sigma_0(X)] = \bigvee f[X].$$

Now, for any updirected $D \subseteq X$, $f(\bigvee D) = \bigvee f[D]$, hence $f(\bigvee D) \leq \bigvee f[X]$, therefore $f[\sigma_0(X)]$ is bounded above by $\bigvee f[X]$, and consequently $\bigvee f[\sigma_0(x)] \leq \leq \bigvee f\{x\}$. This proves the desired identity because the reverse inequality is trivial.

For any family $(A_i)_{i\in I}$ of preframes, let $A \subseteq \prod A_i$ consist of all those $a=(a_i)_{i\in I}$ whose <u>support</u> $\mathrm{spt}(a) = \{i \in I \mid a_i < e_i\}$, e_i the unit of A_i, is finite. A is closed under finitary meet and directed join in $\prod A_i$ and hence a subpreframe of the latter. Also, the maps $k_i : A_i \longrightarrow A$ defined by

$$k_i(x)_j = \begin{cases} x & (j=i) \\ e_j & (j \neq i) \end{cases}$$

are preframe homomorphisms. Their significance lies in

Proposition 2. <u>A is the coproduct of $(A_i)_{i\in I}$ in</u> **PFrm**, <u>with coproduct maps</u> $k_i : A_i \longrightarrow A$.

Proof. Given any family $h_i : A_i \longrightarrow B$ $(i \in I)$ of preframe homomorphisms, we can define $h : A \longrightarrow B$ by

$$h(a) = \bigwedge_{i\in \mathrm{spt}(a)} h_i(a_i).$$

It is a familiar fact, but in any event easily checked, that h preserves all finitary meets and $hk_i = h_i$ for each $i \in I$. Further, if $D \subseteq A$ is updirected we may assume without loss of generality that it has a least element c, and for $E = \mathrm{spt}(c)$ we then have

$$t = \bigwedge_{i\in E} k_i(t_i)$$

for each $t \in D$ since $c \leq t$ and hence $\mathrm{spt}(t) \subseteq \mathrm{spt}(c)$. Therefore

$$\bigvee_{t\in D} h(t) = \bigvee_{t\in D} \bigwedge_{i\in E} h_i(t_i) = \bigwedge_{i\in E} \bigvee_{t\in D} h_i(t_i)$$

$$= \bigwedge_{i\in E} h_i(\bigvee_{t\in D} t_i) = \bigwedge_{i\in E} h_i((\bigvee D)_i) = h(\bigvee D),$$

where the second equality holds since D is updirected, the third because $\{t_i \mid t \in D\}$ is updirected and the fourth by the fact that joins in A, as in $\prod A_i$, are componentwise.

It follows now that $h : A \longrightarrow B$ is a preframe homomorphism, such that $h_i = hk_i$ for each $i \in I$, and since the k_i are preframe homomorphisms as already noted, this proves the assertion.

Now we have the following compactness result:

Proposition 3. Coproduct and frame reflection of preframes preserve compactness.

Proof. Take the coproduct as given in Proposition 2, assuming each A_i compact, that is, its unit e_i is compact - meaning it can only be a directed join for trivial reason. Now, let $D \subseteq A$ be updirected, with least element c, and $E=\mathrm{spt}(c)$. Then $e=\bigvee D$ implies $e_i=\bigvee t_i (t \in D)$ for each $i \in E$, hence there exist $t^{(i)} \in D$ such that $e_i=t_i^{(i)}$, and since E is finite we have some $t \geq t^{(i)}$ $(i \in E)$ in D. Then $t=e$ because $t_i=e_i$ for $i \in E$ by the choice of t and $t_j=e_j$ anyway for $j \notin E$. Hence the coproduct A is compact.

