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ARITHMETIC OF CUTS AND CUTS OF CLASSES

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Abstract: The arithmetical operations on natural numbers are extended to arbitrary cuts and their basic properties are studied. Then to every class its lower and upper cut apprehending the size of its subsets and supersets, respectively, is assigned. The cut arithmetic is applied to derive estimations for lower and upper cuts of classes obtained by various clas-theoretical constructions.

Key words: Alternative set theory, cut, lower cut, upper cut, arithmetic, sum, product, additive, nearly equal, real class.

Classification: 03E70, 03H13, 03H20

The idea of apprehension of a class by means of the size of its subsets and supersets leading to the notions of its lower and upper cut originated from P. Vopěnka some years ago. Some "cut-theoretical" considerations have already appeared implicitly in [Č-V 1979] and [S-Ve 1981]. Both the cuts of a given class can serve as certain "measures" of its size, as well as the gap between them "measures" its vagueness or fuzzyness. The notions of lower (or internal) and upper (or external) cut of a class appeared for the first time in a paper of A. Tzouvaras [Tz 1987], where he used them to develop a kind of measure theory for some classes in the alternative set theory (AST). Some of the results concerning the calculus of cuts which will be stated below are at least partly contained already in his paper, as well.

The present paper begins with the study of cuts on the ordered class N of all natural numbers themselves. The linear order and the arithmetical operations as well as the equivalence of near equality are extended from N to the system of all cuts and some of their basic properties are listed. The equivalence of near equality enables us to refine the arithmetical classification of cuts by topological methods, and, as it is a congruence with respect to some of the operations, it leads to a factorization of the system of all cuts with considerably simplified arithmetic.

Only thereafter the notions of lower and upper cut of a class are introduced. The cut arithmetic is then applied to derive estimations for lower and upper cuts of classes obtained by various class-theoretical constructions. This purpose still requires the introduction of certain infinitary generalizations of sums of cuts, even leading to a new type of cut product. Some of the hitherto stated technical results are utilized in our subsequent paper, devoted to a more detailed study of cuts of real classes. But most of them will be used in our articles (in preparation) investigating Borel classes and developing a different kind (from that of Tzouvaras) of measure theory in the AST by means of cuts.

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1. Preliminaries. The reader is assumed to be familiar with the basic book [V] on the AST. The notions, results and even conventions from it will be used freely without any referring. Some modifications and supplements concerning mainly the notation are stated below.

1.1. The letters a, b, c, d, e (possibly indexed) always denote natural numbers, i.e. the elements of the class N; k, m, n are reserved for finite naturals, i.e. for the elements of the class FN.

The equivalence of infinitesimal nearness on the class Q of all rational numbers is defined by

$$p \doteq q \equiv (\forall n > 0)(|p-q| < 1/n)$$

for $p,q \in Q$. Further we put

$$p <\cdot\, q \equiv (p < q \,\&\, p \not\doteq q),$$

and

$$p \simeq q \equiv (p=0=q \vee (p \neq 0 \neq q \,\&\, p/q \doteq 1)),$$

Obviously, both $\doteq$ and $\simeq$ are π-equivalences on Q. The formula $p \simeq q$ is read "p is nearly equal to q".

For $q \in Q$ we denote by $\lfloor q \rfloor$ the lower and by $\lceil q \rceil$ the upper integer part of q.

The sum operator Σ is defined for each set function f such that $\mathrm{rng}(f) \subseteq Q$ by induction:

$$\Sigma\, \emptyset = 0$$

and

$$\Sigma (f \cup \{\langle q,x \rangle\}) = \Sigma f + q$$

whenever $x \notin \mathrm{dom}(f)$ and $q \in Q$.

1.2. To avoid possible confusions, let us fix the following notation for the class-theoretical difference:

$X \setminus Y = \{x \in X; x \notin Y\}$.

A class X will be called sharp if for each set u also $u \cap X$ is a set. It can be easily seen that the system of all sharp classes is closed with respect to (finite) unions, intersections and class-theoretical differences. All set-theoretically defineable classes (Sd-classes, to be short) are sharp.

Finally, we repeat some notions and results from [V 1979] and [Č-V 1979] introducing some minor notational changes.

For each set w the basic equivalence E_w is defined as follows:

$\langle u,v\rangle \in E_w$ iff for every set-theoretical formula $\varphi(x,y)$ of the language FL_0 it holds

$\varphi(u,w) \equiv \varphi(v,w)$.

1.2.1. Lemma. (a) E_w is an indiscernibility equivalence for each w.

(b) Let Y be a figure in the indiscernibility equivalence E_w and $\varphi(x,X)$ be a normal formula of the language FL_0. Then $\{x; \varphi(x,Y)\}$ is a figure in E_w, too.

(c) For every indiscernibility equivalence R there is a w such that $E_w \subseteq R$.

A class X is called real if there is an indiscernibility equivalence R such that X is a figure in R, i.e. X=R"X. From 1.2.1 (c) it follows that X is a real class iff X is a figure in E_w for some w.

One fact which has to be kept in mind is that the system of all real classes is closed with respect to countable unions and intersections as well as with respect to definitions by normal formulas of the language FL_V. In particular, every Sd-class is real.

2. Cuts and their arithmetic

2.1. Order and operations on cuts. A class A will be called a cut if $A \subseteq N$ and

$(\forall a,b)(a \leq b \,\&\, b \in A \Longrightarrow a \in A)$.

Thus cuts are exactly the transitive subclasses of N.

In particular, all the natural numbers and the class N are cuts. An example of a cut which neither belongs nor is equal to N is the class FN.

Throughout the whole paper the characters A, B, C, D (possibly indexed) always denote cuts.

Obviously, cuts are linearly ordered by inclusion in a way that extends the usual order of the class N. That is why we will write sometimes $A \leq B$ instead of $A \subseteq B$ and $A < B$ instead of $A \subset B$.

The following lemma is a trivial consequence of the linear order of cuts by inclusion.

2.1.1. Lemma. Let $\mathfrak{M}$ be a (not necessarily codable) system of cuts. Then $\bigcup \mathfrak{M}$ and $\bigcap \mathfrak{M}$ are cuts, too.

Similarly, the arithmetical operations, namely the successor, the addition, subtraction, multiplication and division, can be extended from the naturals to all cuts, and, except for the successor, even in two different ways:

$A' = \{a; a \leq A\} = (A \cup \{A\}) \cap N$ (the successor of A);

$A \underset{\cdot}{+} B = \{c; (\exists a \in A', b \in B')(c < a+b)\}$ (the internal sum of A, B);

$A \dot{+} B = \{c; (\forall a \notin A, b \notin B)(c < a+b)\}$ (the external sum of A, B);

$A \underset{\cdot}{-} B = \{c; (\forall b \in B')(c+b \in A)\}$ (the internal difference of A, B);

$A \dot{-} B = \{c; (\exists b \notin B)(c+b \in A)\}$ (the external difference of A, B);

$A \uparrow B = \{c; (\exists a \in A', b \in B')(c < ab)\}$ (the internal product of A, B);

$A \downarrow B = \{c; (\forall a \notin A, b \notin B)(c < ab)\}$ (the external product of A, B);

$A \swarrow B = \{c; (\forall b \in B')(cb \in A)\}$ (the internal quotient of A, B);

$$A \nearrow B = \begin{cases} \{c; (\exists b \notin B)(cb \in A)\} & \text{if } B \neq N, \\ 0 & \text{if } B=N,\ A=0, \\ 1 & \text{if } B=N,\ A \neq 0 \end{cases}$$

(the external quotient of A, B).

