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QUASICONTINUITY AND SOME CLASSES OF BAIRE 1 FUNCTIONS

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Abstract: The paper deals with quasicontinuity of real-valued functions of a real variable. It is shown, e.g., that a Lebesgue measurable function has the set of all its quasicontinuity points Lebesgue measurable. Families of bounded Baire class one functions which are not quasicontinuous almost everywhere are investigated. Such functions are shown to be "typical" in the sense of category in some standard function spaces.

Key words: Quasicontinuity at a point, approximately continuous functions, Baire class one functions, Baire category, Zahorski classes of functions, symmetric derivative.

Classification: 26A15, 26A21

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1. Introduction. Quasicontinuity is a generalization of the notion of continuity. It has been introduced in [Ke] and its basic properties are known (see e.g. [Bl], [LŠ], [Ma], [Th]). The aim of the present paper is to show that the set of all quasicontinuity points of a Lebesgue measurable function is Lebesgue measurable and that there are classes of Baire 1 functions in which a typical function is not quasicontinuous almost everywhere. This last statement improves some of known results (see [BP], [KŠ]).

2. Points of quasicontinuity. Further we shall deal with real-valued functions defined on a real non-degenerate interval I_0. Recall in this case the notion of quasicontinuity of a function at a point.

Definition 1. A function $f:I_0\to R$ (R - the real line) is said to be quasicontinuous at the point $x\in I_0$ if for each $\varepsilon>0$ and each $\delta>0$ there exists a non-void open interval $I\subset(x-\delta, x+\delta)$ such that $|f(t)-f(x)|<\varepsilon$ holds for every $t\in I$. We denote by $Q(f)$ ($R(f)$) the set of all such points of I_0 at which the function f is (is not) quasicontinuous.

Let $f:I_0\to R$ be a function. Put $d_I(f,x)=\sup_{t\in I}\{|f(t)-f(x)|\}$, where $I\subset I_0$ is a non-void open interval and $i_\delta(f,x)=\inf_{I\subset(x-\delta,x+\delta)}\{d_I(f,x)\}$ for $\delta>0$. Obviously $i_\delta(f,x)\ge i_\eta(f,x)$ whenever $\delta<\eta$ and we can define for each $x\in I_0$

$$q_f(x)=\lim_{\delta\to0+} i_\delta(f,x)=\sup_{\delta>0}\{i_\delta(f,x)\}.$$

Theorem 1. (a) A function $f:I_0\to R$ is quasicontinuous at the point x if and only if $q_f(x)=0$.

(b) If $f_n\to f$ uniformly, then also $q_{f_n}\to q_f$ uniformly.

(c) If $f_n\to f$ uniformly, then $\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}Q(f_n)\subset Q(f)$.

Proof. (a) The statement is an immediate consequence of the definition of quasicontinuity.

(b) Let $\varepsilon>0$ and let $|f_n(t)-f(t)|<\varepsilon$ hold for each $t\in I_0$ and $n\ge n_0$. Choose $x\in I_0$. Then $|f_n(t)-f_n(x)|\le|f_n(t)-f(t)|+|f(t)-f(x)|+|f(x)-f_n(x)|<|f(t)-f(x)|+2\varepsilon$. If $I\subset I_0$, then $d_I(f_n,x)\le d_I(f,x)+2\varepsilon$. Analogously $d_I(f,x)\le d_I(f_n,x)+2\varepsilon$. From above inequalities it follows for any $\delta>0$ that $i_\delta(f_n,x)\le i_\delta(f,x)+2\varepsilon$ and $i_\delta(f,x)\le i_\delta(f_n,x)+2\varepsilon$. Hence $q_{f_n}(x)\le q_f(x)+2\varepsilon$, $q_f(x)\le q_{f_n}(x)+2\varepsilon$ and $|q_{f_n}(x)-q_f(x)|\le2\varepsilon$.

