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ON EQUIVALENCE RELATIONS ON A DIFFERENTIAL SPACE

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Abstract: In this paper some properties of equivalence relations on a differential space are studied. Some special examples of equivalence relations are described.

Key words: Differential space, Hausdorff equivalence relation, differential structure.

Classification: 58A40

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1. Introduction. In this paper we consider some properties of equivalence relations on a differential space. In the case of differentiable manifolds the well-known Godement theorem gives a necessary and sufficient condition for the quotient space to have the structure of a differentiable manifold. A generalization of this theorem to the category of differential spaces was given by W. Waliszewski [4]. R.S. Palais [1] gave a necessary and sufficient condition for the quotient space to admit a quotient modulo an equivalence relation in the category of ringed spaces.

In the category of differential spaces the quotient structure always exists. So we may consider the quotient differential structure in many different situations even if Godement's conditions are not fulfilled, e.g. in the theory of homogeneous spaces.

2. Main results. Let (M,C) be a differential space [2],[3]. For an arbitrary mapping $F:M \longrightarrow N$ from M into a set N let $F^{*}:R^N \longrightarrow R^M$ be the map given by the formula

$$F^{*}(\alpha)= \alpha \cdot F \text{ for } \alpha \in R^N.$$

The set $(F^{*})^{-1}(C)$ is a differential structure on N called the differential structure coinduced from C to N by the mapping F [5].

Now let ρ be an equivalence relation on (M,C). A function $f \in C$ is said to be consistent with ρ if $x \rho y$ implies $f(x)=f(y)$ for any $x,y \in M$. We denote by C_ρ the set of all $f \in C$ consistent with ρ . One can easily show that C_ρ is a differential structure on M, i.e. $C_\rho = (\mathrm{sc}\, C_\rho)_M$ [2],[5].

Let now M/ρ denote the set of all equivalence classes of ρ and let $\pi_\rho : M \to M/\rho$, $x \to [x]_\rho$ be the canonical map. We denote by $C/\rho :=$ $:= (\pi_\rho^*)^{-1}(C)$ the differential structure on M/ρ coinduced from C by the mapping π_ρ [4],[5]. Evidently, the diagram

$$\begin{tikzcd} (M,C) \arrow[rr, "\mathrm{id}_M"] \arrow[dr, "\pi_\rho"'] & & (M,C_\rho) \arrow[dl, "\pi_\rho"] \\ & (M/\rho, C/\rho) & \end{tikzcd}$$

is commutative.

It is easy to show that $\pi_\rho^* | (C/\rho) : C/\rho \to C_\rho$ is an isomorphism of algebras. It follows that $(M/\rho, C/\rho)$ has a constant differential dimension if and only if (M, C_ρ) has a constant differential dimension [2],[3].

An equivalence relation ρ on (M,C) is said to be a Hausdorff equivalence relation if for any $x,y \in M$ such that $(x,y) \notin \rho$ there exists a function $f \in C_\rho$ which separates x and y, i.e. $f(x) \neq f(y)$ (cf. [1]).

It is easy to prove that the following conditions are equivalent:

(a) ρ is a Hausdorff equivalence relation on (M,C).

(b) The topology $\tau_{C/\rho}$ in M/ρ is a Hausdorff topology.

(c) For arbitrary $x,y \in M$, if $f(x)=f(y)$ for any $f \in C_\rho$ then $x \rho y$.

Now let (M,C) be a differential space and $D \subset C$ an arbitrary subset of C. Using D we define an equivalence relation ρ_D in M by

(1) $\qquad x \rho_D y$ iff $f(x)=f(y)$ for every $f \in D$.

Obviously, ρ_D is a Hausdorff equivalence relation on (M,C).

Lemma 1. An equivalence relation on a differential space (M,C) is a Hausdorff equivalence relation if and only if there exists a subset $D \subset C$ such that $\rho = \rho_D$.

Proof. If ρ is a Hausdorff equivalence relation, then taking $D = C_\rho$ we obtain $\rho = \rho_D$.

Now we will show that every equivalence relation on a differential space may be extended to a Hausdorff equivalence relation.