Obviously, both types of sum and product satisfy the commutative and associative law, $\uparrow$ is distributive with respect to $\underset{\cdot}{+}$ and so is $\downarrow$ with respect to $\dot{+}$. Sums and products are isotone; differences and quotients are isotone in the first and antitone in the second variable.

Note that all natural numbers a, b satisfy

$a' = a+1$,

$a \underset{\cdot}{+} b = a \dot{+} b = a+b$,

$a \underset{\cdot}{-} b = a \dot{-} b = \max\{0, a-b\}$,

$a \uparrow b = a \downarrow b = ab$,

$$a \swarrow b = a \nearrow b = \begin{cases} \lceil a/b \rceil & \text{if } b \neq 0, \\ N & \text{if } b=0 \neq a, \\ 0 & \text{if } b=0=a. \end{cases}$$

Also, if $a \in N$, then for each A the following identities can be easily verified:

$a \underset{\bullet}{+} A = a \overset{\bullet}{+} A$,

$a \underset{\bullet}{-} A = a \overset{\bullet}{-} A$, $A \underset{\bullet}{-} a = A \overset{\bullet}{-} a$,

$a \uparrow A = a \downarrow A$,

$a \swarrow A = a \nearrow A$, $A \swarrow a = A \nearrow a$.

Thus we can denote $a+A$, $a-A$, $A-a$, $a.A$, a/A, A/a, respectively, the common value of the corresponding operations. The possible ambiguity is excluded by the agreement that a/b always denotes the rational number a/b, and will be not used for its upper integer part $\lceil a/b \rceil = a \swarrow b = a \nearrow b$.

More generally, we will write $A+B$, $A-B$, $A.B$, A/B, respectively, whenever it is assured (with the mentioned exception) that the internal and external version of the corresponding operation coincide.

A cut A will be called successive (additive, multiplicative), if $A'=A$ ($A+A=A$, $A.A=A$, respectively). (Note that $A \underset{\bullet}{+} A = A \overset{\bullet}{+} A$, $A \uparrow A = A \downarrow A$, and also $A' = A + 1$ hold for each A.)

The only nonsuccessive cuts are the naturals; they will be also called principal cuts, i.e. the successive cuts are exactly the nonprincipal ones. According to the axiom of induction, the only set-theoretically definable successive cut is the class N. This has a trivial but rather important consequence.

2.1.2. Proposition. Let A be a cut and $X \subseteq N$ be an Sd-class.

(a) If $A \neq N$ and $A' \subseteq X$, then $X \setminus A \neq \emptyset$.

(b) If $N \setminus A \subseteq X$, then $X \cap A' \neq \emptyset$.

Proof. If $A \in N$, then it is trivial. Otherwise just note that $a' \cap (N \setminus a) = \{a\}$ for each a.

0 is an additive cut; any other additive cut is successive. Similarly, 0 and 1 are multiplicative cuts; any other multiplicative cut is successive and any multiplicative cut except for 1 is additive.

2.1.3. Example. Every cut A satisfies $A \overset{\bullet}{-} A = 0$. On the other hand, if A is additive, then $A \underset{\bullet}{-} A = A$. Thus $A \underset{\bullet}{-} A \neq A \overset{\bullet}{-} A$ for every additive cut $A \neq 0$.

Similarly, for each $A \neq 0$ it holds $A \nearrow A = 1$. On the other hand, if A is multiplicative, then $A \swarrow A = A$. Thus $A \swarrow A \neq A \nearrow A$ for every multiplicative cut $A \neq 0,1$. More generally, for every successive cut A it holds $1 \in A \swarrow A$, hence $A \swarrow A \neq A \nearrow A$.

Further, if A is an additive cut and $0<A<a$, then $(a-A)\underset{\bullet}{+}A=a-A\mathbin{\pmb{+}}a+A=$ $=(a-A)\overset{\bullet}{+}A$.

Similarly, if A is multiplicative and $1<A<a$, then $(a/A)\uparrow A=a/A\mathbin{\pmb{+}}a.A=$ $=(a/A)\downarrow A$.

We record without proof the following inequalities holding for any A, B:

$A\underset{\bullet}{+}B\leq A\overset{\bullet}{+}B,\qquad A\dot{-}B\leq A\underset{\bullet}{-}B,$

$A\uparrow B\leq A\downarrow B,\qquad A\nearrow B\leq A\swarrow B,$

$(A\underset{\bullet}{-}B)\underset{\bullet}{+}B\leq A\leq(A\underset{\bullet}{+}B)\underset{\bullet}{-}B,$

$(A\overset{\bullet}{+}B)\dot{-}B\leq A\leq(A\dot{-}B)\overset{\bullet}{+}B,$

$(((A\swarrow B)-1)\uparrow B)+1\leq A\leq(A\uparrow B)\swarrow B,$

$(A\downarrow B)\nearrow B\leq A\leq(A\nearrow B)\downarrow B.$

The following theorem and its corollary form just a slight extension of a result from [Č-V 1979]; that is why we state them without proof.

2.1.4. Theorem. Let A be a cut. Then A is either a σ-class or a revealed class.

2.1.5. Corollary. Let A be a cut.

(a) If A is a real class, then A is either a σ-class or a π-class.

(b) If A is not real, then both A, $N\setminus A$ are revealed classes.

2.1.4 and 2.1.5 indicate that all the cuts are in some sense "well behaved" classes. Also the following theorem becomes of more interest in view of the results just stated.

2.1.6. Theorem. Let A, B be cuts.

(a) Assume that either A is a revealed class and B is a π-class, or $N\setminus A$ is a revealed class and B is a σ-class (or vice versa). Then

$A\underset{\bullet}{+}B=A\overset{\bullet}{+}B$ and $A\uparrow B=A\downarrow B$.

(b) Assume that either A is a revealed class and B is a σ-class, or $N\setminus A$ is a revealed class and B is a π-class (or vice versa). Then, except for the case $A=B=N$,

$A\underset{\bullet}{-}B=A\dot{-}B$ and $A\swarrow B=A\nearrow B$.

Proof. Since the proofs of all particular cases follow essentially the same pattern, we will present only one of them. So assume that A is revealed and B is a π-class. We are going to show that $A\overset{\bullet}{+}B\leq A\underset{\bullet}{+}B$. If $B=N$, then it is trivial. Otherwise B can be represented in the form $B=\bigcap\{b_n;n\in FN\}$ for some sequence $\{b_n; a \in FN\}$ such that $b_{n+1} \lessdot b_n$ for each n. Let $c \in A \dot{+} B$. Then $(\forall n)(\forall a \notin A)(c < a+b_n)$. By 2.1.2 (a) there is a sequence $\{a_n; n \in FN\}$ such that $a_n \in A'$ and $c < a_n+b_n$ for each n. Obviously, the class A' is revealed, as well. Hence prolonging the sequences $\{a_n; n \in FN\}$, $\{b_n; n \in FN\}$ we can find an $a \in A'$ and a b such that $b \leq b_n$ for each n, i.e. $b \in B'$, satisfying $c < a+b$. Consequently, $c \in A \dot{+} B$.

2.2. The congruence of near equality. Restricting the equivalence of near equality $\doteq$ from Q to N, a π-equivalence on the class N is obtained. In particular, $m \doteq n$ iff m=n for m, $n \in FN$.

Unles otherwise stated, the notion of monad of a point $a \in N$, as well as those of figure, interior and closure of a class $X \subseteq N$ are related to the π-equivalence $\doteq$ on N. To fix the notation we put

$\mathrm{mon}(a) = \{b; b \doteq a\}$,

$\mathrm{fig}(X) = \{b; (\exists c \in X)(b \doteq c)\}$,

$\mathrm{int}(X) = \{b; (\exists Y)(\mathrm{Sd}(Y) \,\&\, \mathrm{mon}(b) \subseteq Y \subseteq X)\}$,

$\mathrm{cl}(X) = \{b; (\forall Y)(\mathrm{Sd}(Y) \,\&\, \mathrm{mon}(b) \subseteq Y \Rightarrow Y \cap X \neq \emptyset)\}$.