(c) Put $Q=\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}Q(f_n)$. If $f_n\to f$ uniformly, then for each $x\in Q$ we have $x\in Q(f_n)$ for infinitely many indices n, and (b) implies $Q \subset Q(f)$.

The set $Q(f)$ of a function $f: I_0 \to R$, in general, need not be Lebesgue measurable. E. g., if $D \subset I_0$ is a nowhere dense Lebesgue non-measurable set, then $\chi_D = q_{\chi_D}$ holds for its characteristic function χ_D, i.e. $Q(\chi_D)$ is Lebesgue non-measurable.

Theorem 2. Let $f: I_0 \to R$ be a Lebesgue measurable function. Then $q_f: I_0 \to R \cup \{+\infty\}$ is Lebesgue measurable.

Corollary 1. The set of quasicontinuity points of a Lebesgue measurable function is a Lebesgue measurable set.

First we shall introduce a class of functions and prove Theorem 2 for this class.

Definition 2. A function $g: I_0 \to R$ is said to be simple, if

(i) g is Lebesgue measurable;

(ii) there exists $\eta > 0$ such that $|g(y) - g(x)| < \eta$ implies $g(y) = g(x)$.

Lemma 1. If $g: I_0 \to R$ is a simple function, then q_g is Lebesgue measurable.

Proof. If I is a non-void open subinterval of I_0, then we can define a function $d_I: I_0 \to R \cup \{+\infty\}$ by $d_I(x) = d_I(g,x)$. For $a \in R$ put $E_I^a = \{x: d_I(x) < a\}$. It follows from the property (ii) of the simple function g that the set $g(I_0)$ is countable and the same holds for the set $d_I(I_0)$. The function d_I is obviously constant on each level set of the function g and each level set of the function d_I is a countable union of level sets of the function g. Consequently level sets of d_I are Lebesgue measurable. Let $\{y_n\}$ be a sequence of all values of the function d_I such that $y_n < a$. Then $E_I^a = \bigcup_n d_I^{-1}(y_n)$ is Lebesgue measurable. Choose $\delta > 0$ and put $i_\delta(x) = i_\delta(f,x)$ and $E_\delta^a = \{x: i_\delta(x) < a\}$ for $a \in R$. Then $E_\delta^a = \bigcup_{r,s}\{x: d_{(r,s)}(x) < a$, r and s are rational numbers, $x - \delta < r < s < x + \delta\}$ and hence E_δ^a is Lebesgue measurable. Consequently, for each $\delta > 0$ the function $i_\delta : I_0 \to R \cup \{+\infty\}$ is Lebesgue measurable and the same holds for the function $q_g = \lim_{m\to\infty} i_{1/m}$.

Proof of Theorem 2. Let $f: I_0 \to R$ be Lebesgue measurable. For each $n = 1, 2, \ldots$ we can define the function $f_n : I_0 \to R$ by the prescription: $f_n(x) = m2^{-n}$ for $x \in f^{-1}([m2^{-n}, (m+1)2^{-n}))$, m - an integer number. Obviously every term of the sequence $\{f_n\}_1^\infty$ is a simple function and $f_n \to f$ uniformly. According to Theorem 1(b) and Lemma 1 $q_{f_n} \to q_f$ uniformly and q_f is Lebesgue measurable.

3. <u>Typical results</u>. In the next text we shall deal with classes of real functions defined on the unit real interval $[0,1]$. We denote by $b\mathcal{A}$ ($b\Delta$, $b\mathcal{B}_1$) the class of bounded approximately continuous functions (bounded derivatives, bounded Baire 1 functions). All these classes are complete metric spaces with the metric $d(f,g) = \sup_{x\in[0,1]} \{|f(x) - g(x)|\}$. There are known some properties which hold for most of the functions of these classes in the sense of the Baire category (see e.g. [BP], [CP], [CP1], [CP2], [EH], [KŠ], [Mu], [Ri], [Ri2]).

Further we shall use the following known notion.

