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ON THE LARGEST GENERALIZED JOINT SPECTRUM

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Abstract. An explicit description of the largest generalized joint spectrum on a Banach algebra is given. It is proved that this spectrum coincides with the rationally convex joint spectrum introduced by Waelbroeck. This answers questions posed in [4].

Key words: Banach algebra, generalized joint spectrum.

Classification: 46H05

Let A be a complex Banach algebra with the unit 1. By $\sigma^A(a)$, or simply $\sigma(a)$ if there is no confusion, we shall denote the usual spectrum of an element $a \in A$. A generalized joint spectrum on A is a function $\tilde{\sigma}$ which assigns to each finite collection $\{a_1,\dots,a_n\}$ of elements in A a compact subset of $\mathbf{C}^n$ (possibly empty) in such a way that the following three conditions are satisfied:

(I) $$\tilde{\sigma}(a_1,\dots,a_n) \subset \prod_{k=1}^{n} \sigma(a_k)$$

(For simplicity we write $\tilde{\sigma}(a_1,\dots,a_n)$ instead of $\tilde{\sigma}(\{a_1,\dots,a_n\})$;

(II) $$p(\tilde{\sigma}(a_1,\dots,a_n)) \subset \tilde{\sigma}(p(a_1,\dots,a_n))$$

where p is an arbitrary m-tuple of polynomials over $\mathbf{C}$ in n non-commutative indeterminates;

(III) $\tilde{\sigma}(a_1,\dots,a_n) \neq \emptyset$ whenever elements $a_1,\dots,a_n$ are pairwise commuting,

The above definition was given in [4]. It was shown that there exists the largest generalized joint spectrum (with respect to the following obvious partial order: $\tilde{\sigma}_1 \le \tilde{\sigma}_2$ if and only if $\tilde{\sigma}_1(a_1,\dots,a_n) \subset \tilde{\sigma}_2(a_1,\dots,a_n)$ for all finite subsets $\{a_1,\dots,a_n\}$ of A).

It was asked if one can give a simple characterization of this spectrum.

The bicommutant joint spectrum was given as a candidate for the largest generalized joint spectrum.

The purpose of the present paper is to give a description of this largest spectrum. We prove that it coincides with the rationally convex joint spectrum introduced by Waelbroeck. We also show (see Example 2 below) that it differs from the bicommutant joint spectrum in general.

Following L. Waelbroeck (see [6] or [7]) we shall give

Definition. Let $a_1,\dots,a_n \in A$ (we do not assume them to commute). The rationally convex joint spectrum of the n-tuple $(a_1,\dots,a_n)$ is the set

$$\bar{\sigma}(a_1,\dots,a_n)=\{(\lambda_1,\dots,\lambda_n)\in C^n : p(\lambda_1,\dots,\lambda_n)\in\sigma(p(A_1,\dots,a_n)) \text{ for every } p\in P_n\}$$

where P_n denotes the set of all polynomials over **C** in n non-commutative indeterminates.

Theorem. The largest generalized joint spectrum and the rationally convex joint spectrum coincide.

Proof. First we show that the rationally convex joint spectrum is a generalized joint spectrum, i.e. it satisfies conditions (I) - (III).

To see that (I) is fulfilled , take $p_j(x_1,\dots,x_n)=x_j$ $(j=1,\dots,n)$. Then $(\lambda_1,\dots,\lambda_n)\in\bar{\sigma}(A_1,\dots,a_n)$ implies

$$\lambda_j=p_j(\lambda_1,\dots,\lambda_n)\in\sigma(p_j(a_1,\dots,a_n))=\sigma(a_j)$$

which gives (I).

It is also clear that (II) holds true. If $(\mu_1,\dots,\mu_m)\in p(\bar{\sigma}(a_1,\dots,a_n))$ then $(\mu_1,\dots,\mu_m)=p(\lambda_1,\dots,\lambda_n)$ for some $(\lambda_1,\dots,\lambda_n)\in\bar{\sigma}(a_1,\dots,a_n)$. Taking an arbitrary $q\in P_m$ we get $q\circ p\in P_n$ and $(q\circ p)(\lambda_1,\dots,\lambda_n)\in\sigma((q\circ p)(a_1,\dots,a_n))$, i.e. $q(\mu_1,\dots,\mu_m)\in\sigma(q(p(a_1,\dots,a_n)))$ which means that $(\mu_1,\dots,\mu_m)\in\bar{\sigma}(p(a_1,\dots,a_n))$.

Finally (III) is trivially satisfied since we always have $\sigma(a_1,\dots,a_n)\subset\bar{\sigma}(a_1,\dots,a_n)$ where $\sigma(a_1,\dots,a_n)$ denotes the Harte´s spectrum (= the union of the left and the right joint spectra) of the n-tuple $(a_1,\dots,a_n)$.

