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ON THE DIFFERENTIAL AND RESIDUAL ENTROPY

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Abstract: We introduce and examine the residual entropy and the regularized residual entropy defined for metric spaces equipped with a finite (respectively, σ-finite) measure and satisfying certain conditions. It is shown that the differential entropy is equivalent, in a specified sense, to the regularized residual entropy.

Key words: Differential entropy, residual entropy, regularized residual entropy, regularized Rényi dimension.

Classification: 94A17

Let $P=\langle Q,\rho,\mu\rangle$ be a metric space endowed with a probability measure μ with respect to which ρ is measurable. We define the residual entropy rE(P) as the "remainder" of the epsilon entropy $H_\varepsilon(P)$, i.e., as the limit (provided it exists) of $H_\varepsilon(P)-RD(P)|\log \varepsilon|$, where RD(P) is a certain modification of the Rényi dimension of P. Based on rE(P), the regularized residual entropy RE(P) and the residual entropy density $\nabla(P)$ are introduced for $P=\langle Q,\rho,\mu\rangle$ with μ σ-finite. It is shown that RE(P) and $\nabla(P)$ do exist for a fairly wide class of spaces. Furthermore, properties of rE, RE and ∇ are examined in some detail.

The concept of the differential entropy, originally defined for probability measures on R^n possessing a density, is examined in a general setting, namely for the case of a pair (μ,ν) of σ-finite measures with μ absolutely continuous with respect to ν. It is proved that the differential entropy and the regularized residual entropy RE are, in a sense, equivalent. Namely, if ν satisfies a separability condition, then the differential entropy of (μ,ν) can be expressed, in a specified sense, by means of RE; on the other hand, RE can be expressed, for a fairly wide class of spaces, by means of the differential entropy.

The article is organized as follows: Section 1 contains preliminaries. In Section 2, σW-spaces are introduced, some concepts previously defined for W-spaces are extended to σW-spaces, and some simple facts are proved. In Section 3, the residual entropy rE is introduced and examined, and partition-regular spaces, on which the behavior of rE is fairly reasonable, are considered. In Section 4, the regularized residual entropy RE is examined. In Section 5 we introduce and examine the residual entropy density. Section 6 contains the theorems on the mutual reducibility of the differential and residual entropy.

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1.1. The terminology and notation is that of [6] with slight modifications (see 2.5). Nevertheless, some definitions and conventions will be restated.

1.2. The symbols N, R, $\overline{R}$, R_+, $\overline{R}_+$ have their usual meaning. The letters m and n (possibly with subscripts) always denote natural numbers. We put 0/0= =0 and, for any $b\in\overline{R}$, 0.b=0. - We write log instead of $\log_2$, and we put log 0= = $-\infty$, L(x)=x log x for all $x\in R_+$. For $x\in R$, we sometimes write exp x instead of 2^x.

1.3. A mapping $f:\mathfrak{X}\rightarrow\overline{R}$, where $\mathfrak{X}$ is a class, is called a function or a functional; as a rule, the word "functional" is preferred if $\mathfrak{X}$ is a proper class or consists of functions or spaces, etc.

1.4. If a set A is given, then, for any $X\subset A$, i_X is the indicator of X, i.e. $i_X(x)=1$ if $x\in X$, $i_X(x)=0$ if $x\in A\setminus X$.

1.5. If $Q\neq\emptyset$ is a set and $\mathcal{A}$ is a σ-algebra of subsets of Q, then a σ-additive function $\mu:\mathcal{A}\rightarrow\overline{R}_+$ satisfying $\mu(\emptyset)=0$ is called a measure on Q (in [3] such functions were called $\overline{R}$-measures, whereas "measure" meant a finite measure). A measure on Q is called finite or bounded if $\mu Q<\infty$, σ-finite or σ-bounded if there are $A_n\in\operatorname{dom}\mu$ such that $\bigcup(A_n:n\in N)=Q$ and $\mu A_n<\infty$ for all n. - The completion of a measure μ is denoted by $\overline{\mu}$ or $[\mu]$. If μ and ν are measures, then $\mu\leq\nu$ means that dom μ=dom ν and $\mu X\leq\nu X$ for all $X\in\operatorname{dom}\mu$, $\mu\subset\nu$ means that dom $\mu\subset$ dom ν and $\nu X=\mu X$ whenever $X\in\operatorname{dom}\mu$.

1.6. Notation (cf. [6], 1.6). A) If $Q\neq\emptyset$ is a set, then $\mathcal{F}(Q)$, $\mathcal{M}(Q)$ and $\mathcal{M}_{\sigma f}(Q)$ will denote, respectively, the set of all $f:Q\rightarrow\overline{R}$, the set of all measures on Q and its subset consisting of σ-finite measures. - B) If $\mu\in\mathcal{M}(Q)$ and $f,g\in\mathcal{F}(Q)$, we write f=g(mod μ) iff there is a set $Z\in\operatorname{dom}\mu$ such that $\mu Z=0$ and f(x)=g(x) whenever $x\in Q\setminus Z$. - C) If $\mu\in\mathcal{M}(Q)$ and $f\in\mathcal{F}(Q)$ is $\overline{\mu}$-measurable, we put $[f]_\mu=\{g\in\mathcal{F}(Q):g=f(\mathrm{mod}\ \mu)\}$ and call $[f]_\mu$ a function (mod μ). We put $\mathcal{F}[\mu]=\{[f]_\mu : f\in\mathcal{F}(Q)$ is $\overline{\mu}$-measurable$\}$. - D) If $\mu\in\mathcal{M}(Q)$, $F,G\in\mathcal{F}[\mu]$, then we put $F\leq G$ (respectively, $F<G$) iff, for

some $f \in F$, $g \in G$, $f(x) \leq g(x)$ (respectively, $f(x) < g(x)$) for all $x \in Q$ (thus, e.g., $-\infty < [f]_{\mu} < \infty$ means that some $g=f$ (mod μ) is finite). - E) If $\mu \in \mathcal{M}(Q)$ and $F \in \mathcal{F}[\mu]$, then sup F denotes the least $b \in \overline{R}$ such that $F \leq b$, and similarly for inf F. - F) If $\mu \in \mathcal{M}(Q)$, $F=[f]_{\mu} \in \mathcal{F}[\mu]$, we put $\int F\, d\mu = \int f\, d\mu$. - G) If $\mu \in \mathcal{M}(Q)$ and $f:Q \rightarrow T$ is a mapping, then $\mu \circ f^{-1}$ denotes the measure $Y \mapsto \mu(f^{-1}Y)$.

1.7. We use the usual convention concerning expressions of the form $\xi \mapsto F(\xi)$. If a term $F(\xi)$ contains a variable ξ, then the expression $\xi \mapsto F(\xi)$ denotes the mapping defined as follows. Let x be an element (from a given class explicitly described or clear from the context). If the term $F(x)$ denotes exactly one element y, we put $f(x)=y$; if not, then $f(x)$ is not defined. Thus, e.g., if $\mu \in \mathcal{M}(Q)$, then the expression $f \mapsto \int f\, d\mu$, where $f \in \mathcal{F}(Q)$, denotes the functional φ such that (1) $\mathrm{dom}\, \varphi = \{f \in \mathcal{F}(Q): \int f\, d\mu$ exists$\}$, (2) if $f \in \mathrm{dom}\, \varphi$, then $\varphi(f) = \int f\, d\mu$.

1.8. Let $\mu \in \mathcal{M}(Q)$. If $F=[f]_{\mu} \in \mathcal{F}[\mu]$, $F \geq 0$, then the function $X \mapsto \int_X f\, d\mu$, defined on $\mathrm{dom}\, \overline{\mu}$, is a measure. Its restriction to $\mathrm{dom}\, \mu$ will be denoted by $f.\mu$ or $F.\mu$. If $X \in \mathrm{dom}\, \overline{\mu}$, we put $X.\mu = i_X.\mu$. - Observe that if $\mu \in \mathcal{M}_{\sigma f}(Q)$ and $0 \leq F < \infty$, then $F.\mu \in \mathcal{M}_{\sigma f}(Q)$.

1.9. If Q is a set, $K \neq \emptyset$ is a countable set, X_k, $k \in K$, are subsets of Q, $\cup X_k = Q$ and $X_i \cap X_j = \emptyset$ if $i,j \in K$, $i \neq j$, then $(X_k : k \in K)$ will be called a partition of the set Q (a μ-measurable partition if $Q \subset T$, μ is a measure on T and all X_k are in $\mathrm{dom}\, \mu$). - Observe that "partition" has a different meaning in the expressions "partition of a σW-space" (see 2.5) and "$\mathfrak{c}$-partition" (see 2.10).

1.10. Conventions and notation. Let τ be a σ-additive function, possibly also assuming the value $-\infty$ or ∞, on a set $Q \neq \emptyset$ (this means that $\mathrm{dom}\, \tau$ is a σ-algebra $\mathcal{A}$ of subsets of Q, $\tau(\emptyset)=0$ and $\tau(A) = \Sigma(\tau(A_n) : n \in N)$ whenever $(A_n : n \in N)$ is a partition of $A \in \mathcal{A}$ and all A_n are in $\mathcal{A}$). Then (1) a set $X \subset Q$ will be called τ-null if there is a set $Y \in \mathrm{dom}\, \tau$ such that $Y \supset X$ and $\tau Z = 0$ whenever $Z \in \mathrm{dom}\, \tau$, $Z \subset Y$, (2) if $X \subset Q$ and, for some $Y \subset \mathrm{dom}\, \tau$, the symmetric difference $X \triangle Y$ is τ-null, we put $\overline{\tau}(X) = \tau(Y)$. The function $\overline{\tau}$, also denoted by $[\tau]$, is σ-additive; it will be called the completion of τ.

1.11. A σ-additive function τ on Q is called bounded (or finite) if $\{\tau X : X \in \mathrm{dom}\, \tau\}$ is bounded; σ-bounded (or σ-finite) whenever there is a partition $(A_n : n \in N)$ of Q such that, for any $n \in N$, $A_n \in \mathrm{dom}\, \tau$ and $\{\tau X : X \in \mathrm{dom}\, \tau,\ X \subset A_n\}$ is bounded.

1.12. **Definition.** There are various slightly differing definitions of absolute continuity (of measures, etc.). We choose a fairly broad one: let μ be a σ-finite measure on Q and let τ be a σ-bounded σ-additive function on Q. Then τ is said to be absolutely continuous with respect to μ if (1) every μ-null set is τ-null, (2) dom $\bar{\mu} \subset$ dom $\bar{\tau}$, and (3) there is a τ-null set A such that if $X \in$ dom $\bar{\tau}$, then $X=Y\cup Z$, where $Y \in$ dom $\bar{\mu}$, $Z \subset A$.

1.13. Fact and notation. If $\mu \in \mathcal{M}_{\sigma f}(Q)$, $f \in \mathcal{F}(Q)$ is $\bar{\mu}$-measurable, $f(Q) \subset \subset R$ and $\int f\, d\mu$ exists, then $X \longmapsto \int_X f\, d\mu$, defined on dom $\bar{\mu}$, is an absolutely continuous (with respect to μ) σ-bounded σ-additive function. Its restriction to dom μ will be denoted by $f.\mu$ or $F.\mu$ where $F=[f]_\mu$.

1.14. We shall need the Radon-Nikodým theorem in the following form.

Theorem. Let μ be a σ-finite measure on Q and let τ be a σ-bounded σ-additive function on Q. If τ is absolutely continuous with respect to μ, then there exists exactly one function (mod μ) F such that $\bar{\tau}X=\int_X F\, d\mu$ for all $X \in$ dom $\bar{\mu}$.

1.15. Notation. The function (mod μ) F from 1.14 will be denoted by $d\tau/d\mu$ or by $D[\tau, \mu]$.

1.16. Fact and notation. Let μ be a measure on Q. If $\emptyset \neq T \subset Q$, then the function $X \longmapsto \inf(\mu Y : Y \in \text{dom}\,\mu, Y\cap T=X)$ defined on $\{Y\cap T : Y \in \text{dom}\,\mu\}$ is a measure on T. It will be denoted by $\mu \restriction T$ provided there is no danger of confusion. - We put $\mu_e(\emptyset)=\emptyset$ and $\mu_e(T)=(\mu\restriction T)(T)$ if $\emptyset \neq T \subset Q$. - Cf. [3], 7.4 and 7.5.

1.17. Fact. If μ is a measure on Q and $\emptyset \neq T \subset Q$, then $[\mu \restriction T] = \bar{\mu} \restriction T$. If, for i=1,2, μ_i is a measure on Q_i and $\emptyset \neq T_i \subset Q_i$, then $\nu = \mu \restriction T$, $\bar{\nu} = \bar{\mu} \restriction T$, where $T=T_1\times T_2$, $\nu = \nu_1 \times \nu_2$, $\nu_i = \mu_i \restriction T$, $\mu = \mu_1 \times \mu_2$. - Cf. [3], 7.6.

1.18. Notation. The Lebesgue measure on R^n, n=1,2,..., will be denoted by λ_n or simply λ. If $Q \subset R^n$, $Q \neq \emptyset$, we often write λ_n or λ instead of $\lambda_n \restriction Q$ provided there is no danger of confusion.

1.19. Conventions and notation. If $\langle Q, \rho \rangle$ is a semimetric space (i.e. ρ is a real-valued function on $Q \times Q$ satisfying $\rho(x,y)=\rho(y,x) \geq 0$, $\rho(x,x)=0$) and $T \subset Q$, then $\langle T, \rho \rangle$ will denote the set T endowed with the semimetric $\rho \restriction (T\times T)$. The symbol R^n, n=1,2,..., will also denote the space $\langle R^n, \rho \rangle$, where ρ is the ℓ_∞-metric, i.e., $\rho((x_i),(y_i))=\max(|x_i-y_i|)$. The ℓ_∞-metric on a set $Q \subset R^n$ will be denoted by ρ (unless explicitly stated that ρ is used in a different sense). If $P_i=\langle Q_i, \rho_i \rangle$ are semimetric spaces, then $P_1 \times P_2$ denotes the space $\langle Q_1 \times Q_2, \rho \rangle$, where $\rho((x_1,x_2),(y_1,y_2))=\max(\rho_1(x_1,y_1), \rho_2(x_2,y_2))$.

1.20. Notation. If S is a set endowed with a topology (in particular, if S is a metric space), then $\mathcal{B}(S)$ will denote the collection of all Borel subsets of S.

1.21. Notation. If $\xi=(x_k:k\in K)$ is a non-void indexed set of nonnegative reals and $\sum x_k<\infty$, we put $H(\xi)=H(x_k:k\in K)=\sum(Lx_k:k\in K)-L(\sum(x_k:k\in K))$. If μ is a finite measure on a countable set Q and $\{x\}\in \operatorname{dom}\mu$ for all $x\in Q$, we put $H(\mu)=H(\mu\{q\}:q\in Q)$.

1.22. The following simple facts concerning the functional H will often be used.

1.22.1. If $x_k\geq 0$, $k=1,\dots,n$, then $H(x_1,\dots,x_n)\leq(\sum x_k).\log n$.

1.22.2. Let $x_{kj}\geq 0$ for $k\in K$, $j\in J_k$ (where K and J_k are non-void sets). Let $\sum(x_{kj}:k\in K,j\in J_k)<\infty$. Then $H(x_{kj}:k\in K,j\in J_k)=H(\sum(x_{kj}:j\in J_k):k\in K)+$ $+\sum(H(x_{kj}:j\in J_k):k\in K)$.

1.22.3. If $K\neq\emptyset$, $x_k\geq 0$ for $k\in K$ and $0<\sum x_k<\infty$, then $H(x_k:k\in K)\geq$ $\geq -L(\sum x_k)-(\sum x_k).\log\sup(x_k:k\in K)$; in particular, if $\sum x_k=1$, then $H(x_k:k\in$ $\in K)\geq -\log\sup(x_k:k\in K)$.