Concerning the compactness of $\mathfrak{F}A$ for compact A, we have that $\mathcal{C}^{\vartheta}A$ is obviously compact so that it will be enough (Lemma 1) to show that the nucleus σ on $\mathcal{C}^{\vartheta}A$ is codense. For this let

$$X_* = \{a \in A \mid \text{ if } c \leq a \text{ is compact then } c \in X\}$$

for each $X \in \mathcal{C}^{\vartheta}A$. We claim this is Scott closed: clearly, it is a downset, and if c is compact and $c \leq \bigvee D$ for some updirected $D \subseteq X_*$ then $c \leq t$ for some $t \in D$ by compactness, and thus $c \in X$ since $t \in X_*$. Also, $X \subseteq X_*$ and hence $\sigma(X) \subseteq X_*$. Therefore, $\sigma(X)=A$, which means $e \in \sigma(X)$, implies $e \in X_*$, and if e is compact this in turn implies $e \in X$, that is, $X=A$.

3. The frame coproduct. It is clear that any reasonable category containing Frm as a reflective subcategory can be used to describe the coproducts in **Frm** as a quotient of the reflection into **Frm** of the coproduct first formed in the larger category; moreover, it is also quite obvious that **PFrm** is a category of this type.

In order to be able to analyze the last step in this procedure we give the following detailed description:

For any family $(L_i)_{i \in I}$ of frames, let $A \subseteq \prod L_i$ be their preframe coproduct as described above and $k_i : L_i \to A$ the preframe coproduct homomorphism. Then, let $\mathcal{L} \subseteq \mathfrak{F}A$ be the closure system defined by the following condition on $X \in \mathfrak{F}A$:

(∗) For all $a \in A$, $i \in I$ and finite $Z \subseteq L_i$, if $a \wedge k_i(t) \in X$ for each $t \in Z$ then $a \wedge k_i(\bigvee Z) \in X$.

Further, define $\lambda_0 : \mathfrak{F}A \to \mathfrak{F}A$ by

$$\lambda_0(X) = \sigma(\{a \wedge k_i(\bigvee Z) \mid a \in A,\ i \in I,\ \text{finite } Z \subseteq L_i,\ a \wedge k_i(t) \in X \text{ for all } t \in Z\}).$$

One easily checks that the indicated set is a down-set in A so that this definition makes sense. Also, $X \subseteq \lambda_0(X)$ since we may take $a \in X$, $i \in I$ arbitrary and $Z = \{e_i\}$.

Clearly, $\mathcal{L} = \mathrm{Fix}(\lambda_0)$. Hence the closure operator λ associated with $\mathcal{L}$ will be a nucleus and $\mathcal{L}$ itself a frame if we show that λ_0 is a prenucleus. Obviously, λ_0 preserves inclusion, so that we only have to check the more subtle third condition. For any $X,Y \in \mathfrak{D}A$, let S be the downset given above such that $\lambda_0(X) = \mathfrak{S}(S)$, and note first that

$$\lambda_0(X) \cap Y = \mathfrak{S}(S) \cap Y = \mathfrak{S}(S) \cap \mathfrak{S}(Y) = \mathfrak{S}(S \cap Y).$$

Now, for any $a \wedge k_i(\bigvee Z) \in S \cap Y$ of the kind involved, $a \wedge k_i(t) \in Y$ for each $t \in Z$ since $a \wedge k_i(t) \leq a \wedge k_i(\bigvee Z)$, hence this element belongs to $X \cap Y$ so that $a \wedge k_i(\bigvee Z) \in \lambda_0(X \cap Y)$. This shows $S \cap Y \subseteq \lambda_0(X \cap Y)$ which implies $\lambda_0(X) \cap Y \subseteq$ $\subseteq \lambda_0(X \cap Y)$ by our first observation.

It follows that we have a frame homomorphism $\lambda : \mathfrak{D}A \longrightarrow \mathcal{L}$, and this is indeed the required quotient map by

Proposition 4. $\mathcal{L}$ <u>is the coproduct of</u> $(L_i)_{i \in I}$ <u>in</u> Frm, <u>with coproduct maps</u> $\lambda {\downarrow} k_i : L_i \to A \to \mathfrak{D}A \to \mathcal{L}$.