Clearly, the operators int and cl share all the formal properties of interior and closure operators in classical topological spaces. Thus they can be used to define the notions of open, closed and clopen class in the common way. They always will refer to the π-equivalence

2.2.1. Lemma. For every cut A the classes fig(A), int(A), cl(A) are also cuts, and it holds

(a) $\mathrm{int}(A) \leq A \lessdot \mathrm{fig}(A) \leq \mathrm{cl}(A)$,

(b) $\mathrm{int}(\mathrm{cl}(A)) = \mathrm{int}(A)$,

(c) $\mathrm{cl}(\mathrm{int}(A)) = \mathrm{cl}(A)$.

Proof. Since the first assertion and (a) are trivial, we will deal only with (b) and (c). It is routine to check that

$\mathrm{int}(A) = \{b; (\exists n > 0)(\lceil b(1+1/n) \rceil \lessdot A)\}$,

$\mathrm{cl}(A) = \{b; (\forall n > 0)(\lceil b(1-1/n) \rceil < A)\}$.

Now, using the inequality $(1+1/n)(1-1/n) < 1$ holding for each $n > 0$, one can immediately verify that

$\mathrm{int}(\mathrm{cl}(A)) \lessdot A \lessdot \mathrm{cl}(\mathrm{int}(A))$.

Applying int to the first and cl to the second inequality, the required conclusions follow.

As it can be easily seen now, for each $a \notin FN$ it holds

$mon(a)=cl(a)\setminus int(a)$,

$fig(a)=cl(a)$.

2.2.2. Theorem. Let A, B be cuts. Then

$int(A)=int(B)$ iff $cl(A)=cl(B)$.

Proof. If $int(A)=int(B)$, then, by 2.2.1 (c), $cl(A)=cl(int(A))=$ $=cl(int(B))=cl(B)$, and vice versa by 2.2.1 (b).

The cuts A, B will be called nearly equal, notation $A \simeq B$, if $int(A)=$ $=int(B)$ (and, of course, $cl(a)=cl(B)$). Obviously, $a,b \in N$ are nearly equal as cuts iff they are nearly equal as natural numbers.

Further we put

$A \lesssim B$ iff $A<B$ or $A\simeq B$

(A is less or nearly equal to B).

As one can easily verify, $\simeq$ is an equivalence and $\lesssim$ is a preorder on cuts, and $A \simeq B$ is equivalent to $A \lesssim B \& B \lesssim A$. Thus $A \lesssim B \lesssim C$ and $A \simeq C$ imply $A \simeq B \simeq C$. Finally, either $A \lesssim B$ or $B \lesssim A$ always holds. In other words, denoting by $\mathcal{L}$ the (not codable) system of all cuts and factorizing it with respect to the equivalence $\simeq$, i.e. regarding $\simeq$ as the equality, the factor system $\mathcal{L}/\simeq$ becomes linearly ordered by $\lesssim$.

A trivial consequence is the following:

2.2.3. Lemma. Let $\mathfrak{M}$, $\mathfrak{N}$ be two (not necessarily codable) systems of cuts such that

$(\forall A \in \mathfrak{M})(\exists B \in \mathfrak{N})(A \lesssim B)$

and

$(\forall B \in \mathfrak{N})(\exists A \in \mathfrak{M})(B \lesssim A)$.

Then

$\cup\mathfrak{M} \simeq \cup\mathfrak{N}$.

2.2.4. Theorem. Let A, B, C, D be cuts. If $A\simeq C$ and $B\simeq D$, then

$A \underset{\cdot}{+} B\simeq C \underset{\cdot}{+} D$, $A \dot{+} B \simeq C \dot{+} D$,

$A \rceil B\simeq C \rceil D$, $A \downarrow B\simeq C \downarrow D$.

Proof. We leave to the reader the straightforward verification of the following equalities, from which the conclusions of the theorem immediately follow:

$\mathrm{int}(A) \underset{\cdot}{+} \mathrm{int}(B)=\mathrm{int}(A \underset{\cdot}{+} B)$,,

$\mathrm{cl}(A) \dot{+} \mathrm{cl}(B)=\mathrm{cl}(A \dot{+} B)$,

$\mathrm{int}(A) \upharpoonleft \mathrm{int}(B)=\mathrm{int}(A \upharpoonleft B)$,

$\mathrm{cl}(A) \downharpoonleft \mathrm{cl}(B)=\mathrm{cl}(A \downharpoonleft B)$.

In other words, $\simeq$ is a congruence of the algebra $\langle \mathcal{L} ; \underset{\cdot}{+} , \dot{+} , \upharpoonleft , \downharpoonleft \rangle$. Hence also the factor system $\mathcal{L}/\simeq$ can be endowed with the corresponding operations in the obvious and natural way (cf. e.g. [Gt 1968]).

2.2.5. Remark. Analogous results can be established for both the divisions $\swarrow$ and $\nearrow$, as well. However, we will not utilize them. On the contrary, neither the internal nor the external difference preserves the near equality of cuts. Indeed, for $a \notin FN$ it holds

$\mathrm{int}(a)=a-a/FN$,

$\mathrm{cl}(a)=a+a/FN$

and

$\mathrm{int}(a)=\mathrm{cl}(a)$.

But, as one can easily verify,

$\mathrm{int}(a) \underset{\cdot}{-} \mathrm{cl}(a)=\mathrm{int}(a) \dot{-} \mathrm{cl}(a)=0$,

$\mathrm{cl}(a) \underset{\cdot}{-} \mathrm{int}(a)=\mathrm{cl}(a) \dot{-} \mathrm{int}(a)=a/FN$.

2.3. Classification of cuts. The dichotomic partition of all cuts into the additive and nonadditive ones will become of substantial significance from the measure-theoretic point of view. Now, we are going to refine this classification using topological methods based on the π-equivalence $\doteq$ on N.

Some of the results stated below could be also proved by another way round , using the fact that the π-equivalence $\simeq$ is compact on the class $a.FN \setminus a/FN$ for each $a \in N$ and referring to some results concerning indiscernibility equivalences from [V]. However, we prefer to give more direct and elementary proofs.

For each cut A we can form the class of rational numbers

$Q(A)=\{a/b; a \in A' \ \& \ 0 \neq b \in N \setminus A\}$.

Since for each $q \in Q(A)$ it holds $0 \leq q \leq 1$, the class $Q(A)$ determines a single <u>real</u> number

$$\theta_A = \sup Q(A)$$

for $A \neq N$. For $A=N$, in which case $Q(N)=\emptyset$, we put

$$\theta_N = 0.$$

Obviously, $Q(0)=\{0\}$, hence $\theta_0=0$. But for any $a \neq 0$ there is $a/a=1 \in Q(a)$, hence $\theta_a=1$. Conversely, if $1 \in Q(A)$, then A is a natural number.

Now, we turn our attention to nonprincipal cuts.

2.3.1. Theorem. For every nonprincipal cut A the following conditions are equivalent:

(a) A is additive;

(b) $\theta_A=0$;

(c) A is clopen.

Proof. (a) $\Rightarrow$ (b): If $A=N$, then $\theta_A=0$ by the definition. Otherwise, for $a \in A=A'$, $b \notin A$ it holds $na<b$ for each n. Consequently $a/b \doteq 0$ and $\theta_A=0$.