1.22.4. Let K be a non-void set. Let x_k, y_k, where $k\in K$, be non-negative reals. Let $\sum y_k=\sum x_k<\infty$. Let $j\in J\subset K$ and let $x_j\geq x_k$ for all $k\in K$. Let $y_k=$ $=x_k$ for $k\in K\setminus J$, $y_j\geq x_j$, $y_k\leq x_k$ for $k\in J$, $k\neq j$. Then $H(y_k:k\in K)\leq H(x_k:k\in K)$.

2

Recall that W-spaces (also called semimetric spaces endowed with a finite measure) are defined as follows: $P=\langle Q,\rho,\mu\rangle$ is a W-space if $Q\neq\emptyset$ is a set, μ is a finite measure on Q and ρ is a $[\mu\times\mu]$-measurable semimetric on Q. - In the present article, we will also consider σW-spaces, obtained by replacing "finite" by "σ-finite" in the above definition. The reason for introducing this broader class of spaces lies in the following facts: (1) the regularized residual entropy (see 4.2) can be defined in a very natural way for σW-spaces, (2) the theorem (see 6.9) on expressing the differential entropy (see 6.1) by means of the regularized residual entropy is valid in full extent only if σW-spaces are taken into consideration, (3) such natural objects as $\langle R^n,\rho,\lambda_n\rangle$ are σW-spaces, not W-spaces.

2.1. **Definition.** Let Q be a non-void set. Let μ be a σ-finite measure on Q and let ρ be a $[\mu\times\mu]$-measurable semimetric on Q. Then $P=\langle Q,\rho,\mu\rangle$ will be called σW-space or a semimetric space endowed with a σ-finite measure. If, in addition, $\mu Q<\infty$, then P is called W-space (or a semimetric space endowed with a finite measure).

2.2. Notation and conventions. If $P=\langle Q,\varphi,\mu\rangle$ is a σW-space, we put $wP=\mu Q$. If $wP=0$, we call P a null space. - The class of all σW-spaces and that of all W-spaces will be denoted, respectively, by $\sigma\mathfrak{M}$ and $\mathfrak{M}$. A σW-space $P=\langle Q,\varphi,\mu\rangle$ will be called metric if φ is a metric (cf. [4], 1.5). If, in addition, every Borel set is in dom $\bar{\mu}$, then P will be called weakly Borel.

2.3. Let $P=\langle Q,\varphi,\mu\rangle$ be a σW-space. If $S=\langle Q,\varphi,\nu\rangle$ and $\nu\leq\mu$, then we call S a subspace of P (a pure subspace if $\nu=X.\mu$ where $X\in$ dom $\bar{\mu}$) and write $S\leq P$. If $F=[f]_{\mu}\in\mathcal{F}[\mu]$ and $0\leq F<\infty$, then $\langle Q,\varphi,f.\mu\rangle$ is a σW-space, which will be denoted by F.P or f.P. If $X\in$ dom $\bar{\mu}$, we put $X.P=i_X.P$. - Cf. [4], 1.6 and 1.7.

2.4. Fact. Let $P=\langle Q,\varphi,\mu\rangle\in\sigma\mathfrak{M}$. Then $S=\langle Q,\varphi,\nu\rangle\leq P$ iff $S=f.P$ for some $\bar{\mu}$-measurable $f\in\mathcal{F}(Q)$ satisfying $0\leq[f]_{\mu}\leq 1$.

2.5. If $K\neq\emptyset$ is a countable set, P_k, $k\in K$, and P are σW-spaces and $\Sigma(P_k:k\in K)=P$ (i.e., $P_k=\langle Q,\varphi,\mu_k\rangle$, $P=\langle Q,\varphi,\mu\rangle$ and $\mu=\Sigma\mu_k$), then we will say that $(P_k:k\in K)$ is a partition of P (a pure partition if all P_k are pure subspaces of P). Cf., e.g., [6],1.12. - Remark. In [2] - [5], the term "ω-partition" was used for what is now called partition, whereas "partition" meant a finite partition.

2.6. Fact. If $P\in\sigma\mathfrak{M}$, $(P_k:k\in K)$ is a partition of P and $S\leq P$, then there are $S_k\leq P_k$ such that $\Sigma(S_k:k\in K)=S$. - Cf. [6], 1.13.

Proof. Let $S=s.P$, $P_k=f_k.P$ (see 1.14). Put $g_k=sf_k$, $S_k=g_k.P\leq P_k$. Clearly, $\Sigma S_k=S$.

2.7. Let $\mathcal{U}=(U_k:k\in K)$ and $\mathcal{V}=(V_j:j\in J)$ be partitions of a σW-space P. If there exists a disjoint collection $(J_k:k\in K)$ such that $\bigcup J_k=J$ and, for each $k\in K$, $\Sigma(V_j:j\in J_k)=U_k$, then $\mathcal{V}$ is said to refine $\mathcal{U}$. - Cf., e.g.,[6], 1.14.

2.8. Fact. If $\mathcal{U}$ and $\mathcal{V}$ are partitions of a σW-space P, then there exists a partition of P refining both $\mathcal{U}$ and $\mathcal{V}$. - Cf. [2], 1.36.

Proof. Let $\mathcal{U}=(U_k:k\in K)$, $\mathcal{V}=(V_j:j\in J)$. By 1.14, there are f_k and g_j such that $U_k=f_k.P$, Put $h_{kj}=f_kg_j$, $\mathcal{T}=(h_{kj}.P:k\in K,j\in J)$. Then $\mathcal{T}$ is a partition of P refining both $\mathcal{U}$ and $\mathcal{V}$.

2.9. Let $P=\langle Q,\varphi,\mu\rangle\in\sigma\mathfrak{M}$ and let $\varepsilon>0$. We put $\varepsilon*P=\langle Q,\varepsilon*\varphi,\mu\rangle$, where $(\varepsilon*\varphi)(x,y)=0$ if $\varphi(x,y)\leq\varepsilon$, and $(\varepsilon*\varphi)(x,y)=1$ if $\varphi(x,y)>\varepsilon$. - Cf. [6], 1.17.

2.10. Let $P=\langle Q,\varphi,\mu\rangle\in\sigma\mathfrak{M}$, $\varepsilon>0$. Then $(X_k:k\in K)$, where $K\neq\emptyset$ is countable, $X_k \in \text{dom}\,\bar{\mu}$, will be called an ε-covering of P if diam $X_k \leq \varepsilon$ for all $k \in K$ and $\bar{\mu}(Q \setminus \bigcup X_k)=0$. If, in addition, $X_i \cap X_j=\emptyset$ for $i \neq j$, then $(X_k : k \in K)$ will be called a disjoint ε-covering of P (an ε-partition if there is no danger of confusion with the partition in the sense of 1.9 or 2.5). - Cf. [4], 1.19.

2.11. If $P=\langle Q,\rho,\mu\rangle$ is a W-space, then the infimum of all $H(\bar{\mu}X_k : k \in K)$, where $(X_k : k \in K)$ is an ε-partition of P, will be denoted by $H_\varepsilon(P)$; if there is no ε-partition of P, we put $H_\varepsilon(P)=\infty$. - Cf. [4], 1.19.

Remark. The functional $H_\varepsilon(P)$, often called the epsilon entropy, has been examined in [7] (to be precise, the $H_\varepsilon(P)$ defined above coincides with the functional in [7] up to a multiplicative constant).

2.12. A functional $\varphi : \mathfrak{M} \to \overline{R}_+$ satisfying the conditions stated in [6], 1.19 is called a Shannon functional (in the broad sense). - The conditions just mentioned include the fundamental equality $\varphi\langle Q,1,\mu\rangle=H(\mu)$ for any finite $\langle Q,1,\mu\rangle \in \mathfrak{M}$. Due to this equality, Shannon functionals (b.s.) have been called extended Shannon semientropies (in the broad sense) in [2],[3] and [5].

2.13. In this article, we consider, in fact, only one Shannon functional, namely C_E, also denoted by E; for its definition see, e.g., [4], 1.13. - The letter E will be sometimes used in a different sense, namely to denote the functional $(P_1,P_2) \mapsto d(P_1+P_2)$ defined on $\mathfrak{A}$, the class of all $(P_1,P_2) \in \mathfrak{M}\times\mathfrak{M}$ such that $P_1 \leq P$, $P_2 \leq P$ for some $P \in \mathfrak{M}$. Recall that if $P=\langle Q,\rho,\mu\rangle \in \mathfrak{M}$, then d(P) denotes the infimum of all $b \in \overline{R}_+$ such that $[\mu\times\mu]\{(x,y) \in Q\times Q : \rho(x,y) > b\}=0$.

2.14. We restate two important properties of E. - A) If (S,T) is a partition of $P \in \mathfrak{M}$, then $E(P) \leq E(S)+E(T)+H(wS,wT)E(S,T)$. If $S \leq P \in \mathfrak{M}$, then $E(S) \leq E(P)$. - See [4], 2.3.

2.15. **Proposition.** If $P=\langle Q,\rho,\mu\rangle$ is a metric W-space, then either (1) $E(\varepsilon * P)=H_\varepsilon(P)$ for all $\varepsilon > 0$, or (2) $E(\varepsilon * P)=H_\varepsilon(P)=\infty$ for all sufficiently small $\varepsilon > 0$. - See [4], 2.18.

2.16. If μ is a σ-finite measure on Q and $\emptyset \neq T \subset Q$, then T is called thick in $\langle Q,\mu\rangle$ if there are $X_n \in \text{dom}\,\mu$, $n \in N$, such that $\mu X_n < \infty$, $\bigcup X_n = Q$ and $(\mu \restriction T)(X_n \cap T)=\mu X_n$ for all $n \in N$.

2.17. Fact. Let T be thick in $\langle Q,\mu\rangle$. If $X \in \text{dom}\,\mu$, $\mu X > 0$, then $T \cap X$ is thick in $\langle X,\nu\rangle$, where ν is the restriction of μ to $\{Y \in \text{dom}\,\mu : Y \subset X\}$.

2.18. Fact and notation. Let $P=\langle Q,\rho,\mu\rangle$ be a σ W-space. Let $\emptyset \neq T \subset Q$.

Then $\langle T, \varphi, \mu\upharpoonright T\rangle$ is a σW-space, which will be denoted by $P\upharpoonright T$.

This follows easily from 1.17.

2.19. Lemma. Let $P=\langle Q,\varphi,\mu\rangle$ be a weakly Borel metric W-space. Let T be thick in $\langle Q,\mu\rangle$. Then, for any $\delta>0$, $H_\delta(P\upharpoonright T)=H_\delta(P)$.

Proof. We can assume $\mu Q>0$. Put $\nu=\mu\upharpoonright T$. Let $\delta>0$. Put $a=H_\delta(P)$, $b=H_\delta(P\upharpoonright T)$. If $(X_n:n\in N)$ is a δ-covering of P, then, clearly, $(X_n\cap T:n\in N)$ is a δ-covering of $P\upharpoonright T$; hence $b\leq a$. Suppose $b<a$ and let $b<c<a$. Then there is a δ-partition $(Y_n:n\in N)$ of $P\upharpoonright T$ such that $H(\bar\nu Y_n:n\in N)<c$. Clearly, there are sets $U_n\in\operatorname{dom}\bar\mu$ such that $Y_n=U_n\cap T$, $\bar\mu U_n=\bar\nu Y_n$. Put $X_n=U_n\cap Y_n$. Since P is weakly Borel, $X_n\in\operatorname{dom}\bar\mu$. It is easy to see that $\bar\mu X_n=\bar\nu Y_n$, $(X_n:n\in N)$ is a δ-covering of P and $H(\bar\mu X_n:n\in N)=H(\bar\nu Y_n:n\in N)<c$. This is a contradiction.

2.20. In [1], [9] and [10], the dimension and the upper (lower) dimension have been introduced for random variables with values in R^n. For W-spaces, dimensions of various kind have been introduced in [5] and [6]; they are, in fact, generalizations of concepts defined in [1], [9] and [10]. We are going to restate (see 2.21 and 2.24) some of the pertinent definitions and some simple facts. Then we introduce (2.26) the regularized Rényi dimension RD(P).

2.21. Let φ be a Shannon functional and let $P\in\mathfrak{M}$. Then φ-uw(P) (respectively, φ-ℓw(P)) denotes the upper (lower) limit of $\varphi(\delta * P)/|\log\delta|$ for $\delta\to 0$. We put φ-ud(P)= φ-uw(P)/wP, φ-ℓd(P)=φ-ℓw(P)/wP. If φ-ud(P)= φ-ℓd(P), we put φ-Rw(P)= φ-uw(P), φ-Rd(P)= φ-Rw(P)/wP. We call φ-Rd(P) and φ-Rw(P) the (exact) Rényi φ-dimension, and the φ-weight of P, respectively. If φ=E, the prefix "φ" is, as a rule, omitted. - See [5], 2.1.

2.22. Fact. If (S,T) is a partition of a W-space, then ℓw(S)+ℓw(T) $\leq$ ℓw(P) $\leq$ ℓw(S)+uw(T) $\leq$ uw(P) $\leq$ uw(S)+uw(T). - See [5], 3.1.

2.23. Lemma. Let P be a W-space and let $b\in R_+$. If ud(S) $\leq$ b for all pure $S\leq P$, then ud(T) $\leq$ b for all $T\leq P$.

Proof. Let T=f.P. Let $m\in N$, $m>1$. Put $V_k=\{x\in Q:(k-1)/m<f(x)\leq k/m\}$ for $k=0,\dots,m$, $S_k=(k/m).V_k.P$, $S=\Sigma(S_k:k=0,\dots,m)$. Then $ud(V_k.P)\leq b$, hence $ud(S_k)\leq b$ and therefore, by 2.2, uw(S) $\leq$ b.wS. Clearly, $T\leq S$, $w(S-T)\leq m^{-1}.wP$. Hence $ud(T)\leq uw(T)/wT\leq b.wS/(wS-m^{-1}.wP)$. Since m=2,3,..., has been arbitrary, we have shown that ud(T) $\leq$ b.

2.24. Let φ be a Shannon functional and let P be a W-space. Then φ-UW(P) (respectively, φ-LW(P)) denotes the infimum of all $b\in\overline{R}_+$ for which there is a partition $\mathcal{U}$ of P such that if $(V_k:k\in K)$ refines $\mathcal{U}$, then $\Sigma(\varphi$-uw(V_k):

$:k \in K) \leq b$ (respectively, $\Sigma(\varphi - \ell w(V_k):k \in K) \leq b$). We put φ-UD(P)= φ-UW(P)/wP, φ-LD(P)= φ-LW(P)/wP. - If φ=E, then the prefix " φ " is, as a rule, omitted. - See [6], 3.1.

2.25. **Proposition.** Let φ be a Shannon functional and let $P=\langle Q,\rho,\mu\rangle$ be a W-space. Then (1) if $(P_k:k \in K)$ is a partition of P, then φ-UW(P)= $=\Sigma(\varphi$-UW$(P_k):k \in K)$, φ-LW(P)= $\Sigma(\varphi$-LW$(P_k):k \in K)$, (2) the functions $X \mapsto \varphi$-UW(X.P) and $X \mapsto \varphi$-LW(X.P) are measures. - See [6], 3.2.

2.26. **Definition.** Let φ be a Shannon functional. Let P be a W-space. If φ-UD(P)= φ-LD(P), we put φ-RD(P)= φ-UD(P), φ-RW(P)= φ-UW(P). We will call φ-RD(P) (respectively, φ-RW(P)) the regularized (exact) Rényi φ-dimension (respectively, the φ-weight) of P. If φ-UD(P) $\neq$ φ-LD(P), we will say that φ-RD(P) does not exist. - If φ=E (which is the case considered in this article), we omit the prefix " φ ".