Proof. To see that these maps are frame homomorphisms we only have to check that they preserve finitary joins since they clearly preserve finitary meets and updirected joins. Now, for any finite $Z \subseteq L_i$,

$$\bigvee_{t \in Z} \lambda \downarrow k_i(t) = \lambda \Big[\bigvee_{t \in Z} \downarrow k_i(t) \Big] = \lambda \mathfrak{S} \Big[\bigcup_{t \in Z} \downarrow k_i(t) \Big] \supseteq \lambda \downarrow k_i(\bigvee Z),$$

the second equality by the definition of join in $\mathfrak{D}A$. This proves that

$$\bigvee_{t \in Z} \lambda \downarrow k_i(t) = \lambda \downarrow k_i(\bigvee Z)$$

since the reverse inclusion is obvious.

To obtain the universality of these frame homomorphisms, consider any family $h_i : L_i \downarrow M$ $(i \in I)$ of frame homomorphisms. Then we have the following diagram

$$\begin{array}{ccccccc} L_i & \xrightarrow{k_i} & A & \xrightarrow{\downarrow} & \mathfrak{D}A & \xrightarrow{\lambda} & \mathcal{L} \\ {\scriptstyle h_i}\downarrow & \swarrow{\scriptstyle \bar{h}} & & & \tilde{h} & & \\ M & & & & & & \end{array}$$

where $\bar{h} : A \longrightarrow M$ is the unique preframe homomorphism, provided by Proposition 2, such that $\bar{h}k_i = h_i$ for all $i \in I$, and $\tilde{h}$ the unique frame homomorphism such that $\tilde{h}\downarrow=\bar{h}$ given by Proposition 1. The question now is to see that $\tilde{h}$ factors through λ : that will establish the desired universality. For this, it will be sufficient (Lemma 1) to show that $\tilde{h}(X)=\tilde{h}(\lambda_0(X))$ for each $X\in\mathfrak{D}A$, the non-trivial part of which is that

$$\tilde{h}(\lambda_0(X))\leq\tilde{h}(X).$$

Since $\tilde{h}$ is already defined on all of $\mathfrak{D}A$, factoring through $\mathfrak{D}A$ in virtue of the condition $\tilde{h}\sigma=\bar{h}$ on $\wp A$, we have

$$\tilde{h}(\lambda_0(X))=\tilde{h}(S)=\bigvee\bar{h}[S]$$

for the downset S introduced earlier. Thus we have to prove that $\tilde{h}(X)$ is an upper bound of $\bar{h}[S]$. Consider then, any $a\wedge k_i(\bigvee Z)\in S$ where $a\in A$, $i\in I$, $Z\subseteq L_i$ and finite, such that $a\wedge k_i(t)\in X$ for each $t\in Z$. Then

$$\bar{h}(a\wedge k_i(\bigvee Z)=\bar{h}(a)\wedge h_i(\bigvee Z)=\bigvee_{t\in Z}\bar{h}(a)\wedge h_i(t)$$

$$=\bigvee_{t\in Z}\bar{h}(a\wedge k_i(t)\leq\bigvee\bar{h}[X]=\tilde{h}(X),$$

which establishes the desired result. In all, this shows that the $\lambda\downarrow k_i:L_i\longrightarrow\mathcal{L}$ $(i\in I)$ are indeed the coproduct maps.

Next, we establish a general property of the nucleus λ :

Lemma 3. <u>For any family $(L_i)_{i\in I}$ of frames, the nucleus λ is finitary.</u>