(b) $\Rightarrow$ (c): N obviously is clopen. Otherwise, $\theta_A=0$ means that $a/b \doteq 0$ for $a \in A$, $b \notin A$. Hence $\mathrm{int}(A)=A=\mathrm{cl}(A)$, so that A is clopen.

(c) $\Rightarrow$ (a): Assume that A is not additive, i.e. there is an $a \in A$ such that $2a \notin A$. We will proceed by induction using the well known idea of Cantor. We put $b_0=a$, $c_0=2a$, $d_n=\lceil (b_n+c_n)/2 \rceil$ and $b_{n+1}=d_n$, $c_{n+1}=c_n$ if $d_n \in A$, or $b_{n+1}=b_n$, $c_{n+1}=d_n$ if $d_n \notin A$. Then $\{b_n; n \in FN\}$, $\{c_n; n \in FN$ are two sequences of natural numbers such that for each n it holds $b_n \leq b_{n+1} < A < c_{n+1} \leq c_n$ and $(c_n-b_n)/a \doteq 2^{1-n}$. Then $B=\bigcup\{b_n; n \in FN\}$, $C=\bigcap\{c_n; n \in FN\}$ are cuts such that $B \leq A \leq C$ and $B \neq C$. But from the construction it follows that $\mathrm{int}(A) \leq B$ and $C \leq \mathrm{cl}(A)$, therefore A is not clopen.

However, note that also each $n \in FN$ is clopen, though, except for $n=0$, not additive and $\theta_n=1$.

2.3.2. Theorem. A cut A is not additive iff $\theta_A=1$.

Proof. It is enough to show that for each nonprincipal, nonadditive, proper cut A it holds $\theta_Q=1$. Assume that $\theta_A<1$, i.e. there is an $m>0$ such that $a/b<1-1/m$ for all $a \in A$, $b \notin A$. Then $\mathrm{cl}(A) \cap \mathrm{cl}(N \setminus A)=\emptyset$, consequently A is clopen and, by the preceding theorem, additive - a contradiction.

Thus Θ_A takes only two values - 0 and 1.

We proceed by classifying the nonadditive cuts.

2.3.3. Theorem. For each cut A the following conditions are equivalent:

(a) $A=\mathrm{int}(a)$ for some $a \notin FN$;

(b) A is a nonadditive σ-cut and

$(\forall q \in Q(A))(q \lessdot 1)$;

(c) A is open but not closed.

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are trivial. In order to show (c) $\Rightarrow$ (a) it suffices to realize that each $a \in \mathrm{cl}(A) \setminus \mathrm{int}(A)$ works.

The next theorem follows by a dual argument.

2.3.4. Theorem. For each cut A the following conditions are equivalent:

(a) $A=\mathrm{cl}(a)$ for some $a \notin N$;

(b) A is a nonadditive π-cut and

$(\forall q \in Q(A))(q \lessdot 1)$;

(c) A is closed but not open.

2.3.5. Theorem. For each cut A the following conditions are equivalent:

(a) $\mathrm{int}(a) < A < \mathrm{cl}(a)$ for some $a \notin N$;

(b) A is nonadditive and not a figure in $\simeq$;

(c) $A \notin FN$ and $(\exists q \in Q(A))(q \doteq 1)$;

(d) A is neither open nor closed.

Proof is trivial.

Theorems 2.3.1 - 5 yield some immediate corollaries.

2.3.6. Corollary. Each cut A is either additive or nearly equal to a natural number, but, with the exception of A=0, not both.

2.3.7. Corollary. The only nonadditive cuts which are figures in $\simeq$ are the positive finite naturals and cuts of form int(a) or cl(a) for $a \notin FN$. Hence, they all are σ- or π-classes.

2.3.8. Corollary. Let A be a nonreal cut. Then A is a figure in $\simeq$ iff A is additive.

For the completeness sake we state without proof a result due to A. Sochor [S 1988].

2.3.9. Theorem. Every real cut A satisfies exactly one of the following four conditions:

(a) $A \in N$;

(b) A is additive and $A \not\simeq 0$:

(c) $A=a-B$ for some $a \notin FN$ and some additive cut B such that $0 < B < a$;

(d) $A=a+B$ for some $a \notin FN$ and some additive cut B such that $0 < B < a$.

Besides, by transfinite induction over the class Ω, Sochor has constructed a nonreal cut omitting each of the conditions (a) - (d).

We close this section, as well as the whole paragraph, with an application of its results to the addition of cuts.

2.3.10. Theorem. Let A, B be cuts. Then $A \underset{\bullet}{+} B \simeq A \overset{\bullet}{+} B$.

Proof. If both A, B are additive, then, as easily seen, $A \underset{\bullet}{+} B = A \cup B = A \overset{\bullet}{+} B$. If one of the cuts, say A, is additive and the other, i.e. B, is not, then there are two possibilities. If $B \leq A$, then obviously $A \underset{\bullet}{+} B = A \overset{\bullet}{+} B = A$. If $A < B$, let us choose a b such that $B \simeq b$. Then $A \leq b/FN$ and $int(b) \leq B \leq A \underset{\bullet}{+} B \leq A \overset{\bullet}{+} B \leq b/FN \overset{\bullet}{+} B = cl(b)$, hence $A \underset{\bullet}{+} B \simeq A \overset{\bullet}{+} B$. Finally, if both A, B are nonadditive, and $A \simeq a$, $B \simeq b$, then $A \underset{\bullet}{+} B \simeq a+b \simeq A \overset{\bullet}{+} B$ obviously holds.

As we have just proved, both the operations induced by $\underset{\bullet}{+}$ and $\overset{\bullet}{+}$ in the factor system $\mathcal{J}/\simeq$ coincide. This justifies the use of the common symbol + when computing the sum of two cuts up the near equality $\simeq$. In order to find a representation of the result, any of the operations $\underset{\bullet}{+}$, $\overset{\bullet}{+}$ can be used. Contrariwise, an analogous result for the multiplication does not hold. Namely, for each $a \notin FN$ it holds

$(a/FN) \uparrow FN = a/FN$ and $(a/FN) \downarrow FN = a.FN$.

The results are even different additive cuts, hence they are very far from being nearly equal.

3. Cuts of classes

3.1. Basic estimations. Each class X determines in a natural way two cuts - the lower cut of X,

$$\underline{X}=\{a;(\exists u)(u\subseteq X \,\&\, a \prec u)\},$$

and the upper cut of X,

$$\overline{X}=\{a;(\forall u)(X\subseteq u \Rightarrow a \prec u)\}.$$

Clearly, for any X both $\underline{X}$ and $\overline{X}$ are cuts, and $\underline{X}\subseteq\overline{X}$. The meaning of the lower and upper cut of a class X becomes perhaps more transparent from the successor of the former

$$\underline{X}' = \{a;(\exists u)(u\subseteq X \,\&\, a \approx u)\}.$$

and the complement of the latter

$$N\setminus\overline{X}=\{a;(\exists u)(X\subseteq u \,\&\, a \succcurlyeq u)\}.$$

The class X is said to have a cut if $\underline{X}=\overline{X}$; we denote by $|X|$ the common value of $\underline{X}$ and $\overline{X}$ in this . Thus the expression $|X|$ is defined and will be used only for classes X having a cut.

Some simple properties of cuts can be verified immediately; we list them consecutively without formulating them as theorems.

For all X, Y from $X\subseteq Y$ it follows $\underline{X}\subseteq\underline{Y}$ and $\overline{X}\subseteq\overline{Y}$.

If X is a real class, then owing to 1.2.1 (b) both $\underline{X}$, $\overline{X}$ are real classes, hence by 2.1.5 (a) each of them is a σ-class or a π-class.