Remark. The properties of RD will not be examined in this article. We state only some simple facts to be used in the sequel.

2.27. Fact. Let (S,T) be a partition of a W-space P. If both RW(S) and RW(T) exist, then RW(P)=RW(S)+RW(T).

This is an immediate consequence of 2.25.

2.28. **Proposition.** Let P be a W-space. If RD(P) exists and is finite, then $RD(S) < \infty$ for all $S \leq P$.

Proof. By 2.25, $UW(T) \leq UW(P) < \infty$ for all $T \leq P$. For any $T \leq P$, put $\sigma(T)$= =UW(T)-LW(T). By 2.25, $\sigma(S)+\sigma(P-S)=\sigma(P)=0$ for all $S \leq P$. This implies $\sigma(S)$= =0, which proves the proposition.

2.29. Lemma. Let $P=\langle Q,\rho,\mu\rangle$ be a W-space. Let $Rd(P)=b < \infty$ and let $ud(T) \leq b$ for all pure $T \leq P$. Then RD(S)=Rd(S)=b for all non-null $S \leq P$.

Proof. By 2.23, $ud(S) \leq b$, hence $uw(S) \leq b.wS$ for all $S \leq P$. Suppose $uw(S_0) < b.wS_0$ for some $S_0 \leq P$. Then, by 2.22, $uw(P) \leq uw(S_0)+uw(P-S_0) < b.wP$, which contradicts Rd(P)=b. Hence Rd(S)=b for all non-null $S \leq P$. Consequently, Rw(S)=b.wS, RW(S)=b.wS for all $S \leq P$.

2.30. **Proposition.** Let $P=\langle R^n,\rho,\mu\rangle$ be a W-space and let μ be absolutely continuous with respect to the Lebesgue measure. Let $wP > 0$. Then (1) RD(P)=n; (2) Rd(P)=n if $H(\vec{\mu}A_z:z \in Z^n) < \infty$, where Z is the set of all integers, $A_z=\{x=(x_1,\dots,x_n) \in R^n:z_i \leq x_i < z_i+1$ for $i=1,\dots,n\}$, (3) Rd(P)= ∞ if $H(\vec{\mu}A_z:z \in Z^n)=\infty$.

Proof. For (2) and (3), see [5], 2.9. To prove (1), consider any partition of P of the form $(X_n.P:n \in N)$ with X_n bounded.

3.1. **Definition.** Let φ be a Shannon functional (b.s.) and let P be a W-space. Assume that φ-RD(P) exists and is finite. Then the limit (provided it exists) of $\varphi(\delta * P)-(\varphi\text{-RW}(P))|\log\delta|$ for $\delta \longrightarrow 0$ will be denoted by $r'\varphi(P)$. We put $r\varphi(P)=r'\varphi(P)+L(wP)$ and call $r\varphi(P)$ the residual φ-entropy of P; if φ=E, then the prefix " φ " in " φ-entropy" will be, as a rule, omitted.

3.2. Remarks. A) In this article, only the case φ=E is examined. - B) Clearly, if $P \in \mathfrak{M}$, wP=1, then $r'\varphi(P)$ and $r\varphi(P)$ coincide (provided they exist). - C) The functional $r'\varphi$ seems to be more natural than $r\varphi$. On the other hand, (1) under certain fairly mild assumptions (see 3.9), $X \longmapsto rE(X.P)$ is additive whereas $r'E$ satisfies the equality $r'E(X \cup Y).P)=r'E(X.P)+$ $+r'E(Y.P)+H(w(X.P),w(Y.P))$ and cannot be additive; (2) in many important cases, $rE\langle Q,\rho,X,\mu\rangle$ can be expressed in the form $\int_X F\, d\mu$, where F depends only on $\langle Q,\rho,\mu\rangle$ (see 5.1, 5.3 and 4.4). - D) It is possible to introduce another kind of residual entropy, say $\hat{r}\varphi(P)$, replacing RD and RW by Rd and Rw in 3.1. This notion, however, is less appropriate since, e.g., there are W-spaces of the form $P=\langle R,\rho,f.\lambda\rangle$ such that RD(P)=1, rE(P)= $-\int f \log f\, d\lambda$ whereas Rd(P)= ∞ and therefore rE(P) does not exist.

3.3. If P is a W-space, rE(P) need not exist, and even if rE(S) exists for all $S \leq P$, the function $X \longmapsto rE(X.P)$ can fail to be additive; for pertinent examples see 3.12 and 3.41. However, under some not too restrictive conditions, $X \longmapsto rE(X.P)$ is additive (see 3.9) and, under certain additional assumptions, even σ-additive (see 3.35).

3.4. Fact. Let $P=\langle Q,\rho,\mu\rangle \in \mathfrak{M}$. Let $(S_1,S_2)=(X_1.P,X_2.P)$ be a pure partition of P. Then, for any $\delta > 0$, $H_\delta(P)+L(wP) \leq H_\delta(S_1)+L(wS_1)+H_\delta(S_2)+$ $+L(wS_2)$.

Proof. We can assume that $H_\delta(S_i) < \infty$. Let $\vartheta > 0$. Choose δ-partitions $(X_{in}:n \in N)$ of S_i, i=1,2, such that $H(\bar{\mu}X_{in}:n \in N) < H_\delta(S_i)+\vartheta/2$. Clearly, $(X_{in}:i=1,2;\ n \in N)$ is a δ-partition of P, hence $H_\delta(P) \leq H(\bar{\mu}X_{in}:i=1,2;n \in N)+$ $+L(wP)=H(\bar{\mu}X_{1n}:n \in N)+L(wS_1)+H(\bar{\mu}X_{2n}:n \in N)+L(wS_2) < H_\delta(S_1)+H_\delta(S_2)+L(wS_1)+$ $+L(wS_2)+\vartheta$. Since $\vartheta > 0$ has been arbitrary, the assertion is proved.

3.5. Notation. If $P \in \mathfrak{M}$, RD(P) exists and is finite, we put, for any $\delta > 0$, $\psi(\delta,P)=E(\delta * P)-RW(P)|\log\delta| +L(wP)$.

3.6. Fact. Let (S,T) be a pure partition of $P \in \mathfrak{M}$ and let RD(P)=RD(S)= =RD(T)=t, $0 < t < \infty$. Let rE(P), rE(S) and rE(T) exist. Assume that the sum

$rE(S)+rE(T)$ exists. Then $rE(P) \leq rE(S)+rE(T)$.

Proof. By 3.4, we have $\psi(\delta,P) \leq \psi(\delta,S)+\psi(\delta,T)$ for all $\delta > 0$. Since $\psi(\delta,P)$, $\psi(\delta,S)$ and $\psi(\delta,T)$ converge to $rE(P)$, $rE(S)$ aand $rE(T)$, respectively, we get $rE(P) \leq rE(S)+rE(T)$.

3.7. **Definition.** A) If μ and ν are measures on Q, $\nu \subset \mu$, and, for any $X \in \mathrm{dom}\,\mu$, there is a set $Y \in \mathrm{dom}\,\nu$ such that the symmetric difference $X \triangle T$ is μ-null, we will say that μ is a faithful extension of ν. - B) A metric σW-space $P=\langle Q,\rho,\mu\rangle$ will be called almost Borel if $\mathcal{B}\langle Q,\rho\rangle \subset \mathrm{dom}\,\bar{\mu}$ and $\bar{\mu}$ is a faithful extension of $\mu \upharpoonright \mathcal{B}\langle Q,\rho\rangle$.

3.8. Lemma. Let $P=\langle Q,\rho,\mu\rangle$ be an almost Borel metric W-space and let $E(\varepsilon * P) < \infty$ for all $\varepsilon > 0$. Let $(S,T)=(X.P,Y.P)$ be a pure partition of P. Then $E(\delta * S)+E(\delta * T)-E(\delta * P) \to H)wS,wT)$ for $\delta \to 0$. If, in addition, RD(P) exists and is finite, then $\psi(\delta,S)+\psi(\delta,T)-\psi(\delta,P) \to 0$.

Proof. I. We can assume that X is Borel and $Y=Q \setminus X$. Let $\vartheta > 0$. By well-known theorems, there is a closed $X^* \subset X$ such that $\bar{\mu}(X \setminus X^*) < \vartheta$. Since X^* is closed, there is an $\alpha > 0$ such that $\mu(Y \setminus Y^*) < \vartheta$, where $Y^* = \{y \in Y: \rho(y,X^*) > \alpha\}$. - Let $0 < \delta < \alpha$. By 2.15 and 2.11, there exists a δ-partition $(U_n:n \in N)$ of P such that $H(\mu U_n:n \in N) < E(\delta * P)+\vartheta$. Put $K_X = \{n \in N:U_n \cap X^* \neq \emptyset\}$ $K_Y=\{n \in N:U_n \cap Y^* \neq \emptyset\}$, $M=N \setminus (K_X \cup K_Y)$. Then $U_n \cap X$, $n \in K_X \cup M$, form a δ-partition of S whereas $U_n \cap Y$, $n \in K_Y \cup M$, form a δ-partition of T. Clearly, $U_n \subset (X \setminus X^*) \cup (Y \setminus Y^*)$ whenever $n \in N$, hence $\Sigma(\mu U_n:n \in M) < 2\vartheta$. For $n \in N$, put $a_n=\bar{\mu}U_n$, $b_n=\bar{\mu}(U_n \cap X)$, $c_n=\bar{\mu}(U_n \cap Y)$. Then we have $\Sigma(Lb_n:n \in M)+ \Sigma(Lc_n:n \in M)= \Sigma(La_n:n \in M)+ \Sigma(H(b_n,c_n):n \in M) \leq \Sigma(La_n:n \in M)+ \Sigma(a_n:n \in M) \leq \Sigma(La_n:n \in M)+2\vartheta$, hence $H(b_n:n \in N)+H(c_n:n \in N) \leq H(a_n:n \in N)-H(wS,wT)+2\vartheta$. Since $H(a_n:n \in N) < E(\delta * P)+\vartheta$, we get $(*)$ $E(\delta * S)+E(\delta * T) \leq E(\delta * P) - H(wS,wT)+3\vartheta$. Thus, for any $\vartheta > 0$ there is an $\alpha > 0$ such that the inequality $(*)$ is satisfied whenever $0 < \delta < \alpha$. On the other hand, by 3.4 and 2.15, $E(\delta * S)+E(\delta * T) \geq E(\delta * P)-H(wS,wP)$. This proves the first assertion. - II. If RD(P) exists and is finite, then, by 2.27, RW(P)=RW(T), which easily implies the second assertion.

3.9. **Proposition.** Let $P=\langle Q,\rho,\mu\rangle$ be an almost Borel metric W-space. Let $E(\varepsilon * P) < \infty$ for all $\varepsilon > 0$. Let (S,T) be a pure partition of P. If both rE(S) and rE(T) exist, then $rE(P)=rE(S)+rE(T)$, unless rE(S) and rE(T) are infinite and $rE(S)= -rE(T)$. If both rE(P) and rE(S) exist, then $rE(T)=rE(P) - rE(S)$ unless $rE(P)=rE(S)= \pm\infty$.

Proof. To prove the first assertion, observe that the existence of rE(S) and rE(T) implies that RD(S) and RD(T) exist and are finite. Hence, by

2.27, RW(P)=RW(S)+RW(T). By 3.8, this implies $\psi(\delta,S)+\psi(\delta,T)-\psi(\delta,P)\to$ $\to 0$, from which the assertion follows at once. The proof of the second assertion is similar.

3.10. We are going to present some examples showing that rE can behave rather irregularly (though being additive in the sense described in 3.9) even on fairly simple almost Borel metric W-spaces. The examples also show that the class of metric W-spaces P satisfying $RD(P)<\infty$ is too broad to allow a sufficiently rich theory of the residual entropy (or of the regularized residual entropy RE, see Section 4). Hence we have to choose a suitable subclass for which a reasonable theory of this kind can be developed. Probably the subclass we introduce (see 3.19 and 3.20) is too narrow, though.

3.11. In 3.12 - 3.14 the following notation will be used. The set $\{0,1\}^\omega$ is denoted by Q. If $p=(p_n:n\in N)$, $1/2\leq p_n<1$, then $S[p]=\langle Q,\mu[p]\rangle$ will denote the product of probability spaces $\langle\{0,1\},\nu_n\rangle$, where $\nu_n\{0\}=p_n$, $\nu_n\{1\}=q_n=1-p_n$. Instead of $\mu[p]$ we often write merely μ. If, in addition, $a=(a_n:n\in N)$, $a_n>0$, $a_n\geq a_{n+1}$, $a_n\to 0$ for $n\to\infty$, then $S[p,a]$ will denote the W-space $\langle Q,\varphi_a,\mu[p]\rangle$ where $\varphi_a(x,y)=a_m$ if $x=(x_n)$, $y=(y_n)$, $x_m\neq y_m$ and $x_i=y_i$ for $i<m$. - If $n\in N$, $z\in\{0,1\}^n$, then A(z) denotes the set $\{x=(x_n)\in Q:x_i=z_i$ for $i<n\}$. The collection of all A(z) will be denoted by $\mathcal{A}$, and that of all A(z), $z\in\{0,1\}^n$, will be denoted by $\mathcal{A}_n$.

3.12. Example. For $n\in N$ let $p_n=1/2$, $a_n=2^{-n}$. Put $p=(p_n:n\in N)$, $a=(a_n:n\in N)$, $P=\langle Q,\varphi,\mu\rangle=S[p,a]$. We are going to show that rE(S) exists for no non-null $S\leq P$.

Let $S\leq P$, $S=f.P$, $wS>0$. Let $2^{-n+1}>\delta\geq 2^{-n}$. Then, clearly, $E(\delta*S)=E(a_n*S)=H(w(A.S):A\in\mathcal{A}_n)$. Hence, by 1.22.1, (1) $E(\delta*S)\leq n.wS$ and, by 1.22.3, (2) $E(\delta*S)\geq n.wS-L(wS)$. This proves that Rd(S)=1. Consequently, RD(T)= =Rd(T)=1 for any non-null $T\leq P$. - If $S=f.P\leq P$, then $E(\delta*S)$ is constant on each interval $(2^{-n+1},2^{-n}]$ and therefore the oscillation of $\psi(\delta,S)$ on $(2^{-n+1},2^{-n}]$ is equal to that of $wS|\log\delta|$, hence to wS. Since, by (1) and (2), $0\leq\psi(\delta,S)\leq L(wS)$, this proves that $\psi(\delta,S)$ has no limit for $\delta\to 0$.

3.13. Example. Let $c>0$. Let P=S[p,b], where p is as in 3.12, $b_n=\exp(-c^{-1}n)$. It is easy to show that, for any non-null $S\leq P$, (1) RD(S)=Rd(S)= =c, (2) $rE(S)=\infty$ if $c>1$, $rE(S)=-\infty$ if $c<1$.

3.14. Example. For $n\in N$, let $q_n=(n+2)^{-1}$, $p_n=1-q_n$, $h(n)=H(p_n,q_n)$, $s_n=\sum(h(m):m<n)$, $a_n=\exp(-s_n)$. Put $P=\langle Q,\varphi,\mu\rangle=S[p,a]$. Then (1) Rd(P)=1, (2) $Rd(S)\leq 1$ for any non-null pure $S\leq P$, and therefore (3) RD(S)=Rd(S)=1 for any non-null $S\leq P$, (4) $rE(A.P)=L(\mu A)-s_n.\mu A$ for any $A\in\mathcal{A}_n$, hence

$rE(P)=0$, (5) there is a disjoint countable collection $\mathfrak{X} \subset \mathcal{R}$ such that $\mu(P\setminus\bigcup\mathfrak{X})=0$, $\Sigma(rE(A.P):A\in\mathfrak{X})<0$.

