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S(n)-SPACES AND H-SETS

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Abstract: Let X be an S(n)-space, $n\in N$. By means of the θ^n-closure operator, introduced in [DG], we define certain subspaces of X, called S(n)-sets, and study their relationships to H-sets and θ-closed sets.

Key words and phrases: θ^n-closure, S(n)-filter, S(n)-cover, S(n)-set.

Classification: 54D25, 54B05, 54D10

Introduction. A Hausdorff space X is said to be H-closed if it is closed in every Hausdorff space in which it can be embedded. Such a property is productive but, in general, it is neither hereditary nor closed hereditary. For an account on H-closed spaces see [PT] and [DP]. A subset M of a topological space X is an H-set if every cover of it by open sets of X has a finite subfamily which covers M with the closures of its members. The concept of H-set was introduced in [V] and, independently, in [PT] under the name of H-closed relative to X. An H-set of an H-closed space need not be H-closed as a space.

Closely related to the study of H-sets is the θ-closure operator, also defined in [V]. The θ-closure of M in X is the set $cl_\theta M=\{x\in X:\overline{V}\cap M\neq\emptyset$, for every open neighborhood V of $x\}$. M is θ-closed iff $M=cl_\theta M$.

The following results are well known:

(a) Every θ-closed subset of an H-closed space is an H-set [V].

(b) If X is H-closed and Urysohn, then $M\subset X$ is θ-closed iff it is an H-set [DP].

(c) M is an H-set of a space X iff, for every filter $\mathcal{F}$ on X, which meets M, $M\cap ad_\theta\mathcal{F}\neq\emptyset$, where $ad_\theta\mathcal{F}=\bigcap\{cl_\theta F:f\in\mathcal{F}\}$ [Ha]. Recently, Dikranjan and Giuli [DG] have introduced a θ^n-closure operator, $n\in N$, for the study of S(n)-closed and S(n)-θ-closed spaces. The S(n)'s, $n\in N$, form a class of quotient reflective subcategories of the category of topological spaces, which includes that of T_1-spaces (n=0), Hausdorff spaces (n=1) and Urysohn spaces (n=2). In [PV] S(α)-spaces were firstly defined, for every ordinal α. In the present note we study H-sets, θ-closed sets and related concepts in the above categories. In particular we give the notions of S(n)-set by means of special filters and covers, and give the correspective of statements (a), (b) and (c) in the categories S(n), $n \in N$.

1. Preliminary notions. Let X be any topological space and let $M \subset X$. The θ^n-closure of M in X [DG], $n > 0$, is the set $cl_{\theta^n} M$ defined by the following property: if $x \in X$, then $x \notin cl_{\theta^n} M$ means that there exists a finite sequence $U_1, \dots, U_n$ of open neighborhoods of x, with

(a) $\overline{U}_i \subset U_{i+1}$, $i=1,\dots,n-1$.

(b) $\overline{U}_n \cap M = \emptyset$.

In such a case x and M are said to be S(n)-separated in X. For n=0 one puts $cl_{\theta^n} M = \overline{M}$, ordinary closure in X. Note that the θ^1-closure coincides with the θ-closure defined in the introduction.

M is θ^n-closed iff $M = cl_{\theta^n} M$. Every θ^n-closed subset of X is closed. Correspondingly, there is a notion of θ^n-interior defined by $int_{\theta^n} M = X - cl_{\theta^n}(X-M)$.

The form of S(n)-separatedness between two distinct points $x, y \in X$ may be simplified as follows [DG], 1.4(b):
x and y are S(n)-separated in X, $n > 0$, iff there are open neighborhoods U, V of x, y, respectively, such that $U \cap \overline{V} = \emptyset$ and $y \in int_{\theta^{n-1}} V$. A topological space X is an S(n)-space if every two distinct points of X are S(n)-separated.

2. Results

2.1. Definitions. Let X be any topological space, M a subset of X, and let $n \geq 0$.

(a) A filter $\mathcal{F}$ on X is an S(n)-filter with respect to M if $M \cap ad\, \mathcal{F} = M \cap ad_{\theta^n} \mathcal{F}$, where $ad_{\theta^n} \mathcal{F} = \bigcap \{ cl_{\theta^n} F : F \in \mathcal{F} \}$.

(b) A cover $\{U_i\}_I$ of M by open sets of X, is an S(n)-cover with respect to M if $M \subset \bigcup \{ int_{\theta^n} U_i : i \in I \}$.

(c) M is an S(n)-set of X if every closed S(n)-filter w.r. to M, which meets M, has adherent points in M.

The former two definitions are taken from [DG], but relativized to the subset M of X. The definition of S(n)-set is clearly inspired to that of

H-set and this will be clear later on.

Let $m \geq n \geq 0$ be integers. It is easy to realize that, for a subset M of X, one has $cl_{\theta^n} M \subset cl_{\theta^m} M$, hence $int_{\theta^m} M \subset int_{\theta^n} M$. From this observation it follows that every S(m)-cover (resp. S(m)-filter) w.r. to M is an S(n)-cover (resp. S(n)-filter) w.r. to M. Then, every S(n)-set of X is an S(m)-set.