<u>Definition 3</u>. ([Br], p. 14) Let $\mathcal{F}$ be a family of functions defined on an interval I. A subfamily $\mathcal{X}$ of $\mathcal{F}$ is called the maximal additive family for $\mathcal{F}$ provided $\mathcal{X}$ is the set of all functions g in $\mathcal{F}$ such that $f + g \in \mathcal{F}$ whenever $f \in \mathcal{F}$.

Definition 4. A family $\mathcal{F}$ of functions $f: [0,1] \to R$ is said to be acceptable, if

(i) $b\mathcal{A} \subset \mathcal{F} \subset b\mathcal{B}_1$;

(ii) $\mathcal{F}$ is uniformly closed;

(iii) $b\mathcal{A} \subset \mathcal{H}$ holds for the maximal additive family $\mathcal{H}$ of $\mathcal{F}$.

We shall also use the following known results.

Theorem Z. ([Za]; [Br], p. 28) Let E be an F_σ set such that the density of E at x $d(E,x) = 1$ for all $x \in E$. Then there exists an approximately continuous function f such that $0 < f(x) \le 1$ for all $x \in E$ and $f(x) = 0$ for all $x \notin E$. The function f is also upper semi-continuous.

Theorem CP. ([CP], Th. 9) Let $f \in b\mathcal{B}_1$ and $\varepsilon > 0$. Then there exists $g \in b\mathcal{A}$ with $\|g\| = \sup_{x \in [0,1]} \{|g(x)|\} < \varepsilon$ such that $h(C(h))$ ($C(h)$ - the set of all continuity points of h) is finite where $h = f - g$.

In what follows λ stands for the Lebesgue measure on $[0,1]$.

Theorem 3. Let $\mathcal{F}$ be an acceptable family of functions endowed with the metric $d(f,g) = \|f - g\| = \sup_{x \in [0,1]} \{|f(x) - g(x)|\}$. Then the family

$$\mathcal{F}^* = \{f \in \mathcal{F} : \lambda(Q(f)) = 0\}$$

is a residual G_δ set in $\mathcal{F}$.

In the proof of Theorem 3 the following preliminary results will be used.

Lemma 2. The class $\mathcal{D}$ of all $h \in \mathcal{F}$ such that $h(C(h))$ is finite is dense in $\mathcal{F}$.

Proof. Let $f \in \mathcal{F}$ and $\varepsilon > 0$. Applying Theorem CP there exists $g \in b\mathcal{A}$ such that $\|g\| < \varepsilon$ and $h(C(h))$ is finite where $h = f - g$. Then $h \in \mathcal{D}$ and $\|f - h\| < \varepsilon$.

Lemma 3. The set $\mathcal{F}^*$ is dense in $\mathcal{F}$.

Proof. It is sufficient to show that for each $f \in \mathcal{D}$ and $\varepsilon > 0$ there exists $h \in \mathcal{F}^*$ such that $\|f - h\| < \varepsilon$. Since $f \in b\mathcal{B}_1$ there exists a countable set D dense in $C(f)$ and also dense in $[0,1]$. Let $f(C(f)) = \{c_1, \dots, c_n\}$ and $S = f^{-1}(f(C(f))) - D$. Obviously S is Lebesgue measurable.

If $\lambda(S) = 0$, then $f(x) \notin f(C(f))$ holds for almost all $x \in [0,1]$ and for such an x we have $d_I(f,x) \geq \min_{j=1,\dots,n} \{|f(x) - c_j|\} > 0$ for each nonvoid open interval I. Consequently $q_f(x) > 0$ and $f \in \mathcal{F}^*$.