Moreover we have $\tilde{\sigma}(a_1,\dots,a_n)\subset\bar{\sigma}(a_1,\dots,a_n)$ for every generalized joint spectrum $\tilde{\sigma}$ on A. Indeed, if $(\lambda_1,\dots,\lambda_n)\in\tilde{\sigma}(a_1,\dots,a_n)$, then by (II) and (I) $p(\lambda_1,\dots,\lambda_n)\in\tilde{\sigma}(p(a_1,\dots,a_n))\subset\sigma(p(a_1,\dots,a_n))$ for every $p\in P_n$.

Hence $(\lambda_1,\dots,\lambda_n)\in\bar{\sigma}(a_1,\dots,a_n)$ and we are done. So, $\bar{\sigma}$ is the largest generalized joint spectrum.

Let K be a compact subset of $\mathbb{C}^n$, $(1 \leq n < \infty)$. Then the rationally convex hull $\widetilde{K}$ of K is defined (see [1] or [7]) as the set of all n-tuples $(\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^n$ such that $|f(\lambda_1,\ldots,\lambda_n)| \leq \sup_{(z_1,\ldots,z_n)\in K} |f(z_1,\ldots,z_n)|$ for every rational function f analytic on the set K. Equivalently, $(\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^n$ belongs to $\widetilde{K}$ if and only if $p(\lambda_1,\ldots,\lambda_n) \in p(K)$ for every polynomial $p \in P_n$. Next corollary shows that if $a_1,\ldots,a_n$ are pairwise commuting elements of a Banach algebra A then $\vec{\sigma}(a_1,\ldots,a_n)$ is equal to the rationally convex hull of $\sigma(A_1,\ldots,a_n)$. Example 1 below will show that this is not the case when $a_1,\ldots,a_n$ do not commute.

Corollary 1. Let $a_1,\ldots,a_n$ be pairwise commuting elements of a Banach algebra A. Then $\vec{\sigma}(a_1,\ldots,a_n)$ is the rationally convex hull of the Harte's spectrum $\sigma(a_1,\ldots,a_n)$.

Proof. If elements $a_1,\ldots,a_n$ are pairwise commuting then the Harte's spectrum has the spectral mapping property. In particular, $\sigma(p(a_1,\ldots,a_n))=$ $=p(\sigma(a_1,\ldots,a_n))$ for all $p \in P_n$ (see [2]). This implies immediately that $\vec{\sigma}(a_1,\ldots,a_n)$ is the rationally convex hull of the Harte's spectrum $\sigma(a_1,\ldots,a_n)$.

Corollary 2. Let $a_1,\ldots,a_n$ be elements of a Banach algebra A. Then

$$\widetilde{\sigma(a_1,\ldots,a_n)} \subset \vec{\sigma}(a_1,\ldots,a_n) \subset \sigma^{[a_1,\ldots,a_n]}(a_1,\ldots,a_n)$$

where $\sigma^{[a_1,\ldots,a_n]}(a_1,\ldots,a_n)$ denotes the Harte's spectrum of the n-tuple $(a_1,\ldots,a_n)$ in the algebra $[a_1,\ldots,a_n]$ generated by $a_1,\ldots,a_n$ and the unit.

Proof. Let $(\lambda_1,\ldots,\lambda_n) \in \widetilde{\sigma(a_1,\ldots,a_n)}$. Then

$$p(\lambda_1,\ldots,\lambda_n) \in p(\sigma(a_1,\ldots,a_n)) \subset \sigma(p(a_1,\ldots,a_n))$$

for every $p \in P_n$ (see [2]). Hence $(\lambda_1,\ldots,\lambda_n) \in \vec{\sigma}(a_1,\ldots,a_n)$ and the rationally convex hull of $\sigma(a_1,\ldots,a_n)$ is contained in $\vec{\sigma}(a_1,\ldots,a_n)$.

Property (II) implies that $\vec{\sigma}$ is translation invariant, i.e. $(\lambda_1,\ldots,\lambda_n) \in \vec{\sigma}(a_1,\ldots,a_n)$ if and only if $(0,\ldots,0) \in \vec{\sigma}(a_1-\lambda_1,\ldots,a_n-\lambda_n)$. Therefore to prove the second inclusion it is sufficient to show that $(0,\ldots,0) \in \vec{\sigma}(a_1,\ldots,a_n)$ implies $(0,\ldots,0) \in \sigma^{[a_1,\ldots,a_n]}(a_1,\ldots,a_n)$.

Suppose $(0,\ldots,0) \in \vec{\sigma}(a_1,\ldots,a_n)$. Then $M=\{p(a_1,\ldots,a_n): p \in P_n,\ p(0,\ldots,0\}^-$ is a linear subspace of codimension 1 in the algebra $[a_1,\ldots,a_n]$

consisting of singular elements in A (and thus singular in $[a_1,\dots,a_n]$). By the Gleason-Kahane-Żelazko theorem (see [8], p. 87) M is a maximal two-sided ideal in $[a_1,\dots,a_n]$ and $(0,\dots,0)\in\sigma^{[a_1,\dots,a_n]}(a_1,\dots,a_n)$.