Obviously, every set u has a cut and $|u|$ is the number of elements of u, i.e. the unique natural number a satisfying $a\approx u$. Conversely, if for a class X either $\underline{X}\in N$ or $\overline{X}\in N$ holds, then X is a set.

X is a semiset iff $\overline{X}<N$.

Every Sd-class X has a cut, and it is a proper class iff $|X|=N$.

Finally, let us note that each cut A has a cut and $|A|=A$.

The classes X, Y are said to have the same cuts if $\underline{X}=\underline{Y}$ and $\overline{X}=\overline{Y}$.

The following lemma is trivial.

3.1.1. Lemma. If F is a one-one set-theoretically definable function, then for each $X\subseteq \mathrm{dom}(F)$ the classes X, F"X have the same cuts.

In particular, for every relation R the classes R, R^{-1} have the same cuts.

3.1.2. Proposition. Let F be an Sd-function and X be a class. Then

(a) $\overline{F"X}\subseteq\overline{X}$,

(b) if moreover $X \subseteq \mathrm{dom}(F)$ and F is one-one on X, then also $\underline{X} \leq \underline{F''X}$.

Proof. (a) Let $a \in \overline{F''X}$. Then for each $u \supseteq X$ it holds $F''X \subseteq F''u$ and $a \precsim F''u \precsim u$. Hence $a \in \overline{X}$.

(b) Let $a \in \underline{X}$ and $u \subseteq X$ be such that $a \precsim u$. Then $a \hat{\precsim} u \hat{\approx} F''u \subseteq F''X$, hence $a \in \underline{F''X}$.

3.1.3. Corollary. (a) Let R be a relation. Then $\overline{\mathrm{dom}(R)} \leq \overline{R}$.

(b) Let G be a function. Then $\underline{G} \leq \underline{\mathrm{dom}(G)}$.

Proof. Just put $F(x,y)=y$ and apply 3.1.2 to the classes $F''R$ and $F''G$.

Given a class X, the cut $\overline{X} \dot{-} \underline{X}$ will be called the gap of X.

3.1.4. Lemma. For each class X it holds

$\overline{X} \dot{-} \underline{X} = \{c; (\forall u,v)(u \subseteq X \subseteq v \Rightarrow c \precnsim v \setminus u)\}$.

Proof. Obviously, $c \in \overline{X} \dot{-} \underline{X}$ iff for each $u \subseteq X$ it holds $c+|u| \in \overline{X}$, i.e. iff for any u, v the inclusions $u \subseteq X \subseteq v$ imply $c+|u| < |v|$.

3.1.5. Proposition. Let F be an Sd-function and X be a class. Then

$\overline{F''X} \dot{-} \underline{F''X} \leq \overline{X} \dot{-} \underline{X}$.

Proof. Let $c \in \overline{F''X} \dot{-} \underline{F''X}$. Then for any u, v such that $u \subseteq X \subseteq v$ it holds $F''u \subseteq F''X \subseteq F''v$, hence $c \precsim F''v \setminus F''u \subseteq F''(v \setminus u) \precsim v \setminus u$.

Consequently, $c \in \overline{X} \dot{-} \underline{X}$.

3.1.6. Proposition. For all classes X, Y the following assertions hold:

(a) $X \cap Y = \emptyset \Rightarrow \underline{X} \dotplus \underline{Y} \leq \underline{X \cup Y}$;

(b) $\overline{X \cup Y} \leq \overline{X} \dotplus \overline{Y}$;

(c) if there is a sharp class S such that $X \subseteq S$, $Y \cap S = \emptyset$, then

$\underline{X \cup Y} = \underline{X} \dotplus \underline{Y}$ and $\overline{X \cup Y} = \overline{X} \dotplus \overline{Y}$.

Proof. (a) is completely trivial.

(b) Let $c \in \overline{X \cup Y}$. Then for any $u \supseteq X$, $v \supseteq Y$ it holds $c \precsim u \cup v \precsim |u| + |v|$. Hence $c \in \overline{X} \dotplus \overline{Y}$.

(c) Owing to (a), (b), each equality requires the proof only for one inclusion. Instead of $\underline{X \cup Y} \leq \underline{X} \dotplus \underline{Y}$, we will show $(\underline{X \cup Y})' \leq (\underline{X} \dotplus \underline{Y})'$, which, obviously, is equivalent to it. So let $c \in (\underline{X \cup Y})'$. Then there is a $w \subseteq X \cup Y$ such that $c \hat{\precapprox} w$. It is enough to put $u = w \cap S \subseteq Y$, $v = w \setminus S \subseteq Y$, in order to see that $c=|u|+|v| \in (\underline{X} \dotplus \underline{Y})'$.

The proof of $\overline{X} \dotplus \overline{Y} \leq \overline{X \cup Y}$ is quite analogous.

3.1.7. Proposition. Let X, Y be classes such that $X \subseteq Y$. Then

(a) $\underline{Y} \dot{-} \overline{X} \leq \underline{Y \setminus X} \leq \overline{Y \setminus X} \leq \overline{Y} \dot{-} \underline{X}$;

(b) if X is a set, then $\underline{Y \setminus X} = \underline{Y} - |X|$ and $\overline{Y \setminus X} = \overline{Y} - |X|$;

(c) if Y is a set, then $\underline{Y \setminus X} = |Y| - \overline{X}$ and $\overline{Y \setminus X} = |Y| - \underline{X}$.

Proof. (a) Let $c \in \underline{Y} \dot{-} \overline{X}$. Then there are u, v such that $X \subseteq u$, $v \subseteq Y$ and $c+|u| < |v|$. Then $c < |v|-|u| \leq |v \setminus u|$ and $v \setminus u \subseteq Y \setminus X$. Hence $c \in \underline{Y \setminus X}$. Now, assume that $c \notin \overline{Y} \dot{-} \underline{X}$. Then there are u, v such that $u \subseteq X$, $Y \subseteq v$ and $c+|u| = |v|$. Hence $Y \setminus X \subseteq v \setminus u \mathrel{\hat{\approx}} c$, therefore $c \notin \overline{Y \setminus X}$.

Owing to (a), it suffices to prove only one inclusion in each of the remaining particular cases. Since the proofs of (b), (c) are very similar, we will show only (c), which is more important and a bit less easy.

(c) Let $c \in \underline{Y \setminus X}$ and $u \subseteq Y \setminus X$ be such that $c \mathrel{\hat{\precsim}} u$. Then $X \subseteq Y \setminus u$ and $c+|Y \setminus u| < c+|Y|-c=|Y|$. Hence $c \in |Y|-\underline{X}$. Now, let $c \in |Y|-\underline{X}$. Then for each $v \supseteq Y \setminus X$ it holds $Y \setminus v \subseteq X$, therefore $c+|Y \setminus v| < |Y|$ and $c \mathrel{\hat{\precsim}} Y \cap v \subseteq v$. Hence $c \in \overline{Y \setminus X}$,

The assertion (c) is a kind of duality enabling us to transform some questions concerning upper cuts of semisets to analogous ones concerning their lower, and conversely.

3.1.8. Proposition. For all classes X, Y the following assertions hold:

(a) $\underline{X \times Y} = \underline{X} \uparrow \underline{Y}$;

(b) $\overline{X \times Y} \in \overline{X} \downarrow \overline{Y}$.