Proposition 1. For any equivalence relation ρ on a differential space (M,C) there exists a unique Hausdorff equivalence relation $\rho_H \supset \rho$ such that $C_{\rho_H} = C_\rho$. Moreover, if ρ is a Hausdorff equivalence relation on (M,C), then

$\rho_H = \rho$.

Proof. Let ρ be an equivalence relation. We put $\rho_H = \rho_{C_\rho}$. By (1) we have

$$x \rho_H y \Longleftrightarrow f(x)=f(y) \text{ for every } f \in C_\rho .$$

It is easy to see that $\rho \subset \rho_H$ and $C_\rho \subset C_{\rho_H}$. Now we must show that $C_{\rho_H} \subset C_\rho$. Indeed, let $\beta \in C_{\rho_H}$. This means that $\beta \in C$ and

$$(\forall f \in C_\rho \; f(x)=f(y)) \Longrightarrow \beta(x)=\beta(y)$$

for $x,y \in M$. Assume that $x \rho y$. Then $f(x)=f(y)$ for any $f \in C_\rho$, and hence $\beta(x)= =\beta(y)$, which means that $\beta \in C_\rho$. This completes the proof.

Corollary 1. A differential space (M,C) is a Hausdorff space if and only if the trivial equivalence relation ρ_0 on (M,C) ($[x]_{\rho_0} = \{x\}$ for any $x \in M$) is a Hausdorff equivalence relation. Moreover, for an arbitrary differential space (M,C) the differential space $(M_H,C_H):=(M/\rho_C, C/\rho_C)$ is a Hausdorff space.

Let us observe that $C_{\rho_C}=C$ and hence C/ρ_C is isomorphic to C. Every smooth mapping $F:(M,C) \longrightarrow (N,D)$ determines the smooth mapping $F_H:(M_H,C_H) \longrightarrow (N_H,D_H)$ given by

$$F_H([x]_{\rho_C}) = [F(x)]_{\rho_D} \quad \text{for } x \in M.$$

The following diagram is commutative:

$$\begin{tikzcd} (M,C) \arrow[r, "F"] \arrow[d, "\pi_{\rho_C}"'] & (N,D) \arrow[d, "\pi_{\rho_D}"] \\ (M_H,C_H) \arrow[r, "F_H"] & (N_H,D_H) \end{tikzcd}$$

Let M be a non-empty set and ρ an equivalence relation on M.

Definition 1. A subset $A \subset M$ is called ρ-saturated if $\pi_\rho^{-1}(\pi_\rho(A))=A$.
It is easy to see that the following conditions are equivalent:

(a) $A \subset M$ is ρ-saturated

(b) $A = \pi_\rho^{-1}(B)$, where $B \subset M/\rho$.

(c) $A = \bigcup_{x \in A} [x]_\rho$.

(d) For any $x,y \in M$, if $x \in A$ and $x \rho y$, then $y \in A$.

The family of all ρ-saturated sets together with the empty set is a topology in M. It is easy to prove

Lemma 2. Let (M,C) be a differential space and ρ an arbitrary equivalence relation on M. Then an arbitrary set $U \in \tau_{C_\rho}$ is ρ-saturated.

Let C_0 be a set of generators of the differential structure C on M. It is known [3] that $\tau_{C_0} = \tau_{scC_0} = \tau_{(scC_0)_M} = \tau_C$.

We shall prove

Lemma 3. If C_0 is a set of generators of a differential structure C on M, then

(2) $$\rho_{C_0} = \rho_{scC_0} = \rho_{(scC_0)_M} = \rho_C,$$

where ρ_{C_0}, ρ_{scC_0}, $\rho_{(scC_0)_M}$, ρ_C are equivalence relations on (M,C), defined by (1).