Now we proceed to the previously mentioned examples.

Example 1 (cf. [5], Example 1). Let A be the algebra $M_5(C)$ of all 5×5 matrices with complex entries. Take the following two elements of A:

$$a_1=\begin{pmatrix}0&1&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix}\text{ and } a_2=\begin{pmatrix}0&0&1&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\end{pmatrix}$$

Then we have $a_1^3=a_2^3=0$. Hence $\sigma^A(a_1)=\sigma^A(a_2)=\{0\}$. This implies $\sigma^A(a_1,a_2)\subset\{(0,0)\}$. Further $a_1a_3+a_2a_1=1$ and $a_2a_1+a_4a_2=1$ where

$$a_3=\begin{pmatrix}0&0&0&0&0\\1&0&0&-1&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}\text{ and } a_4=\begin{pmatrix}0&1&0&0&-1\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix}$$

Hence $\sigma^A(a_1,a_2)=\emptyset$. Let $B=[a_1,a_2]$.

If we assign to each element $b\in B$ the entry of b which is placed in the third row and the third column, then we shall get a linear functional φ on B. We prove that φ is multiplicative on B. By the Gleason-Kahane-Żelazko theorem it is sufficient to show that $\varphi(a_{i_1}a_{i_2}\dots a_{i_k})=0$ for all finite products of a_1 and a_2 i.e. for all $k\in\{1,2,\dots\}$, $i_1,\dots,i_k\in\{1,2\}$. This is clear if $a_{i_1}=a_2$ as the third row is then equal to zero. From the same reason $\varphi(a_{i_1}\dots a_{i_k})=0$ if $a_{i_1}=a_1$, $a_{i_2}=a_2$. The rest follows from the relations $\varphi(a_1^2)=\varphi(a_1^2a_2)=0$, $a_1^3=a_1^2a_2^2=0$ and $a_1^2a_2a_1=a_1^2$ which can be checked directly. Thus $(0,0)=(\varphi(a_1),\varphi(a_2))\in\sigma^B(a_1,a_2)$ and $p(0,0)=p(\varphi(a_1),\varphi(a_2))=\varphi(p(a_1,a_2))\in\sigma^B(p(a_1,a_2))$ for every polynomial $p\in P_2$.

Further $\sigma^B(p(a_1,a_2))=\partial\sigma^B(p(a_1,a_2))\subset\sigma^A(p(a_1.a_2))$ as $\dim B<\infty$.

Hence $(0,0) \in \tilde{\sigma}^A(a_1,a_2)$ and $\tilde{\sigma}^A(a_1.a_2)$ is not the rationally convex hull of $\sigma^A(a_1,a_2)=\emptyset$.

Example 2. Let $K=\{(z_1,z_2)\in C^2,\ |z_2|\leq|z_1|\leq 1\}$. Then K is compact but not rationally convex. Its rationally convex hull $\tilde{K}$ is equal to

$$\tilde{K}=\{(z_1,z_2)\in C^2,\ |z_1|\leq 1,\ |z_2|\leq 1\}$$

(see [1], p. 76).

Let A=C(K) be the algebra of all continuous complex-valued functions on K. Then the bicommutant joint spectrum σ'' (cf. [4]) coincides with the Harte's spectrum on this algebra. Put $\pi_1(z_1,z_2)=z_1$ and $\pi_2(z_1,z_2)=z_2$. Then

$$\tilde{\sigma}(\pi_1,\pi_2)=\widehat{\sigma(\pi_1,\pi_2)}=\tilde{K}\neq K=\sigma'(\pi_1,\pi_2)=\sigma''(\pi_1,\pi_2).$$

Thus we see that the rationally convex joint spectrum is larger than the bicommutant spectrum.

References

[1] T.W. GAMELIN: Uniform algebras, Prentice-hall, Englewood Cliffs, N.J. 1969.

[2] R.E. HARTE: Spectral mapping theorems, Proc. Roy. Irish Acad. Sect. A 72 (1972), 89-107,

[3] V. MÜLLER and A. SOŁTYSIAK: Spectrum of generators of a non-commutative Banach algebra, submitted.

[4] A. SOŁTYSIAK: Joint spectra and multiplicative functionals, it will appear in Coll. Math. vol. 56.

[5] A. SOŁTYSIAK: Approximate point joint spectra and multiplicative functionals, Studia Math. 86(1987), 277-286.

[6] L. WAELBROECK: Le calcul symbolique dans les algèbres commutatives, J. Math. Pures et Appl. 33(9)(1954), 147-186.

[7] L. WAELBROECK: The holomorphic functional calculus as an operational calculus, in: Banach Center Publications, vol. 8, Spectral Theory, PWN, Warszawa 1982, 513-552.

[8] W. ŻELAZKO: Banach algebras, Elsevier, PWN, Amsterdam, Warszawa 1973.

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