Since the example is merely illustrative, we omit the proof (which is rather long and not quite easy) of the facts just mentioned.

3.15. Before introducing partition-regular spaces (see 3.20) we consider (in 3.16 - 3.18) the case of metric W-spaces satisfying $RD(P)=0$, which turns out to be quite simple.

3.16. **Proposition.** If P is a W-space and $RD(P)=0$, then $rE(P)$ exists and is equal to $\lim_{\delta\to 0} E(\delta * P)+L(wP)$.

Proof. Clearly, $E(\delta * P)$ is a non-decreasing function of δ and $\psi(\delta,P)=E(\delta * P)+L(wP)$.

3.17. **Proposition.** Let $P=\langle Q,\rho,\mu\rangle$ be a non-null metric W-space such that $RD(P)=0$ and $\mu\{x\}=0$ for all $x\in Q$. Then $rE(P)=\infty$.

Proof. We can assume that $wP=1$. Suppose that $rE(P)<\infty$, hence $\sup E(\delta * P)<a<\infty$. Then, for any $n=1,2,\ldots$, there is an n^{-1}-partition $(X_{n\,m}:m\in N)$ of P such that $H(\mu X_{n\,m}:m\in N)<a$. By 1.22.3, we have $-\log(\sup \mu X_{m\,n}:m\in N)<a$, and hence, for some $m=m(n)$, $\mu X_{n,m(n)}>2^{-a}$. Put $Y_n=X_{n,m(n)}$. If no $x\in Q$ is in infinitely many Y_n, then $\bigcap(\bigcup(Y_k:k>n):n\in N)=\emptyset$, whereas $\mu(\bigcup(Y_k:k>n))>2^{-a}$ for all n. Hence there is a point $y\in Q$ and an infinite $K\subset N$ such that $y\in Y_k$ for all $k\in K$. For $n\in N$, put $Z_n=\bigcup(Y_k:k\in K,k>n)$. It is easy to see that $\mu Z_n>2^{-a}$ for all $n\in N$ and diam $Z_n\leq 2n^{-1}$, hence $\bigcap Z_n=\{y\}$. Since $\mu\{y\}=0$, we have got a contradiction.

3.18. **Theorem.** Let $P=\langle Q,\rho,\mu\rangle$ be a metric W-space and let $RD(P)=0$. Put $A=\{x\in Q:\mu\{x\}>0\}$, $B=Q\setminus A$. Then (1) for any subspace $S=f.P\leq P$, $rE(S)=H(f(a)\mu\{a\}:a\in A)$ if $w(B.S)=0$, and $rE(S)=\infty$ if $w(B.S)>0$, (2) the function $X\mapsto rE(X.P)$ is a measure defined on dom $\bar{\mu}$.

Proof. If $\mu B>0$, then, by 3.12 and 3.16, $E(\delta *(B.P))\to\infty$, so that $E(\delta * P)\to\infty$, $rE(P)=\infty$. - Let $\mu B=0$. Then $(\{a\}:a\in A)$ is an ε-partition of P for any $\varepsilon>0$ and therefore $\sup E(\delta * P)\leq H(\mu\{a\}:a\in A)$. On the other hand, if $K\subset A$ is finite non-void, choose a positive $\delta<\inf\{\rho(x,y):x\in K,y\in K,x\neq y\}$. It is easy to see that $E(\delta * P)\geq H(\mu\{x\}:x\in K)$. Since clearly $H(\mu\{a\}:a\in A)=\sup(H(\mu\{a\}:a\in K):K\subset A$ finite), this proves the assertion (1) for S=P. The general case of (1) and the assertion (2) are easy consequences.

3.19. Convention. Let t and b be positive reals, let $m\in N$ and let $f:(0,b)\to R_+$ satisfy $f(\delta)\to 1$ for $\delta\to 0$. We will say that a semimetric space

$S=\langle Q,\rho\rangle$ satisfies PR(t,m,b,f) if, for any $X\subset Q$ with diam $X=\delta<b$ and any $n>m$, there are $Y_i\subset X$, $i=1,\ldots,k\leq n^t f(\delta)$, such that $\bigcup Y_i=X$ and diam $Y_i\leq\delta/n$, $i=1,\ldots,k$. We will say that a W-space $P=\langle Q,\rho,\mu\rangle$ satisfies PR(t,m,b,f) if, for any $X\in\operatorname{dom}\bar{\mu}$ with diam $X=\delta<b$ and any $n>m$, there are $Y_i\in\operatorname{dom}\bar{\mu}$, $i=1,\ldots,k\leq n^t f(\delta)$, such that $\bar{\mu}(X\setminus\bigcup Y_i)=0$ and diam $Y_i\leq\delta/n$, $i=1,\ldots,k$.

3,20. **Definition.** Let $0<t<\infty$. A semimetric space S (respectively, a W-space P) will be called partition-regular of order t if, for some m, b and f, S satisfies PR(t,m,b,f) (respectively, P satisfies PR(t,m,b,f) and Rd(T)=t for all non-null $T\leq P$).

3.21. Fact. Let $0<t<\infty$. Let $P=\langle Q,\rho,\mu\rangle$ be a weakly Borel metric W-space and let Rd(P)=t. If, for some b, m and f, there is a $Q'\subset Q$ such that $\bar{\mu}(Q\setminus Q')=0$ and $\langle Q',\rho\rangle$ satisfies PR(t,m,b,f), then P also satisfies TR(t,m,b,f) and is partition-regular of order t.

Proof. Choose a positive $\delta<b$ such that $\log|f(\delta)|<1$. Let $S=\langle Q,\rho,\nu\rangle\leq P$. Since $Rd(P)<\infty$, we have $E(\varepsilon * P)<\infty$ for all $\varepsilon>0$ and therefore, by 2.14 B, $E(\delta * S)<\infty$. Consequently, there is a δ-partition $(X_p:p\in N)$ of S such that $H(\nu X_p:p\in N)=c<\infty$. Since $\langle Q',\rho\rangle$ satisfies PR(t,m,b,f), there exist, for any $p\in N$ and any $n>m$, sets Y_{pi}, $i=1,\ldots,k(p)\leq n^t f(\delta)$, such that, with $K_p=\{1,\ldots,k(p)\}$, we have $Q'\cap X_p=\bigcup(Y_{pi}:i\in K_p)$ and diam $Y_{pi}\leq\delta/n$. Clearly, we can assume that $Y_{pi}\in\operatorname{dom}\bar{\mu}$ (since Y_{pi} can be replaced by the sets $Y_{pi}\cap Q'$). For any $p\in N$, put $Z_{p1}=Y_{p1}$, $Z_{pi}=Y_{pi}\setminus\bigcup(Y_{pj}:j<i)$ for $i=2,\ldots,k(p)$. Clearly, $(Z_{pi}:p\in N,i\in K_p)$ is a (δ/n)-partition of S. By 1.22.2 and 1.22.1, we have $H(\nu Z_{pi}:p\in N,i\in K_p)\leq c+\sum(\nu X_p.\log k(p):p\in N)$, hence $H(\nu Z_{pi}:p\in N,i\in K_p)\leq c+wS.(t\log n+1)$. Consequently, $E((\delta/n)*S)\leq c+wS.(t\log n+1)$ and therefore $\overline{\lim}_{n\to\infty}(E((\delta/n)*S)/wS.|\log(\delta/n)|)\leq t$. It is easy to see that this inequality implies $\overline{\lim}_{\varepsilon\to 0}(E(\varepsilon * S)/wS.|\log\varepsilon|)\leq t$, hence $ud(S)\leq t$. Since $S\leq P$ has been arbitrary, this proves, by 2.29, that Rd(T)=t for all non-null $T\leq P$.

3.22.Facts. A) If a semimetric space (respectively, a W-space) is partition-regular of order t, then each of its subspaces (respectively, each of its non-null subspaces) is partition-regular of order t. - B) If, for i=1,2, S_i is a semimetric space partition-regular of order t, then $S_1\times S_2$ is partition-regular of order t_1+t_2. - C) If, for i=1,2, P_i is a W-space partition-regular of order t_i and $Rd(T)=t_1+t_2$ for each non-null $T\leq P_1\times P_2$, then $P_1\times P_2$ is partition-regular of order t_1+t_2.

3.23. Fact. The space R^n, n=1,2,..., is partition-regular of order n.

3.24. **Proposition.** Let S be an m-dimensional C^1-submanifold of R^n equipped with the ℓ_∞-metric. Then every compact $T \subset S$ is partition-regular of order m.

The proof is straightforward and can be omitted. Observe that S itself need not be partition-regular (however, cf. 4.22).

3.25. Lemma. Let P be a partition-regular W-space of order t. Then there is a function $f:R_+ \to R_+$ such that $f(\delta) \to 1$ for $\delta \to 0$, a positive real b and an $m \in N$ such that if $S \leq P$, $0<\delta< b$, $n \in N$, $n>m$, then $\psi(\delta/n,S) \leq$ $\leq \psi(\delta,S)+wS.\log f(\delta)$.

Proof. Let $P=\langle Q,\rho,\mu\rangle$ satisfy PR(t,m,b,f). Let $S=\langle Q,\rho,\nu\rangle \leq P$ and let $0<\delta< b$, $n>m$. Let $\vartheta>0$. By 2.15, there is a δ-partition $(X_k:k\in N)$ of S such that $H(\bar\nu X_k:k\in N)< E(\delta * S)+\vartheta$. Since diam $X_k \leq \delta$, there are $Y_{kj} \in$ $\in \operatorname{dom}\bar\nu$, $j=1,\dots,p(k)$, $p(k)\leq n^t f(\delta)$, such that diam $Y_{kj} \leq \delta/n$, $\bar\nu(X_k \setminus$ $\setminus \bigcup(Y_{kj}:j=1,\dots,p(k)))=0$. Clearly $(Y_{kj}:k\in N, j=1,\dots,p(k))$ is a (δ/n)-partition of S. By 1.22.2 and 1.22.1, $H(\bar\nu Y_{kj}:k\in N,j=1,\dots,p(k)) \leq H(\bar\nu X_k:k\in N)+$ $+\log(n^t f(\delta))$. $\Sigma(\bar\nu X_k:k\in N)< E(\delta * S)+\vartheta$ +wS.t log n+wS.log $f(\delta)$. Hence $E((\delta/n)*S) \leq E(\delta * S)$+wS.t log n+wS.log $f(\delta)$, and therefore $\psi(\delta/n,S) \leq$ $\leq \psi(\delta,S)$+wS.log $f(\delta)$.

3.26. **Theorem.** If a W-space P is partition-regular, then the residual entropy of P exists.

Proof. Put t=Rd(P)=RD(P). Let P satisfy PR(t,b,m,f). Put $s=\overline{\lim}\ \psi(\delta,P)$. Clearly, we can assume $s>-\infty$. Let $-\infty<u<s$ and choose $\varepsilon>0$ such that $u+2\varepsilon<s$. Choose $c>0$ such that $2c<\min(b,1)$, wP.$|\log f(\delta)|<\varepsilon$ for $\delta\in(0,c)$. - We are going to show that $\psi(\delta,P)\geq u$ whenever $0<\delta<c$. Choose $\tau\in(0,\delta)$ such that $|\eta-\delta|<\tau$ implies $|\log\eta-\log\delta|<\varepsilon.(t.wP)^{-1}$. Since $\overline{\lim}_{z\to 0}\ \psi(z,P)>u+wP.(t+1)\varepsilon$, there is a positive ξ such that $\xi<\tau$, $\delta+\xi<c$, $\psi(\xi,P)>u+2\varepsilon$. Choose $p\in N$ such that $(p-1)\xi\leq\delta$, $p\xi>\delta$. Clearly, $p\xi<c$. By 3.25, we have $\psi(p\xi,P)\geq\psi(\xi,P)$-wP.log $f(p\xi)$. Clearly, $\psi(\delta,P)\geq E((p\xi)*P)$-wP.t$|\log\delta|$+L(wP)= $\psi(p\xi,P)$+wP.t($|\log p\xi|$ - $-|\log\delta|)\geq\psi(p\xi,P)-\varepsilon$. Hence, $\psi(\delta,P)\geq\psi(\delta,P)-2\varepsilon>u$ whenever $\delta\in(0,c)$. Since $u<s$ has been arbitrary, this proves that $\lim\psi(\delta,P)=s$.

Remark. The proof is similar to a part of the proof of Theorem 1 in [8].

3.27. The concept of residual entropy appears implicitly in [8], where the behavior of $\varepsilon\mapsto H_\varepsilon(P)=E(\varepsilon * P)$ is examined for the case $P=\langle R^n,\rho_\tau,\mu\rangle$, $\mu R^n=1$, $\rho_\tau(x,y)=\tau(x-y)$, τ being a norm on R^n. In [8], two theorems are proved, which can be stated, in a modified form and using the terminology of the present article, as follows: (1) If $Q \subset R^n$ is a unit cube, $\mu = Q.\lambda$, then $H_\varepsilon(P) - n|\log \varepsilon|$ converges, for $\varepsilon \to 0$, to $-\log \lambda S_\tau + \varkappa(\tau)$, where $S_\tau = \{x \in R^n : \tau(x) \leq 1/2\}$, $\varkappa(\tau)$ depends on τ , $0 \leq \varkappa(\tau) \leq 1$, and $\varkappa(\tau) = 0$ if τ is the ℓ_∞-norm. - (2) If $\mu = p.\lambda$, p is continuous and satisfies certain conditions (which we do not restate), then $H_\varepsilon(P) - n|\log \varepsilon|$ converges to $-\int p \log p \, d\lambda - \log \lambda S_\tau + \varkappa(\tau)$. - In the terminology of the present article, the theorems assert that, under the assumptions mentioned above, rE(P) exists, and provide a formula for its value.

Observe that, apart from the fact that we explicitly introduce the residual entropy rE, the difference of approach in the present article and in that by Posner and Rodemich lies, among other things, in the following fact. In [8], the class of metric spaces under consideration contains R^n, n=1,2,..., equipped with any metric generated by a norm (and, in fact, all of their subspaces); certain assumptions, not quite weak, are made concerning the measure. In the present article, the class of metric spaces for which a reasonable theory of the entropy rE is available, consists of partition-regular ones, whereas the assumptions on the measure are fairly weak.

3.28. Fact. Let $P = \langle Q, \varphi, \mu \rangle \in \mathcal{M}$ be partition-regular. Then $rE(P) = \infty$ iff $E(\delta * P) = \infty$ for some $\delta > 0$.

Proof. Let P satisfy PR(t,b,m,f). Choose a positive c such that $|\log f(\delta)| < 1$ if $0 \leq \delta < c$. Let $0 < \delta < \min(t,c)$. Then, by 3.25, for any $n \in N$, $n > m$, we have $\psi(\delta/n, P) \leq \psi(\delta, P) + wP$, hence $rE(P) \leq \psi(\delta, P) + wP$. Consequently, if $\psi(\delta, P) < \infty$ for all $\delta > 0$, then $rE(P) < \infty$. - Clearly, if $E(\delta * P) = \infty$ for some $\delta > 0$, then $\psi(\varepsilon, P) = \infty$ for all positive $\varepsilon \leq \delta$.

3.29. Fact. If $x_n \geq 0$, $\sum x_n < \infty$, $H(x_n : n \in N) < \infty$, then, for any $\varepsilon > 0$, there exists a positive δ such that $\sum (Ly_n : n \in N) < \varepsilon$ whenever $0 \leq y_n \leq x_n$ for $n \in N$ and $\sup(y_n : n \in N) < \delta$.

3.30. Fact. Let $P \in \mathcal{M}$, let $\delta > 0$ and let $E(\delta * P) < \infty$. Then, for any $\varepsilon > 0$, there is an $\eta > 0$ such that $E(\delta * S) < \varepsilon$ whenever $wS < \eta$.