Proof. Let $x,y \in M$ and $x \rho_C y$. Hence by (1), $f(x)=f(y)$ for any $f \in C_0$. Let $g \in scC_0$, By definition $g = \omega \circ (f_1^0,\ldots,f_n)$, where $\omega \in C^\infty(R^n)$ and $f_1,\ldots,f_n \in C_0$. Therefore $g(x) = \omega\circ(f_1,\ldots,f_n)(x) = \omega(f_1(x),\ldots,f_n(x)) = \omega(f_1(y),\ldots,f_n(y)) = g(y)$. Hence if $x \rho_{C_0} y$ then $g(x)=g(y)$ for any $g \in scC_0$, i.e. $x \rho_{scC_0} y$. Since the implication $x \rho_{scC_0} y \Rightarrow x \rho_{C_0} y$ is evident, we obtain $\rho_{C_0} = \rho_{scC_0}$. The other equalities may be proved analogously.

Now let (M,C) be a differential space and ρ an equivalence relation on M, Observe that the mapping $M/\rho \supset A \xrightarrow{I} \pi_\rho^{-1}(A) \subset M$ is a bijection between the family of ρ-saturated sets in M and the family of all subsets of M/ρ.

Denote by $\mathfrak{M}_\rho$ the family of all ρ-saturated open sets in τ_C:

$$\mathfrak{M}_\rho := \{U \in \tau_C : U = \pi_\rho^{-1}(\pi_\rho(U))\}.$$

It is easy to see that $\mathfrak{M}_\rho = I(\tau_C/\rho)$, where τ_C/ρ is the quotient topology in the set M/ρ and $\tau_{C_\rho} = I(\tau_{C/\rho})$, where $\tau_{C/\rho}$ is the weakest topology in the set M/ρ and $\tau_{C_\rho} = I(\tau_{C/\rho})$, where $\tau_{C/\rho}$ is the weakest topology on M/ρ such that all functions belonging to C/ρ are continuous.

It is easy to observe that the following diagram is commutative:

$$\begin{tikzcd}
\tau_C/\rho \arrow[r, "I"] & \mathfrak{M}_\rho \\
\tau_{C/\rho} \arrow[r, "I"] \arrow[u, hook] & \tau_{C_\rho} \arrow[u, hook]
\end{tikzcd}$$

From the above diagram we obtain

Corollary 2. $\tau_C/\rho = \tau_{C/\rho} \Longleftrightarrow \mathfrak{M}_\rho = \tau_{C_\rho}$.

Now we will prove

Lemma 4. For an arbitrary set $V \in \tau_{C_\rho}$,

$$(C_\rho)_V = (C_V)_{\rho|V},$$

where $\rho|V$ is the restriction of the relation ρ to the subset V.

Proof. Let $\alpha \in (C_\rho)_V$. Since $C_\rho \subset C$, we have in particular $(C_\rho)_V \subset C_V$ and thus $\alpha \in C_V$. We will prove that $\alpha \in (C_V)_{\rho|V}$. Let $x,y \in V$ with $x \rho y$. There exists an open neighbourhood $U \in \tau_{C_\rho}$ of x and a function $\beta \in C_\rho$ such that $\alpha|V\cap U = \beta|V\cap U$.

Of course $V\cap U \in \tau_{C_\rho}$ and $x,y \in V\cap U$ since $V\cap U$ is ρ-saturated. Hence $\alpha(x) = \beta(x) = \beta(y) = \alpha(y)$. Therefore

$$x(\rho|V)y \Longrightarrow \alpha(x) = \alpha(y)$$

or equivalently $\alpha \in (C_V)_{\rho|V}$.

We now prove the reverse inclusion. Let $\alpha \in (C_V)_{\rho|V}$. Let $p \in V$ be an arbitrary point and $\beta \in C_\rho$ a function separating the point p in the set $V \in \tau_{C_\rho}$, u.e. there exist $V_0, W_0 \in \tau_{C_\rho}$ such that $p \in V_0$, $\beta|V_0 = 1$, $\beta|W_0 = 0$ and $W_0 \cup V = M$.

Consider the function $\bar{\alpha}: M \longrightarrow R$ defined by

$$\bar{\alpha}(x) = \begin{cases} \alpha(x)\ \beta(x) & \text{for } x \in V \\ 0 & \text{for } x \notin V \end{cases}$$

Clearly, $\bar{\alpha} \in C$, since $\alpha\cdot\beta|W_0\cap V = 0|W_0\cap V$, $\{W_0, V\}$ is an open covering of M and $\alpha\cdot\beta|V \in C_V$ and $0 \in C_{W_0}$. We will now verify that $\bar{\alpha} \in C_\rho$. Indeed, let $x \rho y$. Then either $x,y \in W_0$ or $x,y \in V$.