Proof. It has to be shown that $\mathcal{L}$ is closed under directed joins in $\mathfrak{D}A$. Let, then, $\mathcal{Y}\subseteq\mathcal{L}$ be updirected. Its join in $\mathfrak{D}A$ is $\sigma(H)$ for $H=\bigcup\mathcal{Y}$. and by the finitary nature of the condition $(*)$ defining $\mathcal{L}$, H is a downset in A satisfying $(*)$. We have to derive from this that $\sigma(H)$ also satisfies $(*)$. For this, consider the set $\mathfrak{A}$ of all $X\in\wp A$ such that $H\subseteq X\subseteq\sigma(H)$ and $(*)$ holds for X. Then we have that $H\in\mathfrak{A}$, and that the union of any chain in $\mathfrak{A}$ belongs to $\mathfrak{A}$, again by the finitary nature of $(*)$. We claim that, further, $\sigma_0(X)\in\mathfrak{A}$ whenever $X\in\mathfrak{A}$. It is sufficient to verify $(*)$ for the special cases $Z=\emptyset$, and $Z=\{u,v\}$. For $Z=\emptyset$, the condition is clearly satisfied by $\sigma_0(X)$ since $X\subseteq\sigma_0(X)$ satisfies it. Now, take any $a\wedge k_i(u)=\bigvee D$ and $a\wedge k_i(v)=\bigvee E$ for updirected $D,E\subseteq X$. For any $x\in A$, put

$$\bar{x}=\bigwedge_{j\neq i}k_j(x_j).$$

Then $x=\bar{x}\wedge k_i(x_i)$, and for any $t\in D$

$$a\wedge\bar{t}\wedge k_i(t_i\wedge u)=a\wedge t\wedge k_i(u)\in X,$$

and similarly $a\wedge\bar{s}\wedge k_i(s_i\wedge v)\in X$ for each $s\in F$. It follows that

$$a\wedge\bar{t}\wedge\bar{s}\wedge k_i(t_i\wedge u),\ a\wedge\bar{t}\wedge\bar{s}\wedge k_i(s_i\wedge v)\in X$$

and from (*) we obtain that

$$a\wedge\bar{t}\wedge\bar{s}\wedge k_i((t_i\wedge u)\vee(s_i\wedge v))\in X,$$

for all $t\in D$ and $s\in E$. Now, this is an updirected set since D and E are, and hence its join belongs to $\sigma_0(X)$. Further, because directed joins in A are taken componentwise and directed joins in L_i are preserved by k_i, this join is

$$a\wedge\overline{\bigvee D}\wedge\overline{\bigvee E}\wedge k_i\left[((\bigvee D)_i\wedge u)\vee((\bigvee E)_i\wedge v)\right]$$

and comparing the components for i and each $j\neq i$ one sees this is $a\wedge k_i(u\vee v)$. It follows that $\sigma_0(X)\in\mathfrak{A}$.

In all this establishes that $\mathfrak{A}$, partially ordered by inclusion, has the property that every chain in $\mathfrak{A}$ has a join and $\mathfrak{A}$ is mapped into itself by σ_0 such that $X\subseteq\sigma_0(X)$ for each $X\in\mathfrak{A}$. Hence σ_0 has a fixpoint in $\mathfrak{A}$ by Bourbaki's Fixpoint Lemma [1]. But the only fixpoint of σ_0 between H and $\sigma(H)$ is $\sigma(H)$, thus $\sigma(H)\in\mathfrak{A}$, and therefore $\sigma(H)$ satisfies (*).

Now we are in the position to give the promised proof of

Proposition 5. The coproduct of any family of compact frames in compact.

Proof. For any family of compact frames, the preframe coproduct A and its frame reflection $\mathfrak{F}A$ are compact by Proposition 3, and hence the frame coproduct, which is $\mathfrak{L}$ by Proposition 4, is compact by Lemmas 3 and 2.

Remark. Although the above arguments are not constructive, the only non-constructive step required is the disjunction x=y or $x\neq y$ for all elements x, y of some of the sets arising. This clearly enters into the treatment of the preframe coproduct, but it is also used in the proof of Bourbaki's Fixpoint Lemma. Hence, what is presented here is valid in any Boolean topos. It may well be, though, that a different treatment of the preframe coproduct and some alternative proof of Lemma 3 can be given which are entirely constructive, like the proof of the localic Tychonoff Theorem as a whole in Vermeulen [5]. We did not pursue this question at this point.

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(Oblatum 23.6. 1988)