This follows easily from 2.15 and 3.29.

3.31. Fact. Let $P = \langle Q, \varphi, \mu \rangle \in \mathcal{M}$ be partition-regular. Then, for any $\varepsilon > 0$, there is a $\vartheta > 0$ such that, for any positive $\delta < \vartheta$ and any $S \leq P$, $rE(S) \leq \psi(\delta, S) + \varepsilon$.

This immediately follows from 3.25.

3.32. **Proposition.** Let $P = \langle Q, \varphi, \mu \rangle$ be a partition-regular W-space. If $rE(P) < \infty$, then (1) $rE(S) < \infty$ for all $S \leq P$, (2) for any $\varepsilon > 0$, there is an $\eta > 0$ such that $rE(S) < \varepsilon$ whenever $S \leq P$, $wS < \eta$.

Proof. The first assertion immediately follows from 3.26 and 3.28. - Let $\varepsilon > 0$. Choose a $\vartheta > 0$ satisfying the condition stated in 3.31. Choose a positive $\sigma < \vartheta$. By 3.30, there is an $\eta > 0$ such that $E(\sigma * S) < \varepsilon$ whenever $wS < \eta$. Then, for any $S \leq P$ satisfying $wS < \eta$, we have $rE(S) \leq \psi(\sigma, S) + \varepsilon$, $\psi(\sigma, S) \leq E(\sigma * S) + L(wS)$, $E(\sigma * S) < \varepsilon$, hence $rE(S) \leq 2\varepsilon + L(\eta)$. This proves the proposition.

3.33. Lemma. Let $P = \langle Q, \rho, \mu \rangle \in \mathcal{M}$ be partition-regular. If $rE(P) < a < < \infty$, then there is a positive ε such that $rE(S) < a$ whenever $S \leq P$ and $w(P-S) < \varepsilon$.

Proof. Let P satisfy PR(t,b,m,f). Choose $\vartheta > 0$ such that $rE(P) < a - 4\vartheta$. Choose $\sigma > 0$ such that $\sigma < b$, $\psi(\sigma, P) < a - 4\vartheta$ and $wP.\log f(\sigma) < \vartheta$. Choose $\varepsilon > 0$ such that $\varepsilon t |\log \sigma| < \vartheta$, $|L(wS) - L(wP)| < \vartheta$ if $w(P-S) < \varepsilon$. - Let $S \leq P$, $w(P-S) < \varepsilon$. Clearly, $E(\sigma * S) \leq E(\sigma * P)$, and hence $\psi(\sigma, S) \leq$ $\leq E(\sigma * P) - wS.t|\log \sigma| + L(wS) = \psi(\sigma, P) + w(P-S).t|\log \sigma| + L(wS) - L(wP) \leq$ $\leq \psi(\sigma, P) + 2\vartheta$. By 3.25, we have, for any $n > m$, $\psi(\sigma/n, S) \leq \psi(\sigma, S) + wS.$ $.\log f(\sigma)$, so that $\psi(\sigma/n, S) \leq \psi(\sigma, P) + 3\vartheta < a - \vartheta$. Hence $rE(S) = \lim_{n \to \infty} \psi(\sigma/n, S) \leq a - \vartheta < a$.

3.34. Lemma. Let $P = \langle Q, \rho, \mu \rangle$ be a partition-regular almost Borel metric W-space. Let $rE(P) < \infty$. If $S \leq P$, $S_n \leq S$, $n \in N$, and $w(S - S_n) \to 0$, then $rE(S_n) \to rE(S)$.

Proof. By 3.32, $rE(S) < \infty$. If $rE(S) = -\infty$, then the assertion immediately follows from 3.33. Let $rE(P) = a \in R$. Let $\vartheta > 0$. By 3.32 and 3.33, there is an $\varepsilon > 0$ such that (1) if $T \leq S$, $w(S-T) < \varepsilon$, then $rE(T) < a + \vartheta$, (2) if $U \leq S$, $wU < \varepsilon$, then $rE(U) < \vartheta$. Hence, for n sufficiently large, $rE(S_n) < < a + \vartheta$, $rE(S - S_n) < \vartheta$. Since, by 3.9, $rE(S_n) + rE(S - S_n) = rE(S) = a$, we have $rE(S_n) > a - \vartheta$. Since $\vartheta > 0$ has been arbitrary, the lemma is proved.

3.35. **Theorem.** Let $P = \langle Q, \rho, \mu \rangle$ be a partition-regular almost Borel metric space. If $rE(P) < \infty$, then the function $X \mapsto rE(x.P)$, defined on dom $\bar{\mu}$, is σ-additive and bounded from above.

Proof. By 3.9 and 3.34, the function $Y \mapsto rE(X.P)$ is σ-additive. By 3,32, 3.33 and 3.9, it is bounded from above.

3.36. **Definition.** A metric σW-space P will be called totally bounded if, for any $\varepsilon > 0$, there is a finite ε-covering of P.

3.37. Proposition. If a metric W-space is totally bounded, then $E(\delta * P) < \infty$ for all $\delta > 0$. If, in addition, P is partition-regular, then $rE(P) < \infty$.

Proof. The first assertion is obvious. The second assertion follows from 3.26 and 3.28.

3.38. Proposition. Let $P = \langle R^n, \rho, \lambda \rangle$ and let $X \subset R^n$, $X \in \operatorname{dom} \lambda$, $\lambda X < \infty$. Let $\mathcal{A}$ consist of all sets of the form $\{x \in R^n : z_i \leq x_i < z_i + 1 \text{ for } i = 1, \dots, n\}$ where $(z_1, \dots, z_n) \in Z^n$, Z is the set of all integers. Then $rE(X.P) = 0$ if $H(\lambda(A \cap X) : A \in \mathcal{A}) < \infty$, $rE(X.P) = \infty$ if $H(\lambda(A \cap X) : A \in \mathcal{A}) = \infty$.

Proof. I. Let X be a cube $[a,b]^m$, Put $S = X.P$, $c = b - a$. Let $m = 1, 2, \dots$ and let $\delta = c/m$. If $Y \subset R^n$, $Y \in \operatorname{dom} \lambda$, $\operatorname{diam} Y \leq \delta$, then evidently $\lambda Y \leq \delta^n$. Hence, by 1.22.3, for any δ-partition $(Y_k : k \in K)$ of S, we have $H(\lambda Y_k : k \in K) \geq -L(c^n) - c^n \log \delta^n$. On the other hand, clearly, there is a δ-partition $(U_k : k \in K)$ of S such that $H(\lambda U_k : k \in K) = -c^n \log \delta^n - L(c^n)$. This proves that $E(\delta * S) = -c^n \log \delta^n - L(c^n)$ and therefore $\psi(\delta, S) = 0$ for $\delta = c/m$, $m = 1, 2, \dots$. By 3.26, this implies $rE(S) = 0$. - II. Let X be bounded. Let Q be a cube containing X. Let $\mathcal{K}$ be the collection of all $Y \subset Q$ such that $Y \in \operatorname{dom} \lambda$, $rE(Y.P) = 0$. By 3.35, the function $Y \mapsto rE(Y.P)$ is σ-additive; hence $\mathcal{K}$ is a σ-algebra of subsets of Q. By I, $rE(Y.P) = 0$ whenever Y is a cube contained in Q. Consequently, $\mathcal{K}$ contains all λ-measurable subsets of Q and therefore $rE(X.P) = 0$. - III. Let $X \subset R^n$ be an arbitrary λ-measurable set satisfying $H(\lambda(A \cap X) : A \in \mathcal{A}) < \infty$. For $m = 1, 2, \dots$, let $\mathcal{A}_m$ denote the collection of all cubes of the form $\{x \in R^n : 2^m x \in A\}$, where $A \in \mathcal{A}$. Clearly, $H(\lambda(A \cap X) : A \in \mathcal{A}_m) < \infty$. Hence, $E(\delta * (X.P)) < \infty$ whenever $\delta = 2^{-m}$, $m \in N$, and therefore $E(\delta * (X.P)) < \infty$ for all $\delta > 0$. By 3.26 and 3.28, this implies $rE(X.P) < \infty$. Consequently, by 3.35, the function $Y \mapsto rE(Y.P)$, where $Y \in \operatorname{dom} \lambda$, $Y \subset X$, is σ-additive. Combined with II, this implies $rE(X.P) = 0$. - IV. Let $X \subset R^n$, $X \in \operatorname{dom} \lambda$, $H(\lambda(A \cap X) : A \in \mathcal{A}) = \infty$. Put $S = X.P$, Suppose that $rE(S) < \infty$. Then, by 3.28, $E(\delta * S) < \infty$ for all $\delta > 0$. Let $\delta = 1/3$. By 2.15, there is a δ-partition $(X_k : k \in K)$ of S such that $H(\lambda X_k : k \in K) < \infty$. Clearly, for any $k \in K$, there are at most 2^n sets $A \in \mathcal{A}$ such that $X_k \cap A \neq \emptyset$. By 1.22.2 and 1.22.1, this implies that $H(\lambda(X_k \cap A) : k \in K, A \in \mathcal{A}) \leq H(\lambda X_k : k \in K) + n \lambda X < \infty$. Since $H(\lambda A : A \in \mathcal{A}) \leq H(\lambda(X_k \cap A) : k \in K, A \in \mathcal{A})$, we have got a contradiction.

4

4.1. Fact. Let $P = \langle Q, \rho, \mu \rangle$ be a σW-space. Then there exists at most one $b \in \overline{R}$ such that the following condition holds: (*) for any neighborhood G of b in $\overline{R}$, there exists a pure partition $\mathcal{P}$ of P consisting of W-spaces and such that if $(S_k:k\in K)$ is a pure partition refining $\mathcal{P}$, then all $rE(S_k)$ exist, the sum $\Sigma(rE(S_k):k\in K)$ exists and is in G.

Proof. Suppose that ($*$) is satisfied for $b=b_1$ and $b=b_2$ where $b_1\neq b_2$. Let G_i be a neighborhood of b_i, i=1,2, and let $G_1\cap G_2=\emptyset$. Then there are pure partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ consisting of W-spaces and such that $\Sigma(rE(S_k):k\in K)\in G_i$ whenever $(S_k:k\in K)$ is a pure partition refining $\mathcal{P}_i$. Let $\mathcal{P}_i=(X_{ik}.P:k\in K_i)$, i=1,2. Put $Y_{kj}=X_{1k}\cap X_{2j}$ for $k\in K_1$, $j\in K_2$. Then $\Sigma(rE(Y_{kj}.P):k\in K_1, j\in K_2)\in G_1$, $\Sigma(rE(Y_{kj}.P):k\in K_1,j\in K_2)\in G_2$, which is a contradiction.

4.2. **Definition.** Let $P=\langle Q,\rho,\mu\rangle$ be a σW-space. If there exists a $b\in\overline{R}$ such that the condition ($*$) from 4.1 holds, then this b will be denoted by RE(P) and called the regularized residual entropy of P.

4.3. **Proposition.** Let P be a metric W-space. Assume that either (1) RD(P)=0, or (2) P is almost Borel partition-regular and $rE(P)<\infty$. Then RE(S)=rE(S) for all $S\preceq P$ and the function $X\mapsto RE(X.P)$ is σ-additive.

Proof. In the case (1), the assertion follows from 3.18. In the case (2), rE(S) exists for all $S\preceq P$ by 3.26, and $X\mapsto RE(X.P)$ is σ-additive by 3.35. Let $S\preceq P$. It is easy to see that S is almost Borel partition-regular. By 3.35, $rE(S)<\infty$. Hence $X\mapsto rE(X.S)$ is σ-additive and therefore RE(S)=rE(S).

4.4. **Proposition.** If $P=\langle R^n,\rho,\lambda\rangle$ and $X\in\mathrm{dom}\,\lambda$, then RE(X.P)=0.

Proof. Put S=X.P. Consider the pure partition $\mathcal{S}=([n,n+1].S:n=0,\pm 1,\dots)$ of S. By 3.38, we get $\Sigma(rE(U_k):k\in K)=0$ for any pure partition $(U_k:k\in K)$ refining $\mathcal{S}$.

4.5. Example. Choose $a_n>0$ such that $a_n<1$, $\Sigma(a_n:n\in N)<\infty$, $\Sigma a_n|\log a_n|=\infty$: Put $X=\{x\in R_+:n\leq x<n+a_n$ for some $n\in N\}$, $P=X.\langle R^n,\rho,\lambda\rangle$. By 3.38, $rE(P)=\infty$ whereas, by 4.4, RE(P)=0.

4.6. Remark. By 4.3, there is a lot of spaces for which RE and rE coincide. On the other hand, 4.5 provides a very simple example of a W-space for which RE and rE are distinct and the behavior of RE is more reasonable than that of rE. These facts, and even more the connection (see Section 6) with the differential entropy (see 6.1) provide the motivation for introducing the regularized residual entropy.

4.7. **Proposition.** Let $P=\langle Q,\rho,\mu\rangle$ be a σW-space. If RE(P) exists and is finite, then all RE(S), $S\preceq P$ pure, exist and are finite.

Proof. If $T\in\sigma\mathcal{W}$, then $\Phi(T)$ will denote the collection of all pure partitions $(T_k:k\in K)$ of T (where, to avoid proper classes, K is taken from a fixed collection of indexing sets). If $\mathcal{T}_0 \in \Phi(T)$, then $\Phi(T,\mathcal{T}_0)$ will denote the collection of all $\mathcal{T}\in\Phi(T)$ refining $\mathcal{T}_0$. - Let $S=X.P\leq P$. Let $\mathcal{G}$ denote the collection of all $G\subset\overline{R}$ such that, for some $\mathcal{S}\in\Phi(S)$, $\Sigma(rE(U_k:k\in K)$ exists and is in G whenever $(U_k:k\in K)\in\Phi(S,\mathcal{S})$. Evidently, all $G\in\mathcal{G}$ are non-void. It is easy to see that if $G_1,G_2\in\mathcal{G}$, then $G_1\cap G_2\in\mathcal{G}$. We are going to show that $\mathcal{G}$ contains sets of arbitrarily small diameter. This will imply that $\bigcap(G:\overline{G}\in\mathcal{G})$ is a one-point set.

Put a=RE(P). Let $\varepsilon>0$ and let A be a neighborhood of a, diam $A<\varepsilon$. There is a $\mathcal{P}\in\Phi(P)$ such that if $(U_k:k\in K)\in\Phi(P,\mathcal{P})$, then $\Sigma(rE(U_k):k\in K)\in A$. Put $\mathcal{S}=(X.U_k:k\in K)$. If, for i=1,2, $(V_k^{(i)}:k\in K_i)\in\Phi(S,\mathcal{S})$, then $V_k^{(i)}$, $k\in K_i$, and $(Q\setminus X).U_k$, $k\in K$, form, for i=1,2, a pure partition of P refining $\mathcal{P}$. Hence, $\Sigma(rE(V_k^{(i)}):k\in K_i)+\Sigma\, rE((Q\setminus X).U_k:k\in K)\in A$ for i=1,2, and therefore $|\Sigma(rE(V_k^{(1)}):k\in K_1)-\Sigma(rE(V_k^{(2)}):k\in K_2)|<\varepsilon$. We have shown that $\Sigma(rE(V_k):k\in K)$ exists for any $(V_k:k\in K)\in\Phi(S,\mathcal{S})$ and that the set of these $\Sigma(rE(V_k):k\in K)$ is of diameter $\leq\varepsilon$. Thus, we have proved that $\bigcap(\overline{G}:G\in\mathcal{G})$ contains exactly one point, say b. It is easy to prove that RE(S)=b.