Proof. (a) For $c \in \underline{X \times Y}$ there is a set $w \subseteq X \times Y$ such that $c \mathrel{\hat{\precsim}} w$. Then $\mathrm{rng}(w) \subseteq X$, $\mathrm{dom}(w) \subseteq Y$ and $c \mathrel{\hat{\precsim}} w \subseteq \mathrm{rng}(w) \times \mathrm{dom}(w) \subseteq X \times Y$. Hence $c \in \underline{X} \uparrow \underline{Y}$. Conversely, if $c \in \underline{X} \uparrow \underline{Y}$, then $c < |u|.|v|$ for some $u \subseteq X$, $v \subseteq Y$, hence $c \mathrel{\hat{\precsim}} u \times v \subseteq X \times Y$ and $c \in \underline{X \times Y}$.

(b) Let $c \in \overline{X \times Y}$. Then for all $u \supseteq X$, $v \supseteq Y$ it holds $X \times Y \subseteq u \times v$, hence $c < |u \times v| = |u|.|v|$. Consequently, $c \in \overline{X} \downarrow \overline{Y}$.

However, as we shall see later, the external product $\overline{X} \downarrow \overline{Y}$ can be very far from both the cuts $\overline{X} \uparrow \overline{Y}$ and $\overline{X \times Y}$, even if X, Y are cuts themselves. Thus just in the opposite to the situation with the union of disjoint classes X, Y, where

$$\underline{X} \dotplus \underline{Y} \leq \underline{X \cup Y} \leq \overline{X \cup Y} \leq \overline{X} \dotplus \overline{Y},$$

so that $\underline{X} \simeq \overline{X}$ and $\underline{Y} \simeq \overline{Y}$ imply the near equality

$$\underline{X} \dot{+} \underline{Y} \simeq \underline{X \cup Y} \simeq \overline{X \cup Y} \simeq \overline{X} \dot{+} \overline{Y},$$

the estimation given in (b) is rather loose.

3.2. Infinitary sums of cuts and the subexternal product. In order to be able to grasp also the cuts of unions of some codable infinite systems of classes in terms of cuts of their members, we generalize the internal and external sum of two cuts to certain infinitary operations.

From 3.1.6 (c) it follows that

$$A \underset{\bullet}{+} B = \underline{(A \times \{0\}) \cup (B \times \{1\})}$$

and

$$A \dot{+} B = \overline{(A \times \{0\}) \cup (B \times \{1\})}$$

for any A, B. A straightforward generalization leads to the following definitions of the internal and external sum, respectively, of a codable system $\{A_y; y \in Y\}$ of cuts:

$$\underset{\bullet}{\Sigma}\{A_y; y \in Y\} = \underline{\bigcup\{A_y \times \{y\}; y \in Y\}},$$

$$\dot{\Sigma}\{A_y; y \in Y\} = \overline{\bigcup\{A_y \times \{y\}; y \in Y\}}.$$

Obviously, $\underset{\bullet}{\Sigma}\{A_0, A_1\} = A_0 \underset{\bullet}{+} A_1$ and $\dot{\Sigma}\{A_0, A_1\} = A_0 \dot{+} A_1$ for any A_0, A_1. The meaning of our definitions can be visualized by the following:

3.2.1. Proposition. Let $\{A_y; y \in Y\}$ be a system of cuts and $a \in N$. Put $Y_0 = \{y \in Y; A_y \neq 0\}$. Then

(a) $a \in \underset{\bullet}{\Sigma}\{A_y; y \in Y\}$ iff there is a function f such that $dom(f) \subseteq Y$, $(\forall y \in dom(f))(f(y) \in A_y')$ and $a < \Sigma f$.

(b) $a \in \dot{\Sigma}\{A_y; y \in Y\}$ iff for each function f from $Y_0 \subseteq dom(f)$, $rng(f) \subseteq N$ and $(\forall y \in Y_0)(f(y) \notin A_y)$ it follows $a < \Sigma f$.

Proof. Let us denote $K = \bigcup\{A_y \times \{y\}; y \in Y\}$. Then $\underset{\bullet}{\Sigma}\{A_y; y \in Y\} = \underline{K}$ and $\dot{\Sigma}\{A_y; y \in Y\} = \overline{K}$.

(a) Let $a \in \underline{K}$. Then $a \precsim u$ for some $u \subseteq K$. Then the function f, such that $dom(f) = dom(u) \subseteq Y$ and $f(y) = |u''\{y\}| \in A_y'$ for $y \in dom(f)$, satisfies $a < |u| = \Sigma f$. Conversely, let $a < \Sigma f$ for some function f satisfying the conditions reuired. Then for the set

$$u = \bigcup\{f(y) \times \{y\}; y \in dom(f)\} \subseteq K$$

also $a < \Sigma f = |u|$ holds, i.e. $a \in \underline{K}$.

(b) Let $a \in \overline{K}$. Take an f satisfying the stated conditions. Then $u = \bigcup\{f(y)\times\{y\};y\in \mathrm{dom}(f)\}\supseteq K$, hence $a < |u| = \sum f$. Conversely, let $a < \sum f$ for every such an f. Take a $u \supseteq K$. Again, putting $\mathrm{dom}(f)=\mathrm{dom}(u)\supseteq Y_0$ and $f(y) = |u''\{y\}|$ for $y \in \mathrm{dom}(f)$, we obtain $f(y) \notin A_y$ for $y \in Y_0$. Hence $a < \sum f = |u|$, i.e. $a \in \overline{K}$.

3.2.2. Theorem. Let R be a relation, $Y=\mathrm{dom}(R)$. Then

(a) $\underline{R} \leq \underset{\bullet}{\sum}\{\underline{R''\{y\}};y\in Y\}$,

(b) $\dot{\sum}\{\overline{R''\{y\}};y\in Y\}\leq \overline{R}$.

Proof. Let us denote $A_y = \underline{R''\{y\}}$, $B_y = \overline{R''\{y\}}$, $K = \bigcup\{A_y\times\{y\};y\in Y\}$, $M = \bigcup\{B_y\times\{y\};y\in Y\}$.

(a) Let $a \in \underline{R}$. Then $a \precsim u$ for some $u \subseteq R$. We put $v = \bigcup\{|u''\{y\}|\times\{y\}; y\in\mathrm{dom}(u)\}\subseteq K$. Then $a \precsim u \mathrel{\hat{\approx}} v$, hence $a \in \underline{K}$.

(b) Let $a \in \overline{M}$. Take a $u \supseteq R$ and put $v = \bigcup\{|u''\{y\}|\times\{y\};y\in\mathrm{dom}(u)\}\supseteq M$, then $a \precsim v \mathrel{\hat{\approx}} u$, hence $a \in \overline{R}$.

For relations with countable domain even more can be proved.

3.2.3. Theorem. Let R be a relation and $Y=\mathrm{dom}(R)$ be a countable class. Then

(a) $\underline{R} = \underset{\bullet}{\sum}\{\underline{R''\{y\}};y\in Y\}$,

(b) $\overline{R} = \dot{\sum}\{\overline{R''\{y\}};y\in Y\}$.

3.2.3 is in fact a special case of the following generalization of 3.1.6.

3.2.4. Theorem. Let $\{X_n;n\in FN\}$ be a sequence of classes, and $X = \bigcup\{X_n;n\in FN\}$.

(a) If $X_m \cap X_n = \emptyset$ for $m \neq n$, then $\underset{\bullet}{\sum}\{\underline{X_n};n\in FN\}\leq \underline{X}$.

(b) $\overline{X}\leq \dot{\sum}\{\overline{X_n};n\in FN\}$.