If $x,y \in V$, then

$$\bar{\alpha}(x) = \alpha(x)\ \beta(x) = \alpha(y)\ \beta(y) = \bar{\alpha}(y)$$

and if $x,y \in W_0$, then

$$\bar{\alpha}(x) = 0 = \bar{\alpha}(y).$$

Thus for any $p \in V$ we have found a neighbourhood $V_0 \subset V$ of p, $V_0 \in \tau_{C_\rho}$, and a function $\bar{\alpha} \in C_\rho$ such that $\alpha|V_0 = \bar{\alpha}|V_0$. This means that $\alpha \in (C_\rho)_V$, which completes the proof of the lemma.

Now let (M,C) be a differential space and $D \subset C$ an arbitrary subset. The sets D, scD, $(scD)_M$ and C_{ρ_D} are sets of functions consistent with the equivalence relation ρ_D defined by (1). We have the inclusions

$$D \subset scD \subset (scD)_M \subset C_{\rho_D}.$$

Moreover, $(scD)_M$ and C_{ρ_D} are differential substructures of the differential structure C. Evidently, by definition, the differential structure C_{ρ_D} is the maximal set of functions consistent with the equivalence relation ρ_D. We now show that in general $(scD)_M \neq C_{\rho_D}$.

Example 1. Consider the differential space $(R, C^{\infty}(R))$. Of course the function $f: R \to R$ given by the formula $f(x)=x^4$ belongs to $C^{\infty}(R)$. This function defines by (1) the Hausdorff equivalence relation $\rho_{\{f\}}$ on R. It is easy to observe that the function given by $g(x)=x^2$ is consistent with $\rho_{\{f\}}$ and $g \notin (sc\{f\})_R$. Therefore $C_{\rho_{\{f\}}} \neq (sc\{f\})_R$.

The above considerations suggest the following definition.

Definition 2. Let (M,C) be a differential space. A differential substructure D of the differential structure C is said to be saturated if $D=C_{\rho_D}$.

Evidently, the differential structure C itself is saturated because $C=C_{\rho_C}$.

We will prove

Proposition 2. The mapping $\rho \longmapsto C_{\rho}$ defines a one-to-one correspondence between the set of Hausdorff equivalence relations on a differential space (M,C) and the family of saturated differential substructures of the differential structure C.

Proof. Let ρ_1 and ρ_2 be Hausdorff equivalence relations on (M,C) such that $C_{\rho_1}=C_{\rho_2}$. We shall prove that $\rho_1=\rho_2$. Indeed,

$$x\rho_1 y \Longleftrightarrow (\forall f \in C_{\rho_1}\ f(x)=f(y)) \Longleftrightarrow (\forall f \in C_{\rho_2}\ f(x)=f(y)) \Longleftrightarrow x\rho_2 y.$$

On the other hand, if $D \subset C$ is a saturated differential substructure, then by definition $F=C_{\rho_D}$, where ρ_D is the Hausdorff equivalence relation on (M,C) defined by (1).

Now we may prove

Proposition 3. Let ρ be a Hausdorff equivalence relation on a differential space (M,C). Then the quotient differential structure C/ρ on M/ρ is generated by a set $\overline{D}$ of real functions if and only if the differential structure C_ρ is generated by the set $D=\pi_\rho^*(\overline{D})$. Moreover, if C/ρ is generated by $\overline{D}$ then $\rho=\rho_D$.

Proof. Since $\pi_\rho:(M,C)\longrightarrow(M/\rho,C/\rho)$ is a smooth mapping such that $\pi_\rho^*\,|(C/\rho):C/\rho\longrightarrow C_\rho$ is an isomorphism of linear rings, $\overline{D}$ is a set of generators of the differential structure C/ρ iff $D=\pi_\rho^*(\overline{D})$ is a set of generators of the differential structure C_ρ.