4.8. **Proposition.** Let $P=\langle Q,\rho,\mu\rangle$ be a σW-space. Let $(X_n.P:n\in N)$ be a pure partition of P. Assume that, for any $n\in N$, $RE(X_n.P)$ exists. Then RE(P)= $=\Sigma(RE(X_n.P):n\in N)$, unless neither RE(P) ncr $\Sigma(RE(X_n.P):n\in N)$ exists.

Proof. Put $a_n=RE(X_n.P)$. - Assume that $\Sigma(a_n:n\in N)$ exists and put $a=\Sigma a_n$. Let G be a neighborhood of a in $\overline{R}$. Clearly, there are neighborhoods G_n of a_n, $n\in N$, such that if $x_n\in G_n$, then $\Sigma x_n\in G$. For any $n\in N$, let $(Y_{nk}:k\in N)$ be a pure partition of $X_n.P$ such that if a pure partition $(Z_j:j\in N)$ refines $(Y_{nk}:k\in N)$, then $\Sigma(rE(Z_j):j\in N)$ exists and is in G. Then $\mathcal{U}=(Y_{nk}:n\in N,k\in N)$ is a pure partition of P, and it is easy to prove that, for any pure partition $(T_k:k\in K)$ refining $\mathcal{U}$, we have $\Sigma(rE(T_k):k\in K)\in G$. This proves that a=RE(P). - Assume that RE(P) exists and put a=RE(P). We are going to show that $\Sigma(a_n:n\in N)$ exists and is equal to a. Let G be a neighborhood of a; for any $n\in N$, let G_n be a neighborhood of a_n. Then there are pure partitions $\mathcal{B}_n=(B_{nk}.P:k\in N)$ of $X_n.P$, $n\in N$, and $\mathcal{A}_n=(A_k.P:k\in N)$ of P such that $\Sigma(rE(U_n):n\in N)\in G_n$ for any $(U_n:n\in N)$ refining $\mathcal{B}_n$, $n\in N$, and $\Sigma(rE(V_n):n\in N)\in G$ for any $(V_n:n\in N)$ refining $\mathcal{A}$. Clearly, there is a pure partition $(Z_k.P:k\in N)$ of P refining $\mathcal{A}$ and such that, for any $n\in N$, $(Z_k.P:k\in N,Z_k\subset X_n)$ is a pure partition of $X_n.P$ refining $\mathcal{B}_n$. Put $y_n=\Sigma(rE(Z_k.P):k\in N,Z_k\subset X_n)$, $y=\Sigma(rE(Z_k.P):k\in N)$. Then $y=\Sigma y_n$, $y\in G$, $y_n\in G_n$. Since the neighborhood G_n (of a_n) and G (of a) have been arbitrary, this proves $\Sigma a_n=a$.

4.9. **Proposition.** Let $P=\langle Q,\varphi,\mu\rangle$ be a σW-space. If RE(S) exists for each pure $S\leq P$ (in particular, if RE(P) exists and is finite), then the function $X\mapsto RE(X.P)$, defined on dom $\bar{\mu}$, is σ-additive and absolutely continuous with respect to μ.

This follows at once from 4.8 and 4.7.

4.10. **Definition.** A σW-space $P=\langle Q,\varphi,\mu\rangle$ will be called (1) RE-regular if there are pure subspaces P_n such that $\Sigma(P_n:n\in N)=P$ and, for each $n\in N$, RE(S) exists for all pure $S\leq P_n$, (2) strongly RE-regular if all $S\leq P$ are RE-regular and the following continuity condition is satisfied:

($*$) if $S\leq P$, $S_n\leq S_{n+1}\leq S$ for all $n\in N$ and $w(S-S_n)\rightarrow 0$, then there are $X_k\in$ $\in$ dom $\bar{\mu}$, $k\in N$, such that $\cup X_k=Q$, $X_i\cap X_j=\emptyset$ for $i\neq j$, and, for any $k\in N$, $RE(X_k.S)$ and all $RE(X_k.S_n)$ exist, $RE(X_k.S_n)\rightarrow RE(X_k.S)$.

4.11. Fact. Let $P\in\sigma\mathcal{W}$. If P is RE-regular, then so is each of its pure subspaces. If P is strongly RE-regular, then so is each of its subspaces.

4.12. Fact. Let $P\in\sigma\mathcal{W}$ and let $(P_n:n\in N)$ be a pure partition of P. If each P_n is RE-regular (strongly RE-regular), then so is P.

Proof. The assertion concerning RE-regularity is evident. - Let P_n be strongly RE-regular. Then, clearly, each subspace of P is RE-regular. Let $P_n=$ $=Y_n.P$; we can assume that $Y_i\cap Y_j=\emptyset$ for $i\neq j$. Let $S\leq P$, $S_n\leq S_{n+1}\leq S$ for all $m\in N$, $w(S-S_m)\rightarrow 0$. Then, for each n, $Y_n.S_m\leq Y_n.S_{m+1}\leq Y_n.S$, $w(Y_n.S-Y_n.S_m)\rightarrow$ $\rightarrow 0$ for $m\rightarrow\infty$, and therefore there are $X_{nk}\in$ dom $\bar{\mu}$, $k\in N$, such that $\cup(X_{nk}:k\in N)=Q$, $X_{ni}\cap X_{nj}=\emptyset$ for $i\neq j$, and $RE((X_{nk}\cap Y_n).S_m)\rightarrow RE((X_{nk}\cap Y_n),S)$ for $m\rightarrow\infty$. Put $Z_{nk}=X_{nk}\cap Y_n$. Clearly, $\cup(Z_{nk}:(n,k)\in N\times N)=Q$, $Z_{nk}\cap Z_{ij}=\emptyset$ for $(n,k)\neq(i,j)$. This proves that the continuity condition from 4.10 is satisfied.

4.11. **Proposition.** If $P=\langle Q,\varphi,\mu\rangle$ is a strongly RE-regular σW-space and $f:Q\rightarrow R_+$ is $\bar{\mu}$-measurable, then f.P is strongly RE-regular.

Proof. By 2.3, f.P is a σW-space. For $n\in N$ put $X_n=\{x\in Q:n\leq f(x)<n+1\}$. Put S=f.P. Clearly, $X_n.S\leq n.(X_m.P)$. By 4.11, $X_m.P$, hence also $n.(X_n.P)$ is strongly RE-regular. Therefore, by 4.11, $X_m.S$ is strongly RE-regular. Since $S=\Sigma(Y_n:n\in N)$, where $Y_0=X_0$, $Y_{n+1}=X_{n+1}\setminus\cup(X_k:k\leq n)$, S is strongly RE-regular by 4.12.

4.14. **Proposition.** Let $P=\langle Q,\varphi,\mu\rangle$ be a σW-space. If there are strongly RE-regular subspaces $P_n\leq P$ such that $\Sigma(P_n:n\in N)=P$, then P is strongly RE-regular.

Proof. Let $P_n=f_n.P$. Put $X_n=\{x\in Q:f_n(x)>0\}$. Put $g_n(x)=1/f_n(x)$ if $x\in X_n$,

$g_n(x)=0$ if $x \in Q \setminus X_n$. Then $X_n.P=g_n.P_n$, hence, by 4.13, each $X_n.P$ is strongly RE-regular. Put $Y_0=X_0$, $Y_{n+1}=X_{n+1} \cdot \bigcup(X_k:k \leq n)$. Then $P=\Sigma(Y_n.P:n \in N)$, $Y_n.P= =Y_n.P)$, hence $Y_n.P$ are strongly RE-regular. By 4.12, this implies that P is strongly RE-regular.

4.15. Lemma. Let $P=\langle Q,\varrho,\mu\rangle$ be a metric W-space. If $RD(P)<\infty$, then there is a pure partition $(P_n:n \in N)$ of P such that all P_n are totally bounded.

Proof. Since $RD(P)<\infty$, there is a pure partition $(S_n:n \in N)$ of P such that the Rényi dimensions $Rd(S_n)$ exist and are finite. Let $n \in N$. Since $Rd(S_n)< <\infty$, there exists, for any $k=1,2,\dots$, a (2^{-k})-partition $(X(n,k,j):j \in N)$ of S_n such that $H(\bar{\mu} X(n,k,j):j \in N)<\infty$. For any n and k choose $m(n,k)$ such that $\bar{\mu}(Y(n,k))>\mu Q-2^{-k}$, where $Y(n,k)=\bigcup(X(n,k,j):j \leq m(n,k))$. For $t \in N$ put $Z(n,t)=\bigcap(Y((n,k):k \geq t)$. It is easy to see that all $Y(n,k).P$ are totally bounded and $\bar{\mu}(\bigcup(Y(n,k):n \in N,k \in N))=\mu Q$. From this, the assertion follows at once.

4.16. **Proposition.** Let $P=\langle Q,\varrho,\mu\rangle$ be a metric W-space. If P is almost Borel partition-regular or $RD(P)=0$, then P is strongly RE-regular.

Proof. By 4.14 and 4.15, it is sufficient to prove the proposition under the assumption that P is totally bounded. - Under this assumption, the continuity condition from 4.10 is satisfied; this follows from 3.34 and 4.3 if P is partition-regular, and is an easy consequence of 3.28 if $RD(P)=0$. Since, by 4.3, $RE(S)$ exists for all $S \leq P$, we have shown that P is strongly RE-regular.

4.17. Remarks. A) I do not know whether every RE-regular σW-space is strongly RE-regular. - B) If a σW-space is given, it can be quite difficult to decise whether it is strongly RE-regular. Therefore we introduce (see 4.18) a fairly wide class of σW-spaces contained in that of strongly RE-regular ones and defined in terms not involving the behavior of RE.

4.18. **Definition.** A metric σW-space P will be called piecewise partition-regular if it has a partition $(P_n:n \in N)$ such that all P_n are partition-regular W-spaces.

4.19. **Theorem.** Every piecewise partition-regular metric σW-space is strongly RE-regular.

This is an immediate consequence of 4.16 and 4.14.

4.20. Fact. Every subspace of a piecewise partition-regular σW-space is piecewise partition-regular. If P is a σW-space, $P=\Sigma(P_n:n \in N)$ and all P_n are piecewise partition-regular, then so is P.

4.21. Fact. If $P=\langle Q,\rho,\mu\rangle$ is a piecewise partition-regular σW-space and $f:Q\to R_+$ is $\bar{\mu}$-measurable, then f.P is piecewise partition-regular.

Proof. Put $X_n=\{x\in Q:n\leq f(x)<n+1\}$. Clearly, $X_n.(f.P)\leq(n+1)P$, $\Sigma(X_n.(f.P))=f.P$. By 4.20, this proves the assertion.

4.22. **Proposition.** Let $\langle Q,\rho\rangle$ be an m-dimensional C^1-submanifold of some R^n (endowed with the ℓ_∞-metric). If $P=\langle Q,\rho,\mu\rangle$ is a σW-space and $RD(S)=m$ for all non-null pure $S\leq P$, then P is piecewise partition-regular.

This is an easy consequence of 3.24 and 4.20.

4.23. We conclude this section with some simple facts which will be used later and an example of a partition-regular space P for which $X\mapsto rE(X.P)$ is not additive.

4.24. Fact. Let $\langle Q,\mu\rangle$ be a σ-bounded measure space and let T be thick in $\langle Q,\mu\rangle$. If ν_n, $n\in N$, are measures on T and $\Sigma\nu_n=\mu\restriction T$, then there exist measures μ_n on Q such that $\Sigma\mu_n=\mu$, $\nu_n=\mu_n\restriction T$ for all $n\in N$.

Proof. If $X\in\operatorname{dom}\mu$, put $\mu_nX=\nu_n(X\cap T)$. It is easy to see that $\Sigma\mu_n=\mu$, $\mu_n\restriction T=\nu_n$.

4.25. **Proposition.** Let $P=\langle Q,\rho,\mu\rangle$ be a W-space and let T be thick in $\langle Q,\mu\rangle$. Let φ be one of the functionals Rd, RD, rE. Then $\varphi(P)=\varphi(P\restriction T)$ unless neither $\varphi(P)$ nor $\varphi(P\restriction T)$ exists.

Proof. I. If $\varphi=Rd$, then the assertion follows from 2.18. - II. Let $\varphi=RD$ and assume that $\varphi(P)$ exists. For any partition $(P_n:n\in N)$ of P, $(P_n\restriction T:n\in N)$ is a partition of $P\restriction T$; by I, $Rd(P_n\restriction T)=Rd(P_n)$ whereas Rd(P) exists. This proves $RD(P\restriction T)=RD(P)$. - III. Let $\varphi=RD$ and assume that $\varphi(P\restriction T)$ exists. If $(S_n:n\in N)=(\langle T,\rho,\nu_n\rangle:n\in N)$ is a partition of $P\restriction T$, then, by 4.24, there is a partition $(P_n:n\in N)$ of P such that $S_n=P_n\restriction T$, hence, by I, $Rd(P_n)=Rd(S_n)$ provided $Rd(S_n)$ exists. This proves that $RD(P)=RD(P\restriction T)$. - IV. The assertion concerning rE is an immediate consequence of II, III and 2.18.

4.26. **Proposition.** Let $P=\langle Q,\rho,\mu\rangle$ be a σW-space and let T be thick in $\langle Q,\mu\rangle$. Then (1) $RE(P)=RE(P\restriction T)$ unless neither RE(P) nor $RE(P\restriction T)$ exists, (2) P is RE-regular (respectively, strongly RE-regular) if and only if so is $P\restriction T$.

Proof. The assertion (1) follows from 4.25 and 4.24. The assertion (2) is an easy consequence of (1) and 4.24.

4.27. **Proposition.** Let $P=\langle Q,\rho,\mu\rangle$ and $S=\langle Q,\rho,\nu\rangle$ be weakly Borel metric W-space. Assume that there is a measure η such that both $\bar{\mu}$ and $\bar{\nu}$ are faithful extensions of η. Let φ be one of the functionals Rd, RD, rE.

Then (1) for any $\delta>0$, $H_\delta(P)=H_\delta(S)$, (2) $\varphi(P)=\varphi(S)$ unless neither $\varphi(P)$ nor $\varphi(S)$ exists.

Proof. I. Clearly, it is sufficient to consider the case $\bar{\mu}=\mu$, $\bar{\nu}=\nu$. Furthermore, if we put $\eta'X=\mu X$ whenever $X\in \mathrm{dom}\,\mu\cap\mathrm{dom}\,\nu$, then η' is a measure, $\eta'\supset\eta$ and both μ and ν are faithful extensions of η'. Hence we can assume that $\mathrm{dom}\,\eta=(\mathrm{dom}\,\mu)\cap(\mathrm{dom}\,\nu)$. - II. If $(X_n:n\in N)$ is a δ-partition of P, then there are $Y_n\in\mathrm{dom}\,\eta$ such that $\mu(X_n\triangle Y_n)=0$, hence $\mu X_n=\eta Y_n$. Put $V_n=Y_n\cap\overline{X}_n$. Clearly, $\mathrm{diam}\,V_n\leq\delta$, $V_n\in\mathrm{dom}\,\eta$. It is easy to see that $\eta V_n=\mu X_n$ and $\eta(V_i\cap V_j)=0$ whenever $i\neq j$. Put $Z_n=V_n\setminus\bigcup(V_k:k\neq n)$. Then $(Z_n:n\in N)$ is a δ-partition of S, $\nu Z_n=\mu X_n$. This proves that $H_\delta(P)\geq H_\delta(S)$. The proof of $H_\delta(S)\geq H_\delta(P)$ is analogous. - III. The proof of (2) is analogous to that of 4.25 and can be omitted.

4.28. **Proposition.** Let $P=\langle Q,\rho,\mu\rangle$ and $S=\langle Q,\rho,\nu\rangle$ be weakly Borel metric W-spaces. Assume that there is a measure η such that both μ and ν are faithful extensions of η. Then (1) RE(P)=RE(S) unless neither RE(P) nor RE(S) exists, (2) P is RE-regular (respectively, strongly RE-regular) if and only if so is S.