(c) If there is a sequence $\{S_n;n\in FN\}$ of sharp classes such that $S_m\cap S_n = \emptyset$ and $X_n\subseteq S_n$ for all $m\neq n$, then

$$\underline{X} = \underset{\bullet}{\sum}\{\underline{X_n};n\in FN\}$$

and

$$\overline{X} = \dot{\sum}\{\overline{X_n};n\in FN\}.$$

Proof. Let us denote $K = \bigcup\{\underline{X_n}\times\{n\};n\in FN\}$, $M = \bigcup\{\overline{X_n}\times\{n\};n\in FN\}$.

(a) Let $a \in \underline{K}$. Then $a \precsim u$ for some $u \subseteq K$. Then $\mathrm{dom}(u)\subseteq FN$ is finite so that we can find a set $v_n \subseteq X_n$ such that $v_n \approx u''\{n\}$ for each $n \in \mathrm{dom}(u)$. Since $v_m \cap v_n = \emptyset$ for $m, n \in \mathrm{dom}(u)$, $m \neq n$, the set $v = \bigcup\{v_n; n \in \mathrm{dom}(u)\} \subseteq X$ satisfies $a \precsim u \approx v$. Hence $a \in \underline{X}$.

(b) Let $a \in \overline{X}$. Take a $u \supseteq M$. Since those n for which $X_n = \emptyset$ do not matter, we can assume without loss of generality that $FN \subseteq \mathrm{dom}(u)$. Then for each $n \in FN$ there is a $v_n \supseteq X_n$ such that $v_n \approx u''\{n\}$. By the axiom of prolongation there is a function g such that $FN \subseteq \mathrm{dom}(g) \subseteq \mathrm{dom}(u)$, $g(n) = v_n$ for each $n \in FN$ and $g(x) \approx u''\{x\}$ for each $x \in \mathrm{dom}(g)$. We put $v = \bigcup\{g(x) \times \{x\}; x \in \mathrm{dom}(g)\} \supseteq X$. Then $a \succsim v \succsim u$, hence $a \in \overline{M}$.

(c) According to (a), (b), we have to prove only one inclusion in each case. So let $a \in \underline{X}$. Then $a \precsim u$ for some $u \subseteq X$. We put $u_n = u \cap S_n \subseteq X_n$ for each n. Then $u_m \cap u_n = \emptyset$ for $m \neq n$, and $u = \bigcup\{u_n; n \in FN\}$. There has to be an m such that $u = u_0 \cup \ldots \cup u_m$. Then $a < |u| = |u_0| + \ldots + |u_m|$, hence $a \in \underline{K}$. Now, let $a \in \overline{M}$. Take a $u \supseteq X$ and put $u_n = u \cap S_n \supseteq X_n$. By the axiom of prolongation there is a function g such that $FN \subseteq \mathrm{dom}(g)$, $g(n) = u_n$ for each n, $g(x) \cap g(y) = \emptyset$ for all $x, y \in \mathrm{dom}(g)$, $x \neq y$, and $\bigcup\{g(x); x \in \mathrm{dom}(g)\} \subseteq u$. We put $v = \bigcup\{|g(x)| \times \{x\}; x \in \mathrm{dom}(g)\} \supseteq M$. Then $a \succsim v \succsim u$, hence $a \in \overline{X}$.

Let A be a cut and $\{A_y; y \in Y\}$ be the constant system of cuts such that $A_y = A$ for each $y \in Y$. We put

$$A \intercal Y = \underset{\cdot}{\Sigma}\{A_y; y \in Y\}$$

(the internal product of A, Y),

and

$$A \odot Y = \dot{\Sigma}\{A_y; y \in Y\}$$

(the subexternal product of A, Y).

By the definition $A \intercal Y = \underline{A \times Y}$. Therefore our notation is justified by 3.1.8. Summarizing:

3.2.5. Theorem. For arbitrary classes X, Y it holds

$$\underline{X \times Y} = \underline{X} \intercal \underline{Y} = \underline{X} \intercal Y.$$

Proof. $\underline{X} \intercal Y = \underline{\underline{X} \times Y} = \underline{\underline{X}} \intercal \underline{Y} = \underline{X} \intercal \underline{Y} = \underline{X \times Y}$.

However, the subexternal product still does not behave so smoothly. Let us start with some trivial observstions.

For any cuts A, B it follows directly from the definition $\overline{A \times B} = A \odot B$. Now, 3.1.8 yields

$$A \intercal B \leq A \odot B \leq A \rfloor B,$$

Hence 2.1.6 (a), (b) give also some sufficient conditions for the coincidence of the subexternal product $\odot$ both with $\upharpoonleft$ and $\downharpoonleft$. In such a case $A \bullet B$ denotes any of $A \upharpoonleft B$, $A \odot B$ or $A \downharpoonleft B$.

3.1.1 implies the commutativity and associativity of the operation $\odot$ on cuts. Then 3.1.1 and 3.1.6 (c) suffice to establish its distributivity with respect to the external sum $\dotplus$.

3.2.6. Example. If $a \notin FN$, then $\underline{\mathrm{int}(a) \times \mathrm{cl}(a)} \neq \overline{\mathrm{int}(a) \times \mathrm{cl}(a)}$. Indeed, a simple computation gives

$$\underline{\mathrm{int}(a) \times \mathrm{cl}(a)} = (a-a/FN) \upharpoonleft (a+a/FN) =$$
$$= (a-a/FN) \upharpoonleft a \underset{\bullet}{+} (a-a/FN) \upharpoonleft (a/FN) = (a^2-a^2/FN) \underset{\bullet}{+} a^2/FN = \mathrm{int}(a^2),$$
$$\overline{\mathrm{int}(a) \times \mathrm{cl}(a)} = (a-a/FN) \odot (a+a/FN) =$$
$$= (a-a/FN) \odot a \dotplus (a-a/FN) \odot (a/FN) = (a^2-a^2/FN) \dotplus a^2/FN = \mathrm{cl}(a^2).$$

3.2.7. Theorem. Let A, B, C, D be cuts such that $A \simeq C$ and $B \simeq D$. Then

$A \odot B \simeq C \odot D$.

Proof. According to the commutativity of $\odot$, it is enough to consider the case B=D. If A=C, there is nothing to prove, so we can exclude the case of additivity of A and/or C. Let a satisfy $A \simeq a \simeq C$. If B is additive, then obviously $A \odot B = a \bullet B = C \odot B$. Otherwise, $B \simeq b$ for some b. Then $A \odot B \simeq a \bullet b \simeq$ $\simeq C \odot B$.

Thus the equivalence $\simeq$ is a congruence with respect to the operation $\odot$ on $\mathcal{L}$, as well.

3.2.8. Theorem. If A, B are cuts and not both of them are additive and nonreal, then

$A \odot B \simeq A \upharpoonleft B$.

Proof. It suffices to consider the following particular cases:

(a) A, B are not additive, $A \simeq a$, $B \simeq b$. Then $A \odot B \simeq a \bullet b \simeq A \upharpoonleft B$.

(b) A is additive and $B \simeq b$ is not. Then $A \odot B = A \bullet b = A \upharpoonleft B$.

(c) A, B are additive and either exactly one of them is not real, or both are σ-classes, or both are π-classes. Then $A \upharpoonleft B = A \odot B = A \downharpoonleft B$ follows from 2.1.5 (b) and 2.1.6 (a).

(d) A, B are additive cuts, A is a π-class and B is a σ-class. Then the conclusion follows from the following more general result.

3.2.9. Lemma. Let A be an additive cut and B a σ-cut. Then

$A \odot B = A \mathbin{\dot{\uparrow}} B$.