Let $\overline{D}$ be a set of generators of C/ρ. Then D is a set of generators of C_ρ, i.e. $C_\rho=(scD)_M$. From Proposition 2 it follows that $\rho=\rho_{C_\rho}$. Lemma 3 gives $\rho_{C_\rho}=\rho_{(scD)_M}=\rho_D$. Hence $\rho=\rho_D$.

Corollary 3. The quotient differential space $(M/\rho,C/\rho)$ is a Hausdorff differential space generated by a set of n real valued functions if and only if there exists a smooth mapping $F:(M,C)\longrightarrow(R^n,\mathcal{E}_n)$ such that $C_\rho=(scF_0)_M$, where F_0 is the set of coordinate functions of the mapping F.

Now consider a smooth mapping $f:(M,C)\longrightarrow(N,D)$ between differential spaces. The mapping f determines the equivalence relation ρ_f in M defined by

$$x\rho_f y \Longleftrightarrow f(x)=f(y), \text{ for } x,y\in M.$$

The map $g:M/\rho_f\longrightarrow N$ defined by

$$g([x]\rho_f)=f(x) \text{ for } x\in M \tag{3}$$

is a bijection onto $f(M)$.

It is easy to observe that the following diagram is commutative:

$$\begin{tikzcd}
(M,C) \arrow[r, "id_M"] \arrow[dr, "\pi_\rho"'] & (M,C_\rho) \arrow[r, "f"] \arrow[d, "\pi_\rho"] & (N,D) \\
& (M/\rho,C/\rho) \arrow[ur, "g"'] &
\end{tikzcd}$$

We will prove

Lemma 5. Let $f:(M,C)\longrightarrow(N,D)$ be a smooth surjection. Then the homomorphism $f^*\,|D:D\longrightarrow C_{\rho_f}$ is an isomorphism of linear rings if and only if $(f^*)^{-1}(C)=D$.

Proof. $\Leftarrow$ Assume that $(f^*)^{-1}(C)=D$. Since f is a surjection, f^* is an injection. We now show that $f^*|D:\longrightarrow C_{\wp_f}$ is "onto". Let $\gamma\in C_{\wp_f}$. Thus $\gamma\in C$ and for any $x,y\in M$, if $f(x)=f(y)$ then $\gamma(x)=\gamma(y)$.

Consider the function $\overline{\gamma}:N\longrightarrow R$ defined by $\overline{\gamma}(q)=\gamma(x)$ for $q\in N$, where $x\in M$ is any element such that $q=f(x)$. Since $\gamma\in C_{\wp_f}$, the definition of $\overline{\gamma}$ is correct. Moreover, $\overline{\gamma}\circ f=\gamma$. Hence $\overline{\gamma}\in(f^*)^{-1}(C)$ because $\gamma\in C$. Therefore $\overline{\gamma}\in D$. We have shown that $f^*|D:D\longrightarrow C_{\wp_f}$ is an isomorphism of linear rings.

The implication $\Longrightarrow$ is obvious.

Corollary 4. Let $f:(M,C)\longrightarrow(N,D)$ be a smooth mapping of a differential space (M,C) onto a differential space (N,D).

Then the smooth bijection $g:(M/\wp_f,C/\wp_f)\longrightarrow(N,D)$ defined by (3) is a diffeomorphism if and only if $(f^*)^{-1}(C)=D$.

Now we will give a sufficient condition for a mapping $f:(M,C)\longrightarrow(N,D)$ to satisfy the condition $(f^*)^{-1}(C)=D$. In [4] it is proved that if the mapping $f:(M,C)\longrightarrow(N,D)$ is weak coregular, then $(f^*)^{-1}(C)=D$. In particular, we have

Lemma 6. Let $f:(M,C)\longrightarrow(N,D)$ be a smooth surjection. If there exists a smooth map $i:(N,D)\longrightarrow(M,C)$ such that $f\circ i=id_N$, then $(f^*)^{-1}(C)=D$.