Suppose $\lambda(S) > 0$. According to the Lebesgue Density Theorem ([Br], p. 18) $d(S,x) = 1$ holds for almost every $x \in S$. Put $T = \{x \in S: d(S,x) = 1\}$. The set T is of the form $E \cup N$, where E is an F_σ set, $\lambda(N) = 0$ and $d(E,x) = 1$ for each $x \in E$. Let g be a function, the existence of which is quaranteed by Theorem Z for the above set E. Choose $\eta > 0$ such that $\eta < \varepsilon$ and $\eta < \min_{1 \leq j < k \leq n} \{|c_j - c_k|\}$. We show that the function $h = f + \eta g$ has desired properties. Obviously $h \in \mathcal{F}$ and $\|f - h\| < \varepsilon$. Since $E \subset f^{-1}(f(C(f)))$ and $\lambda(E) = \lambda(f^{-1}(f(C(f))))$ we have $h(x) \notin f(C(f))$ for almost all $x \in [0,1]$. It is sufficient to show $\{x: h(x) \notin f(C(f))\} \subset R(h)$. If $h(x) \notin f(C(f))$ and $I \subset [0,1]$ is a non-void open interval, then $d_I(h,x) \geq \min_{j=1,\dots,n} \{|h(x) - c_j|\} > 0$. This follows from the fact that there is a point $y \in I \cap D$ and $h(y) = f(y) = c_k$ for some k, $1 \leq k \leq n$. Hence $q_h(x) > 0$ and $x \in R(h)$.

Lemma 4. Let $\delta > 0$. The family

$$\mathcal{F}_\delta = \{f \in \mathcal{F}: \ \lambda(Q(f)) \geq \delta\}$$

is uniformly closed.

Proof. Suppose that $f_n \in \mathcal{F}_\delta$, $f_n \to f$ uniformly. According to Theorem 1(c) we have $\lambda(Q(f)) \geq \lambda(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} Q(f_n)) \geq \delta$, hence $f \in \mathcal{F}_\delta$.

Proof of Theorem 3. Put $\delta(n) = n^{-1}$ ($n = 1, 2, \ldots$). According to Lemma 3 and Lemma 4 each of the sets $\mathcal{F}_{\delta(n)}$ is closed and nowhere dense in $\mathcal{F}$. Consequently $\mathcal{F}^* = \mathcal{F} - \bigcup_{n=1}^{\infty} \mathcal{F}_{\delta(n)}$ is a residual G_δ set in $\mathcal{F}$.

Let $b\Delta^s$ ($b\mathcal{M}_i$, $i = 1, \ldots, 5$) denote the class of bounded symmetric derivatives (bounded functions of the i^{th} Zahorski class (see [Za])) on $[0,1]$. Note that $b\mathcal{A} = b\mathcal{M}_5$ and that f is a symmetric derivative on $[0,1]$ if and only if there is a function $F:(a,b) \to R$, $[0,1] \subset (a,b)$, such that

$$f(x) = \lim_{h \to 0} (F(x+h) - F(x-h))/2h$$

holds for every $x \in [0,1]$.

Corollary 2. The family of all $b\Delta$ ($b\Delta^s$, $b\mathcal{B}_1$, $b\mathcal{M}_i$; $i = 4,5$) functions which are not quasicontinuous almost everywhere is a residual G_δ set in the space $b\Delta$ ($b\Delta^s$, $b\mathcal{B}_1$, $b\mathcal{M}_i$; $i = 4,5$) with the metric d.

Proof. It is sufficient to prove that each of the above classes is acceptable in the sense of Definition 4. The fact that the classes $b\Delta$ and $b\mathcal{M}_i$, $i = 4,5$, fulfil the condition (i) is well known. The inclusion $b\Delta^s \subset b\mathcal{B}_1$ is proved in [La]. Every of the Zahorski classes is uniformly closed. It is shown in [Ri]. The uniform closednees of $b\Delta^s$ is proved in [Ko]. The condition (iii) for classes $b\mathcal{M}_4$ and $b\mathcal{M}_5$ is proved in [Za], p. 45.

Remark. In [Za], p. 45, it is also proved that the condition (iii) of Definition 4 is not fulfilled for classes $b\mathcal{M}_i$, i = 1, 2, 3. It is an open problem wether, or not, Corollary 2 holds also for these classes.

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