Proof. Excluding the trivial case $B \in N$ or $B=N$, B can be written in the form $B=\bigcup\{b_n; n \in FN\}$ for some sequence of naturals such that $b_0=0$ and $b_n<b_{n+1}$ for each n. It suffices to show $A \odot B \leq A \mathbin{\dot{\uparrow}} B$. Let $c \notin A \mathbin{\dot{\uparrow}} B$. Then $(\forall a \in A')(\forall n)(a \cdot b_{n+1} \leq c)$. By 2.1.2 (a), for each n there is a $d_n \notin A$ such that $d_n \cdot b_{n+1} \leq c$. As A is additive, for each n even $\lfloor d_n/2^{n+1} \rfloor \notin A$. By the axiom of prolongation there are functions g, h and an $e \in N \setminus FN$ such that $\mathrm{dom}(g)=e+1$, $\mathrm{dom}(h)=e$, $g(n)=b_n$, $h(n)=\lfloor d_n/2^{n+1} \rfloor$ for each n, and $2^{j+1} \cdot h(j) \cdot g(j+1) \leq c$ for each $j<e$. A function f with domain g(e) will be defined by $f(i)=h(\tilde{i})$, where $\tilde{i}$ denotes the unique natural number $j<e$ such that $g(j) \leq i < g(j+1)$. Obviously, $B \leq \mathrm{dom}(f)$ and $f(i) \notin A$ for $i \in B$. A simple computation gives

$$\Sigma f = \sum_{j<e} h(j)(g(j+1)-g(j)) \leq \sum_{j<e} h(j)\, g(j+1) \leq \sum_{j<e} c/2^{j+1} < c.$$

Owing to 3.2.1 (b), $c \notin A \odot B$.

The question whether there are two additive nonreal cuts A, B such that $A \odot B \neq A \mathbin{\dot{\uparrow}} B$, i.e. $A \odot B \not\simeq A \mathbin{\dot{\uparrow}} B$, remains open.

Nevertheless, restricting our attention to the (codable) system $\mathcal{L}_0$ of all real cuts (i.e. to σ- and π-cuts only),
$\langle \mathcal{L}_0; \underset{\bullet}{+}, \overset{\bullet}{+}, \underset{\bullet}{-}, \overset{\bullet}{-}, \mathbin{\dot{\uparrow}}, \mathbin{\dot{\downarrow}}, \odot, \swarrow, \nearrow \rangle$ becomes a subalgebra of
$\langle \mathcal{L}; \underset{\bullet}{+}, \overset{\bullet}{+}, \underset{\bullet}{-}, \overset{\bullet}{-}, \mathbin{\dot{\uparrow}}, \mathbin{\dot{\downarrow}}, \odot, \swarrow, \nearrow \rangle$ and $\simeq$ is still a congruence of
$\langle \mathcal{L}_0; \underset{\bullet}{+}, \overset{\bullet}{+}, \mathbin{\dot{\uparrow}}, \mathbin{\dot{\downarrow}}, \odot \rangle$. Excluding the operation $\mathbin{\dot{\downarrow}}$, which will play only an auxiliar role in what follows, not only $\underset{\bullet}{+}$ and $\overset{\bullet}{+}$, but also $\mathbin{\dot{\uparrow}}$, $\odot$ coincide on $\mathcal{L}_0/\simeq$. In other words, the factor algebra $\langle \mathcal{L}_0; \underset{\bullet}{+}, \overset{\bullet}{+}, \mathbin{\dot{\uparrow}}, \odot \rangle/\simeq$ can be regarded as endowed solely with two basic operations + (corresponding to $\underset{\bullet}{+}$ and $\overset{\bullet}{+}$) and $\cdot$ (corresponding to $\mathbin{\dot{\uparrow}}$ and $\odot$), i.e. as $\langle \mathcal{L}_0/\simeq; +, \cdot \rangle$.

3.2.10. Example. As we have already seen,

$(a/FN) \mathbin{\dot{\uparrow}} FN = a/FN \neq (a/FN) \mathbin{\dot{\downarrow}} FN$

for $a \notin FN$. But from 3.2.9 it follows that

$(a/FN) \odot FN = (a/FN) \mathbin{\dot{\uparrow}} FN = a/FN$.

Hence the class $(a/FN) \times FN$ has a cut, namely a/FN, not coinciding with the estimation $a \cdot FN$ arising from 3.1.8 (b).

3.2.11. Theorem. Let $\{A_n; n \in FN\}$, $\{B_n; n \in FN\}$ be two sequences of cuts such that $A_n \simeq B_n$ for each n. Then

(a) $\dot{\Sigma}\{A_n; n\in FN\} \simeq \dot{\Sigma}\{B_n; n\in FN\}$,

(b) $\dot{\Sigma}\{A_n; n\in FN\} \simeq \dot{\Sigma}\{B_n; n\in FN\}$.

Proof. (a) Obviously,

$\dot{\Sigma}\{A_n; n\in FN\} = \bigcup\{A_0 \dotplus \ldots \dotplus A_n; n\in FN\}$,

and analogously for the other sequence. By 2.2.4

$A_0 \dotplus \ldots \dotplus A_n \simeq B_0 \dotplus \ldots \dotplus B_n$

for each n. The conclusion follows from 2.2.3.

(b) is a consequence of (a) and of the following result.

3.2.12. Theorem. Let $\{A_n; n\in FN\}$ be a sequence of cuts. Then

$\dot{\Sigma}\{A_n; n\in FN\} \simeq \dot{\Sigma}\{A_n; n\in FN\}$.

Proof. Let us denote $B = \dot{\Sigma}\{A_n; n\in FN\}$, $C = \dot{\Sigma}\{A_n; n\in FN\}$. If B is additive, then, as for each n obviously $A_n \preccurlyeq B$ holds, using 3.2.9 we obtain $B \preccurlyeq C \preccurlyeq B \odot FN = B$. If $B \simeq b$ is not additive, we put $X = \{n\in FN;\ A_n$ is additive$\}$. Then, obviously,

$B = \dot{\Sigma}\{A_n; n\in X\} \dotplus \dot{\Sigma}\{A_n; n\in FN\setminus X\}$,

$C = \dot{\Sigma}\{A_n; n\in X\} \dotplus \dot{\Sigma}\{A_n; n\in FN\setminus X\}$.

As both the first summands are additive, it suffices to prove the theorem under the assumption that $A_n \simeq a_n$ is not additive for each n. It is just enough to show that $C \preccurlyeq cl(b)$. Let $c \notin cl(b)$. Then there is a $k\in FN$, $k>0$, such that $b(1+1/k) < c$. Without loss of generality we can assume that $a_0+\ldots+a_n \preccurlyeq b$ for each n (cl(b) is revealed!). We put $c_n = \lfloor a_n(1+1/k\cdot 2^{n+1})\rfloor$. Then $A_n \preccurlyeq c_n$ and $c_0+\ldots+c_n \preccurlyeq b\cdot(1+1/k) \preccurlyeq c$ for each n. By the axiom of prolongation there is a function g such that $dom(g)\in N\setminus FN$, $rng(g) \preccurlyeq N$, $g(n)=c_n$ for each $n\in FN$, and $\Sigma g \preccurlyeq c$. Hence $c \notin C$.

We have proved that not only finite but also countable internal and external sums of cuts preserve the equivalence $\simeq$ and coincide with respect to it. Hence the "FN-ary" operation Σ can be introduced on the factor system $\mathcal{L}/\simeq$, as well as on $\mathcal{L}_0/\simeq$, and both $\dot{\Sigma}$, $\dot{\Sigma}$ can be used to compute its value. For reasons that will come out later the (infinitary) algebra

$\langle \mathcal{L}_0/\simeq\ ; +, \cdot, \Sigma\rangle$ will be called the algebra of Borel cardinals. As a trivial consequence of 3.1.1, the distributivity of the product, even with respect to the countable sum Σ is obtained.

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