Example 2. Let (M,C) be a differential space, (TM,TC) the tangent differential space to (M,C), $\pi:(TM,TC)\longrightarrow(M,C)$ the canonical projection. It is easy to observe that the zero section $0:M\longrightarrow TM$ is smooth. By Lemma 6 it follows that $(\pi^*)^{-1}(TC)=C$, and Corollary 4 yields that the quotient space $(TM/\wp_\pi,TC/\wp_\pi)$ is diffeomorphic to (M,C).

Now we will prove

Proposition 4. Let $f:(M,C)\longrightarrow(N,D)$ be a smooth surjection such that $(f^*)^{-1}(C)=D$. Then $\tau_{C/\wp_f}=\tau_C/\wp_f$ if and only if f is an open mapping.

Proof. $\Leftarrow$ Let f be an open mapping. In view of Corollary 2 it suffices to prove that $\mathfrak{M}_{\wp_f}=\tau_{C_{\wp_f}}$. The inclusion $\tau_{C_{\wp_f}}\subset\mathfrak{M}_{\wp_f}$ is evident. Now we will show the inclusion $\mathfrak{M}_{\wp_f}\subset\tau_{C_{\wp_f}}$. Let $U\in\mathfrak{M}_{\wp_f}$. By definition U is an open $\wp_f$-saturated set. To prove that $U\in\tau_{C_{\wp_f}}$ it suffices to show that for any $p\in U$ there exists a function $\beta\in C_{\wp_f}$ such that $\beta(p)=1$ and $\beta(q)=0$ for

$q \in U$. Since f is an open mapping, F(U) is an open set in τ_D. Let $\varphi \in D$ be a function separating the point f(p) in the set f(U), i.e. $\varphi(f(p))=1$ and $\varphi(s)=$ $=0$ for $s \notin f(U)$. Consider the function $\beta = \varphi \circ f$. Evidently, $\beta \in C_{\rho_f}$ has the required properties: $\beta(p)= \varphi(f(p))=1$ and $\beta(q)= \varphi(f(q))=0$ for $q \in U$.

$\Longrightarrow$ Suppose $\tau_C/\rho_f = \tau_{C/\rho_f}$. Let $U \in \tau_C$. Since $\pi_{\rho_f}:(M, \tau_C) \longrightarrow$ $\longrightarrow (M/\rho_f, \tau_C/\rho_f)$ is an open mapping, it follows that $\pi_{\rho_f}(U) \in \tau_{C/\rho_f}=$ $= \tau_C/\rho_f$.

On the other hand, $f(U)=g(\pi_{\rho_f}(U)$, where g is defined by (3). Since g is a diffeomorphism, $g(\pi_{\rho_f}(U)) \in \tau_D$. Therefore $f(U) \in \tau_D$, and the proof is complete.

Now we consider families of equivalence relations given on elements of some covering of a set M. We would like to give a sufficient condition for such a family to generate an equivalence relation on M.

Definition 3. Let A and B be subsets of a set M and ρ_1, ρ_2 equivalence relations on A and B respectively. The relations ρ_1 and ρ_2 are said to be compatible on $A \cap B$ if $\rho_1|A\cap B= \rho_2|A\cap B$.

The following proposition is obvious:

Proposition 5. Let $(A_i)_{i\in I}$ be a covering of a non-empty set M and $(\rho_i)_{i\in I}$ a family of equivalence relations given on the sets A_i respectively, satisfying the following conditions:

(a) $A_i \cap A_j$ is ρ_i-saturated in A_i and ρ_j-saturated in A_j for any $i,j \in I$.

(b) $\rho_i|A_i\cap A_j= \rho_j|A_i\cap A_j$ for any $i,j \in I$.

Then there exists a unique equivalence relation ρ on M such that $\rho|A_i=$ $= \rho_i$ and A_i is ρ-saturated in M for $i \in I$.

Proof. Let ρ be the relation on M defined in the following way:

(5) $\quad x \rho y$ iff there exists $i \in I$ such that $x,y \in A_i$ and $x \rho_i y$.

It is easy to show that ρ is a unique equivalence relation satisfying the assumptions of the proposition.