Proof. The first assertion follows easily from 4.27 and the fact (which is easy to prove) that every pure partition $(T_n:n\in N)$ of P or of S is of the form $(X_n.P:n\in N)$ or, respectively, $(X_n.S:n\in N)$ where $X_n\in\mathrm{dom}\,\eta$. The assertion (2) is an easy consequence of (1).

4.29. Fact. Let $P=\langle Q,\rho,\mu\rangle\in\mathcal{M}$ and let $b\in R_+$. If rE(P) exists, then rE(b.P)=b.rE(P)+wP.L(b). If RE(P) exists, then RE(b.P)=b.RE(P)+wP.L(b).

Proof. For any $\delta>0$, we have $\psi(\delta,b.P)=E(\delta*(b.P))-RW(b.P)|\log\delta|+L(b.wP)=b.E(\delta*P)-b.RW(P)|\log\delta|+b.L(wP)+wP.L(b)=b\psi(\delta,P)+wP.L(b)$. This proves the first assertion. The second assertion is an easy consequence of the first.

4.30. Example. Let $Q=[0,1]$, $P=\langle Q,\rho,\lambda\rangle$. Let $S\subset Q$ and let both S and $T=Q\setminus S$ be thick in $\langle Q,\lambda\rangle$. Define μ as follows: if $X\subset Q$, $Y\cap S\in\mathrm{dom}(\lambda\restriction S)$ and $X\cap T\in\mathrm{dom}(\lambda\restriction T)$, put $\mu X=((\lambda\restriction S)(X\cap S)+(\lambda\restriction T)(X\cap T))/2$. Clearly, μ is a measure, $\mu\supset\lambda\restriction Q$, $P'=\langle Q,\varphi,\mu\rangle$ is a partition-regular weakly Borel metric W-space. Obviously, $H_\delta(P')\leq H_\delta(P)$ for all $\delta>0$; consequently, $\psi(\delta,P')\leq\psi(\delta,P)$ for all $\delta>0$. Since, by 3.26, rE(P′) exists, and, by 3.38, rE(P)=0, we get rE(P′)≤0. On the other hand, since both S and T are thick in $\langle Q,\lambda\rangle$, we have, by 4.25 and 3.38, $rE(P\restriction S)=0$, $rE(P\restriction T)=0$. Since $\mu\restriction S=(\lambda\restriction S)/2$, $\mu\restriction T=(\lambda\restriction T)/2$, we get, by 4.29, $rE(P'\restriction S)=w(P\restriction S).L(1/2)=.(1/2)=1/2$, and similarly $rE(P'\restriction T)=1/2$. Since, by 4.25, $rE(S.P')=rE(P'\restriction S)$, $rE(T.P')=rE(P'\restriction T)$, we get $rE(S.P')+rE(T.P')=1$ whereas $rE(P')\leq 0$.

5.1. Fact. Let $P=\langle Q,\varrho,\mu\rangle$ be a σW-space. Then there is at most one function (mod μ) $F=[f]_\mu$ such that (*) f is $\bar{\mu}$-measurable and the functions $X\mapsto RE(X.P)$ and $X\mapsto\int_X F\,d\mu$ coincide.

Proof. Suppose that both $F=[f]_\mu$ and $G=[g]_\mu$ satisfy (*) and $F\neq G$. Clearly, either (1) $\bar{\mu}\{x\in Q:f(x)>g(x)\}>0$ or (2) $\bar{\mu}\{x\in Q:f(x)<g(x)\}>0$. It is sufficient to consider the case (1). Then there are reals r and s and an $X\in$ $\in\operatorname{dom}\bar{\mu}$ such that $0<\mu X<\infty$ and $f(x)>r>s>g(x)$ whenever $x\in X$. Clearly, both $\int_X fd\mu$ and $\int_X gd\mu$ exist, hence, $\int_X fd\mu=RE(X.P)=\int_X gd\mu$. This is a contradiction since $\int_X fd\mu\geq r.\mu X$, $\int_X gd\mu\leq s.\mu X$.

5.2. **Definition.** If $P=\langle Q,\varrho,\mu\rangle$ is a σW-space, then $F\in\mathcal{F}[\mu]$ satisfying the condition (*) from 5.1 will be called the residual entropy density or the RE-density of P and will be denoted by $\nabla(P)$ (or $\nabla^{res}(P)$ if there is a danger of confusion with the dimensional densities introduced in [6], 4.1 and 4.9). If no F satisfying (*) exists, we will say that $\nabla(P)$ does not exist.

5.3. **Proposition.** A σW-space P is RE-regular if and only if $\nabla(P)$ exists.

Proof. I. Let $P=\langle Q,\varrho,\mu\rangle$ be RE-regular. Then there are pure subspaces $P_n=A_n.P$, $n\in N$, such that $\Sigma P_n=P$ and, for each $n\in N$, RE(S) exists for all pure $S\leq P_n$. *We can assume that* $\bigcup A_n=Q$, $A_i\cap A_j=\emptyset$ *for* $i\neq j$. By 4.9, *for any* $n\in$ $\in N$, the function $X\mapsto RE(x.P_n)$, defined on $\operatorname{dom}\bar{\mu}$, is σ-additive and absolutely continuous with respect to μ. Hence, by 1.14, there are $\bar{\mu}$-measurable functions $f_n:Q\to R$ such that $RE(X.P_n)=\int_X f_n d\mu$ for all $X\in\operatorname{dom}\bar{\mu}$. Since $X.P_n=(X\cap A_n).P$, we can assume that, for each n, $f_n(x)=0$ if $x\in Q\setminus A_n$. For any $x\in Q$, put $f(x)=\Sigma(f_n(x):n\in N)$. Clearly, for any $X\in\operatorname{dom}\bar{\mu}$, $\int_X f\,d\mu=$ $=\Sigma(RE(X.P_n):n\in N)$ provided either the sum or the integral exists. Since, by 4.8, RE(X.P) exists iff the sum $\Sigma(RE(X.P_n):n\in N)$ exists, we have shown that $\nabla(P)=[f]_\mu$. - II. Assume that $\nabla(P)$ exists; let $\nabla(P)=[g]_\mu$. Put $K=\{k\in$ $\in\bar{R}:|k|\in N\cup\{\infty\}\}$. If $k\in K\cap R$, put $B_k=\{x\in Q:k\leq g(x)<k+1\}$; if $k=\pm\infty$, put $B_k=\{x\in Q:g(x)=k\}$. It is easy to see that, for each $k\in K$, RE(S) exists for all pure $S\leq B_k.P$.

5.4. We use the following conventions (cf. [6], 4.2). - A) If $\mu\in\mathcal{M}(Q)$, $f\in\mathcal{F}(Q)$ and $g\in\mathcal{F}(Q)$ are $\bar{\mu}$-measurable, $F=[f]_\mu$, $G=[g]_\mu$, we put $f.G=$ $=F.G=[fg]_\nu$, where $\nu=f.\mu$. - B) Let $\mu\in\mathcal{M}(Q)$, $\mu(n)\in\mathcal{M}(Q)$, $n\in N$. Let $\mu=\Sigma\mu(n)$. Assume that, for each $n\in N$, $\mu(n)=Y_n.\mu$ for some $Y_n\in\operatorname{dom}\bar{\mu}$. If $F_n\in\mathcal{F}[\mu(n)]$, $n\in N$, then ΣF_n is defined as follows. Choose $X(n)\in\operatorname{dom}\bar{\mu}$, $n\in N$, such that $\bigcup X(n)=Q$, $X(i)\cap X(j)=\emptyset$ if $i\neq j$, and $\mu(n)=$ $=X(n).\mu$ for all n. Choose f_n such that $F_n=[f_n]_{\mu(n)}$; for

$x \in X(n)$, put $f(x)=f_n(x)$. Put $\Sigma F_n = [f]_{\mu}$.

5.5. Fact. Let $P=\langle Q,\wp,\mu\rangle \in \sigma\mathfrak{W}$ be RE-regular. Then (1) for any $X \in \operatorname{dom}\bar{\mu}$, $\nabla(X.P)=i_X.\nabla(P)$. (2) If $(P_n : n \in N)$ is a pure partition of P, then $\nabla(P)=\Sigma(\nabla(P_n):n \in N)$.

5.6. Fact. Let $P=\langle Q,\wp,\mu\rangle$ be an RE-regular σW-space and let $b \in R$ be positive. Then $\nabla(b.P)=\nabla(P)-\log b$.

Proof. Clearly, there are $X_n \in \operatorname{dom}\bar{\mu}$, $n \in N$, such that $\bigcup X_n = Q$ and all $RE(X_n.P)$ exist, hence RE(S) exists whenever S is a pure subspace of some $X_n.P$. Let $Y \in \operatorname{dom}\mu$, $Y \subset X_n$ for some n. By the definition of the RE-density, $RE(b.Y.P)=\int_Y \nabla(b.P)d(b\mu)$, $RE(Y.P)=\int_Y \nabla(P)d\mu$. By 4.29, $RE(b.Y.P)=$ $=b.RE(Y.P)+\mu Y.L(b)$. Hence $\int_Y \nabla(b.P)d(b\mu)=b\int_Y \nabla(P)d\mu=\int_Y L(b)d\mu$ and therefore $\int_Y \nabla(b.P)d\mu=\int_Y(\nabla(P)-\log b)d\mu$. Since $Y \subset X_n$ and $n \in N$ have been arbitrary, this proves the assertion.

5.7. Fact. Let $P=\langle Q,\wp,\mu\rangle$ be an RE-regular σW-space. Let $f:Q \to R_+$ be $\bar{\mu}$-measurable and let f(Q) be countable. Then $\nabla(f.P)=(\operatorname{sgn} f).\nabla(P)-$ $-(\operatorname{sgn} f).\log f$, $RE(f.P)=\int(f.\nabla(P)+L\circ f)d\mu$.

This follows easily from 5.5 and 5.6.

5.8. Lemma. Let $P=\langle Q,\wp,\mu\rangle$ be a strongly RE-regular W-space. Let $S=$ $=f.P \leq P$, $0 \leq f(x) \leq 1$ for all $x \in Q$. Assume that RE(S) exists and one of the following conditions is satisfied: (a) $\nabla(P)$ is bounded, (b) $\nabla(P)=\infty$, (c) $\nabla(P)=-\infty$. Then

(1) $RE(S)=\int(f.\nabla(P)+L\circ f)d\mu$,

(2) $\nabla(S)=(\operatorname{sgn} f).\nabla(P)-(\operatorname{sgn} f).\log f$.

Proof. Clearly, we can assume that $f(x)>0$ for all $x \in Q$. It is easy to see that there are $\bar{\mu}$-measurable functions f_n, $n \in N$, such that all $f_n(Q)$ are countable, $0 \leq f_n \leq f_{n+1} \leq f$ for all $n \in N$, and $\int(f-f_n)d\mu \to 0$ for $n \to \infty$. Put $S_n=f_n.P$. Since $w(S-S_n) \to 0$, and P satisfies the continuity condition from 4.10, there is a partition $(X(k):k \in N)$ of Q such that all X(k) are in dom μ and, for any k, $RE(X(k).S_n) \to RE(X(k).S)$ for $n \to \infty$. By 5.7, we have, for any $k \in N$, $n \in N$, $RE(X(k).S_n)=\int_{X(k)}(f_n.\nabla(P)+L\circ f_n)d\mu$. - Consider the case (a). Since $\nabla(P)$ is bounded, it is easy to see that, for any k, $\int_{X(k)}(f_n.\nabla(P)+L\circ f_n)d\mu \to \int_{X(k)}(f.\nabla(P)+L\circ f)d\mu$ for $n \to \infty$, hence $RE(X(k).S)=\int_{X(k)}(f.\nabla(P)+L\circ f)d\mu$. Since RE(S) exists, we have $RE(S)=$ $=\Sigma(RE(X(k).S):k \in N)=\int(f.\nabla(P)+L\circ f)d\mu$. - Consider the case (b). Then, for any $k \in N$, we have, for large n, $\int_{X(k)}(f.\nabla(P)+L\circ f)d\mu=\infty$, $RE(X(k).$ $.S_n)=\infty$. This implies $RE(X(k).S)=\infty$. Since RE(S) exists, we get $RE(S)=\infty$.

Clearly, $\int (f.\ \nabla(P)+L\circ f)d\mu = \infty$ =RE(S). In the case (c), the proof is analogous. - We have proved the formula (1). The formula (2) is an easy consequence.

5.9. **Theorem.** Let $P=\langle Q,\rho,\mu\rangle$ be a strongly RE-regular σW-space. Let $f:Q\to R_+$ be $\bar{\mu}$-measurable. Then

(1) $RE(f.P)=\int (f.\ \nabla(P)+L\circ f)d\mu$, unless neither RE(f.P) nor the integral exists,

(2) $\nabla(f.P)=(\mathrm{sgn}\ f).\ \nabla(P)-(\mathrm{sgn}\ f).\log f$.

Proof. I. Consider the case of $0\leq f(x)\leq 1$ for all $x\in Q$. Let $\nabla(P)=[g]_\mu$. Put $K=\{k\in\bar{R}:|k|\in N\cup\{\infty\}\}$. If $k\in K$, $|k|\in N$, put $A_k=\{x\in Q:k\leq g(x)<k+1\}$; if $k=\pm\infty$, put $A_k=\{x\in Q:g(x)=k\}$. Choose a partition $(B_n:n\in N)$ of Q such that all $B_n.P$ are W-spaces. If $u=(k,n)\in N\times N$, put $V(u)=A_k\cap B_n$. By 5.8, for any $u\in N\times N$, $\nabla(V(u).P)=i_{V(u)}.(\mathrm{sgn}\ f).\ \nabla(P)-i_{V(u)}.(\mathrm{sgn}\ f)\log f$. This implies, by 5.5, the formula (2); the formula (1) is an immediate consequence. - II. Consider the general case. For $n\in N$, put $X(n)=\{x\in Q:n\leq f(x)<n+1\}$. By I and 5.5, we have $\nabla(X(n).f.P)=i_{X(n)}.(\mathrm{sgn}\ f).\ \nabla(P)-i_{X(n)}.(\mathrm{sgn}\ f).\log f$. By 5.5, this implies $\nabla(f.P)=(\mathrm{sgn}\ f).\ \nabla(P)-(\mathrm{sgn}\ f).\log f$.

5.10. Corollary. Let μ be a measure on R^n, n=1,2,..., absolutely continuous with respect to the Lebesgue measure λ. If $f=d\mu/d\lambda$ and, in accordance with 1.19, ρ is the ℓ_∞-metric on R^n, then

$$RE\langle R^n,\rho,\mu\rangle = -\int f\log f\ d\lambda,$$

unless neither $RE\langle R^n,\rho,\mu\rangle$ nor the integral exists.

6

In the classical setting, which stems from C.E. Shannon [11], the differential entropy is defined for probability measures μ on R^n possessing a density p and is equal to $-\int p\log p\ d\lambda$. This concept can be easily extended to a considerably more general situation (see 6.1 below). We intend to show that the differential entropy and the regularized residual entropy are equivalent in a sense made precise in 6.9 and 6.10 below. Roughly speaking, under certain conditions, (1) if μ and ν are measures on Q, then there is a metric τ on Q such that, for any measurable $g:Q\to R_+$, the differential entropy of the pair $\langle g.\mu,\nu\rangle$ is equal to $RE\langle Q,\tau,g.\mu\rangle$, (2) if $\langle Q,\rho,\mu\rangle$ is a W-space, then, for any measurable $g:Q\to R_+$, $RE\langle Q,\rho,g.\mu\rangle$ is equal to the differential entropy of $\langle g.\mu,\nu\rangle$ where ν does not depend on g.