Now we will prove

Proposition 6. Let (M,C) be a differential space. Let $(V_i)_{i\in I}$ be an open covering of (M, τ_C) such that:

1^0 on each set V_i, $i\in I$, there is given an equivalence relation ρ_i,
2^0 $V_i\cap V_j$ is ρ_i-saturated in V_i and ρ_j-saturated in V_j, $i,j\in I$,
3^0 $\rho_i|V_i\cap V_j=\rho_j|V_i\cap V_j$ for $i,j\in I$,
4^0 $V_i\in\tau_{C_0}$ for any $i\in I$, where $C_0:=\{\alpha:M\to R|\ \alpha|V_i\in(C_{V_i})_{\rho_i}\}$.

Then there exists a unique equivalence relation ρ on M such that $\rho|V_i=\rho_i$, the sets V_i are ρ-saturated in M, $C_\rho=C_0$ and $(C_\rho)_{V_i}=(V_{V_i})_{\rho_i}$.

Proof. Let ρ be the equivalence relation defined by (5). By Proposition 5, to complete the proof it suffices to show that $C_\rho=C_0$.

We first prove that $C_0\subset C_\rho$. Let $\alpha\in C_0$. Since $\alpha|V_i\in(C_{V_i})_{\rho_i}\subset C_{V_i}$ for any $i\in I$ and $(V_i)_{i\in I}$ is an open covering of (M,τ_C), it follows that $\alpha\in C$. We have to verify that α is consistent with the equivalence relation ρ. Indeed, if $x\,\rho\,y$, then there exists $i\in I$ such that $x,y\in V_i$ and $x\,\rho_i\,y$. Hence $\alpha|V_i\in(C_{V_i})_{\rho_i}$ yields $(\alpha|V_i)(x)=(\alpha|V_i)(y)$ or equivalently $\alpha(x)=\alpha(y)$.

Conversely, let $\alpha\in C_\rho$. By Lemma 4, $\alpha|V_i\in(C_\rho)_{V_i}=(C_{V_i})_{\rho_i}$, i.e. $\alpha\in C_0$, which completes the proof.

Example 3. Let $(M,C)=(R^2\setminus 0,(\mathcal{E}_2)_{R^2\setminus 0})$ be the plane with the origin removed and with the natural differential structure. Let $V_k=\{(x_1,x_2):x_k\neq 0\}$ for k=1,2. Of course $\{V_1,V_2\}$ is an open covering of (M,τ_C). Consider the mapping $F_k:V_k\to R$ for k=1,2 defined by the formulas:

$$F_1(x_1,x_2):=\frac{x_2}{x_1}\ \text{ for } (x_1,x_2)\in V_1 \text{ and } F_2(x_1,x_2):=\frac{x_1}{x_2}\ \text{ for } (x_1,x_2)\in V_2.$$

The relations $\rho_1=\rho_{F_1}$ and $\rho_2=\rho_{F_2}$ are compatible on $V_1\cap V_2$.

The mappings $i_1:R\to R^2\setminus 0$ and $i_2:R\to R^2\setminus 0$ defined by

(6) $\quad i_1(t)=(1,t)$ and $i_2(t)=(t,1)$ for $t\in R$

are smooth and satisfy $F_1\circ i_1=id_R$ and $F_2\circ i_2=id_R$. By Corollary 4 and Lemma 6 it is easy to see that the bijections

$$\hat{F}_1:(V_1/\rho_{F_1},C_{V_1}/\rho_{F_1})\to(R,\mathcal{E}) \text{ and } \hat{F}_2:(V_2/\rho_{F_2},C_{V_2}/\rho_{F_2})\to(R,\mathcal{E})$$

defined by (3) are diffeomorphisms and the sets V_1 and V_2 belong to τ_{C_0}. So there exists a unique equivalence relation ρ on M such that $\rho|V_i=\rho_{F_i}$, the sets V_1 and V_2 are ρ-saturated, and $C_\rho = C_0$. One can prove that the quotient space $(M/\rho, C/\rho)$ is the one-dimensional projective manifold $P_1(R^2)$ and the maps $\hat{F}_1$ and $\hat{F}_2$ are its charts.

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