6.1. **Definition.** If μ and ν are σ-bounded measures and μ is absolutely continuous with respect to ν , then the integral $\int L \circ D[\mu, \nu]\, d\nu$, provided it exists, will be denoted by $DE\langle \mu, \nu \rangle$ and will be called the differential entropy of μ with respect to ν (or of the pair $\langle \mu , \nu \rangle$).

Remark. If μ is a probability measure on R^n, possessing a density p with respect to λ , then the differential entropy $DE\langle \mu , \lambda \rangle$ is equal to $-\int p \log p \, d\lambda$, i.e. to the differential entropy in the usual sense.

6.2. **Definition.** Let $\langle Q, \mu \rangle$ be a measure space. The space $\langle Q, \mu \rangle$ and the measure μ will be called strongly separable if there exists a countably generated σ-algebra $\mathcal{A} \subset \mathrm{dom}\, \mu$ satisfying the following conditions: (1) $Q \setminus \{x \in Q : \{x\} \in \mathcal{A}\}$ is a μ-null set, (2) μ is a faithful extension of $\mu \restriction \mathcal{A}$.

6.3. We are going to show that a strongly separable σ-bounded measure space $\langle Q, \mu \rangle$ can be equipped with a metric τ such that $\nabla\langle Q, \tau , \mu \rangle = 0$. To this end, we shall need some lemmas.

6.4. Lemma. Let $M \subset R$ be bounded. Let ν be a finite measure on M such that $\mathfrak{B}(M) \subset \mathrm{dom}\, \nu$, $\nu M > 0$, $\nu\{x\} = 0$ if $x \in M$. Then there exist sets $T \subset M$, $S \subset R$ and a bijective mapping $f : T \to S$ such that (1) $T \in \mathrm{dom}\, \nu$, $\nu(M \setminus T) = 0$, (2) $\overline{S} = [0, M]$, S is thick in $\langle \overline{S}, \lambda \rangle$, (3) $Y \in \mathfrak{B}(S)$ iff $f^{-1}Y \in \mathfrak{B}(T)$, (4) if $Y \in \mathfrak{B}(S)$, then $\nu(f^{-1}Y) = (\lambda \restriction S)(Y)$.

Proof. Let G be the largest open (in R) set such that $\nu(G \cap M) = 0$. Let $(J_k : k \in K)$ be the partition of G into open intervals. Put $T = M \setminus \bigcup(\overline{J}_k : k \in K)$. Clearly, $\nu(M \setminus T) = 0$. Put $a = \inf T$. For $x \in T$ let $f(x) = \nu((a,x) \cap M)$. Put $S = \{f(x) : x \in T\}$. - Suppose $f(x) = f(y)$ for some $x, y \in T$, $x \neq y$. Then $\nu((a,x) \cap M) = \nu((a,y) \cap M)$, hence $\nu((x,y) \cap M) = 0$ and therefore $x, y \in V \setminus T$, which is a contradiction. We have shown that $f : T \to S$ is bijective. It is easy to prove that $\overline{S} = [0, ..M]$. - Now we are going to show that ($*$) if $J = (u,v) \subset \overline{S}$, then $\nu(f^{-1}(J)) = \lambda J$. Clearly, it is sufficient to prove ($*$) for the case when $u, v \in S$. Let $u = f(b)$, $v = f(c)$. Obviously, $\nu(f^{-1}J) = \nu((b,c) \cap T) = \nu((a,x) \cap T) - \nu((a,b) \cap T) = f(x) - f(b) = v - u = \lambda J$. - Suppose that S is not thick in $\overline{S}$. Then $\lambda_e(S) < \nu M$, hence there is an open set $G \subset \overline{S}$ such that $G \supset S$, $\lambda G < \nu M$. By ($*$), we get $\nu(f^{-1}G) < \nu M$. Since $f^{-1}G = T$, this is a contradiction, which proves that S is thick in $\langle \overline{S}, \lambda \rangle$. - Clearly, $f : T \to S$ is continuous and therefore $Y \in \mathfrak{B}(S)$ implies $f^{-1}Y \in \mathfrak{B}(T)$. On the other hand, if $U \subset T$ is open, then f(U) is Borel in S; this proves that $Y \in \mathfrak{B}(S)$ whenever $f^{-1}Y \in \mathfrak{B}(T)$. - To prove (4), it is sufficient to show that if $J \subset \overline{S}$ is an open interval, then $\nu(f^{-1}J) = (\lambda \restriction S)(J \cap S)$. By ($*$), we have $\nu(f^{-1}J) = \lambda J$. Since S is thick, we have, by 2.17, $\lambda J = (\lambda \restriction S)(J \cap S)$.

6.5. Lemma. Let $\langle Q, \mu \rangle$ be a strongly separable bounded measure space.

Assume that $\mu\{x\}=0$ for all $x\in Q$. Then there are sets $Q'\subset Q$, $M\subset R$ and a bijective mapping $\varphi:Q'\longrightarrow M$ such that, with $\nu=(\mu\restriction Q')\circ\varphi^{-1}$, we have (1) $Q'\in \operatorname{dom}\mu$, $\mu(Q\setminus Q')=0$, (2) $\mathcal{B}(M)\subset \operatorname{dom}\nu$, (3) M is bounded, (4) ν is a faithful extension of $\nu\restriction\mathcal{B}(M)$.

Proof. Let $\mathcal{A}\subset\operatorname{dom}\mu$ be a countably generated σ-algebra satisfying (1) and (2) from 6.2. Let $X(n)$ be a sequence of sets generating $\mathcal{A}$. Let $U\in\operatorname{dom}\mu$, $\mu U=0$, be such that $\{x\}\in\mathcal{A}$ whenever $x\in Q\setminus U$. Put $Q'=Q\setminus U$. For $n\in N$, put $g_n=i_{X(n)}$. For $x\in Q'$ put $g(x)=(g_n(x):n\in N)$. It is easy to show that g is an injective mapping of Q' into the topological space 2^{ω}. Let $\mathcal{A}^*$ denote the σ-algebra consisting of all $A\cap Q'$, where $A\in\mathcal{A}$. Clearly, for any $B\subset g(Q')$, $g^{-1}B\in\mathcal{A}^*$ iff $B\in\mathcal{B}(g(Q'))$. Let $h:2^{\omega}\longrightarrow R$ be a homeomorphism; put $\varphi=h\circ g$, $M=\varphi(Q')$. If $B\in\mathcal{B}(M)$, then $h^{-1}B\in\mathcal{B}(g(Q'))$, hence $\varphi^{-1}B\in\mathcal{A}^*\subset$ $\subset\operatorname{dom}\mu$ and therefore $B\in\operatorname{dom}\nu$. If $Y\in\operatorname{dom}\nu$, then $\varphi^{-1}Y\in\operatorname{dom}(\mu\restriction Q')$, hence there is a set $V\in\mathcal{A}^*$ such that $(\varphi^{-1}Y)\vartriangle V$ is μ-null. Clearly, $Y\vartriangle\varphi(V)$ is ν-null and $\varphi(V)$ is Borel in M. Thus, the condition (4) is satisfied. This proves the lemma since, evidently, M is bounded.

6.6. **Proposition.** Let $\langle Q,\mu\rangle$ be a strongly separable σ-bounded measure space. Assume that $\mu\{x\}=0$ for all $x\in Q$. Then there exists a set $Q^*\subset Q$, a set $S\subset R$ and a bijective mapping $\Phi:Q^*\longrightarrow S$ such that, with $\eta=(\mu\restriction Q^*)\circ$ $\circ\,\Phi^{-1}$, we have (1) $Q^*\in\operatorname{dom}\mu$, $\mu(Q\setminus Q^*)=0$, (2) S is thick in $\langle\overline{S},\lambda\rangle$, and if $\mu Q<\infty$, then $\overline{S}$ is an interval of length μQ, (3) $\mathcal{B}(S)\subset\operatorname{dom}\eta$, (4) $\eta B=(\lambda\restriction S)(B)$ whenever $B\in\mathcal{B}(S)$, (5) η is a faithful extension of $\eta\restriction\mathcal{B}(S)$.

Proof. I. Assume that $\mu Q<\infty$. Let Q', M, φ and ν be as in 6.5. Then, by 6.4, there are sets $T\subset M$, $S\subset R$ and a bijective mapping $f:T\longrightarrow S$ with properties described in 6.4. Put $Q^*=\varphi^{-1}(T)$. For $x\in Q^*$, put $\Phi(x)=f(\varphi(x))$. Put $\eta=(\mu\restriction Q^*)\circ\Phi^{-1}$. It is easy to see that the conditions (1) - (5) are satisfied. - II. Consider the general case. Let $(Q_n:n\in N)$ be a μ-measurable partition of Q such that all μQ_n are finite. Choose disjoint closed intervals $J_n\subset R$ such that $\lambda J_n>\mu Q_n$. It follows easily from I that there are $Q_n^*\subset Q_n$, $S_n\subset J_n$ and $\Phi_n:Q_n^*\longrightarrow S_n$ such that, for each $n\in N$, Q_n^*, S_n and Φ_n satisfy, with respect to $\mu\restriction Q_n$, the conditions (1), (3) - (5) as well as the condition (2') S_n is thick in $\langle\overline{S}_n,\lambda\rangle$. Put $Q^*=\bigcup Q_n^*$, $S=\bigcup S_n$, $\Phi(x)=\Phi_n(x)$ for $x\in Q_n^*$. It is easy to prove that Q^*, S, Φ satisfy (1) - (5).

6.7. **Theorem.** Let $\langle Q,\mu\rangle$ be a strongly separable σ-bounded measure space and let $\mu\{x\}=0$ for all $x\in Q$. Then there exists a metric τ on Q such that $P=\langle Q,\tau,\mu\rangle$ is a strongly RE-regular σW-space and $\nabla(P)=0$.

Proof. Let Q^*, S, Φ and η be as in 6.6. Choose an $a\in Q^*$. If $x,y\in Q^*$, put $\tau(x,y)=\rho(\Phi x,\Phi y)$. If $x,y\in Q\setminus Q^*$, put $\tau(x,y)=1$ if $x\neq y$, $\tau(x,y)=0$ if $x=y$. If $x\in Q^*$, $y\in Q\setminus Q^*$, put $\tau(x,y)=\tau(y,x)=\rho(\Phi a,\Phi x)+1$. Clearly, τ is a metric on Q. By 4.4 and 4.26, $P_1=\langle S,\rho,\lambda\restriction S\rangle$ is strongly RE-regular, $\nabla(P_1)=0$. Hence, by 4.28, $P_2=\langle S,\rho,\eta\rangle$ is strongly RE-regular, $\nabla(P_2)=$ $=0$. Since $\Phi:\langle Q^*,\tau,\mu\restriction Q^*\rangle\rightarrow\langle S,\rho,\eta\rangle$ preserves both metric and measure, $P^*=\langle Q^*,\tau,\mu\restriction Q^*\rangle$ is strongly RE-regular and $\nabla(P^*)=0$. Since $\mu(Q\setminus$ $\setminus Q^*)=0$, this proves the theorem.

6.8. **Proposition.** Let μ and ν be σ-finite measures on Q and let μ be absolutely continuous with respect to ν. Let $\langle Q,\rho,\nu\rangle$ be a strongly RE-regular σW-space and let $\nabla\langle Q,\rho,\nu\rangle=0$. Let $P=\langle Q,\rho,\mu\rangle$. Then, for any $\bar{\mu}$-measurable $g:Q\rightarrow R_+$,

$$DE\langle g.\mu,\nu\rangle=RE(g.P),$$

unless neither $DE\langle g.\mu,\nu\rangle$ nor $RE(g.P)$ exists.

Proof. Let $[f]_\nu=D[\mu,\nu]$. Put $P'=\langle Q,\rho,\nu\rangle$. If $\int L\circ(gf)d\nu$ exists, then (1) evidently, $DE\langle g.\mu,\nu\rangle=\int L\circ(gf)d\nu$, (2) due to $\nabla(P')=0$, we have, by 5.10, $RE(gf.P')=\int L\circ(gf)d\nu$, hence $RE(g.P)=\int L\circ(gf)d\nu$. If $\int L\circ(gf)d\nu$ does not exist, then it is easy to see (using 5.10) that neither $DE\langle g.\mu,\nu\rangle$ nor $RE(g.P)$ exists.

6.9. **Theorem.** Let μ and ν be σ-finite measures on a set Q and let μ be absolutely continuous with respect to ν. Let ν be strongly separable and let $\nu\{x\}=0$ for all $x\in Q$. Then there exists a metric τ on Q such that $P=\langle Q,\tau,\mu\rangle$ is a strongly RE-regular σW-space and, for any $\bar{\mu}$-measurable $g:Q\rightarrow R_+$,

$$DE(g.\mu,\nu)=RE(g.P),$$

unless neither $DE(g.\mu,\nu)$ nor $RE(g.P)$ exists.

This follows easily from 6.7 and 6.8.

6.10. **Theorem.** Let $P=\langle Q,\rho,\mu\rangle$ be a strongly RE-regular σW-space and let $-\infty<\nabla(P)<\infty$. Put $\nu=2^{\nabla(P)}.\mu$. Then, for any $\bar{\mu}$-measurable $g:$ $:Q\rightarrow R_+$,

$$RE(g.P)=DE\langle g.\mu,\nu\rangle,$$

unless neither $RE(g.P)$ nor $DE\langle g.\mu,\nu\rangle$ exists.

Proof. Put $f=2^{\nabla(P)}$, $P'=\langle Q,\rho,\nu\rangle$. By 5.10, we have $\nabla(P')=\nabla(P)-$ $-\log f=0$. Hence, again by 5.10, if $RE(g.P)$ exists, it is equal to $RE((g/f).P')=$ $=\int L\circ(g/f)d\nu$. On the other hand, if $DE\langle g.\mu.\nu\rangle$ exists, then it is equal to $\int L\circ D[g.\mu,\nu]\,d\nu=\int L\circ(g/f)d\nu$.

References

[1] J. BALATONI, A. RÉNYI: On the notion of entropy (Hungarian), Publ. Math. Inst. Hungarian Acad. Sci. 1(1956), 5-40. - English translation: Selected papers of Alfred Rényi, vol. I, pp. 558-584, Akadémiai Kiadó, Budapest, 1987.

[2] M. KATĚTOV: Extended Shannon entropies I, Czechoslovak Math. J. 33(108) (1983), 564-601.

[3] M. KATĚTOV: Extended Shannon entropies II, Czechoslovak Math. J. 35 (110) (1985), 565-616.

[4] M. KATĚTOV: On extended Shannon entropies and the epsilon entropy, Comment. Math. Univ. Carolinae 27(1986), 519-543.

[5] M. KATĚTOV: On the Rényi dimension, Comment. Math. Univ. Carolinae 27 (1986), 741-753.

[6] M. KATĚTOV: On dimensions of semimetrized measure spaces, Comment. Math. Univ. Carolinae 28(1987), 399-411.

[7] E.C. POSNER, E.R. RODEMICH, H. RUMSEY, Jr.: Epsilon entropy of stochastic processes, Ann. Math. Statist. 38(1967), 1000-1020.

[8] E.C. POSNER, E.R. RODEMICH: Differential entropy and tiling, J. Statist. Phys. 1(1969), 57-69.

[9] A. RÉNYI: On the dimension and entropy of probability distributions, Acta Math. Acad. Sci. Hung. 10(1959), 193-215.

[10] A. RÉNYI: Dimension, entropy and information, Trans. 2nd Prague Conf. Information Theory, pp. 545-556, Prague, 1960.

[11] C.R. SHANNON: A mathematical theory of communication, Bell System Tech. J. 27(1948), 379-423, 623-656.

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