The S(0)-sets of X are exactly the compact subsets. Hence, a compact subset M of X is (an H-set and) an S(n)-set of X, for every $n \geq 0$.

2.2. Proposition. M is an S(n)-set of X, $n \geq 0$, iff every S(n)-cover w.r. to M has a finite subcover.

Proof. Let M be an S(n)-set of X, $n \geq 0$, and let $\{U_i\}_I$ be an S(n)-cover w.r. to M which has no finite subcover. For every finite subset $\alpha \subset I$, let $F_\alpha = X - \bigcup_{i \in \alpha} U_i$. The closed filter $\mathcal{F}$ generated by the F_α's is then a closed S(n)-filter w.r. to M which meets M and has no adherent points in M, in fact $M \cap ad\,\mathcal{F} = M \cap ad_{\theta^n}\mathcal{F} = \emptyset$, since $M \cap ad_{\theta^n}\mathcal{F} = M \cap (\bigcap_\alpha cl_{\theta^n} F_\alpha) = M \cap (\bigcap_\alpha cl_{\theta^n}(X - \bigcup_{i \in \alpha} U_i)) = M \cap (\bigcap_\alpha (X - int_{\theta^n} \bigcup_{i \in \alpha} U_i)) \subset M \cap (\bigcap_\alpha (X - \bigcup_{i \in \alpha} int_{\theta^n} U_i)) = M \cap (X - \bigcup_{i \in I} int_{\theta^n} U_i) = \emptyset$.

Conversely, suppose that $\mathcal{F} = \{F_i\}_I$ is a closed S(n)-filter w.r. to M which meets M and such that $M \cap ad_{\theta^n}\mathcal{F} = \emptyset$. Let us define $U_i = X - F_i$, for every $i \in I$. Then $\{U_i\}_I$ is a cover of M by open sets of X. Moreover, it is an S(n)-cover w.r. to M which has no finite subcover.

The following results give the relations of the concepts of H-sets, S(n)-sets, θ-closed and θ^n-closed subsets of a given space.

2.3. Proposition. Every H-set of a space X is an S(n)-set, for every $n > 0$.

Proof. Let M be an H-set of X; by the remark above, in order to prove the proposition, it is sufficient to show that M is an S(1)-set of X.

Let $\{U_i\}_I$ be an S(1)-cover w.r. to M. For every $x \in M$ there is an index $i(x) \in I$ such that $x \in int_\theta U_{i(x)}$. Then x and $X - int_\theta U_{i(x)}$ are S(1)-separated in X, hence there is an open neighborhood $V_{i(x)}$ of x with $\overline{V}_{i(x)} \cap (X - int_\theta U_{i(x)}) = \emptyset$, that is $\overline{V}_{i(x)} \subset int_\theta U_{i(x)}$. Since M is an H-set, the cover $\{V_{i(x)}\}_{x \in M}$ of M admits a finite subfamily $\{V_{i(x_1)}, \ldots, V_{i(x_m)}\}$ with $M \subset \bigcup_{k=1}^{m} \overline{V_{i(x_k)}}$. It follows that $\{U_{i(x_k)}\}_{k=1}^{k=m}$ is a finite subcover of $\{U_i\}_I$, so that M is an S(1)-set of X.

2.4. Proposition. Let M be an S(n)-set, $n>0$, of a space (X,τ). M is compact in (X, τ_{θ^n}), where τ_{θ^n} is the topology generated on X by the θ^n-closure.

Proof. If $\{U_i\}_I$ is a cover of M with τ_{θ^n}-open sets of X, then $\{U_i\}_I$ is an S(n)-cover w.r. to M, so it admits a finite subcover.

2.5. Proposition. Let X be an S(n)-space, $n>0$. If M is an S(n-1)-set of X, then M is θ-closed in X.

Proof. The proof goes almost on the same line of that of Th. 2.2 of [DG]. We give it for sake of completeness.

Suppose there is a point $x \in cl_\theta M-M$. Then, for every $m \in M$, x and m are S(n)-separated. This means that there are open neighborhoods U_m and V_m of m, x, respectively, such that $m \in int_{\theta^{n-1}} U_m$ and $U_m \cap \overline{V}_m=\emptyset$. $\{U_m\}_{m\in M}$ is an S(n-1)-cover w.r. to M, hence it has a finite subcover $\{U_{m_1},\dots,U_{m_k}\}$. Setting $V= \bigcap_{i=1}^{k} V_{m_i}$, then $\overline{V}\cap M=\emptyset$, by hypothesis. Since V is an open neighborhood of x, this is a contradiction to $x \in cl_\theta M-M$, hence M has to be θ-closed.

In [DG] an S(n)-space M, $n>0$, is defined to be S(n)-θ-<u>closed</u> if it is closed in every S(n)-space in which it can be embedded. By Th. 2.2 of [DG], X is S(n)-θ-closed, $n>1$, if and only if it is an S(n-1)-set of itself. Every S(n)-θ-closed space is S(n)-closed. A space X which is H-closed and Urysohn is S(n)-θ-closed, for every $n>1$.

Also in [DG], Ex. 4.4, there is exhibited a space X which is Urysohn (= S(2))-θ-closed and not H-closed. This can be read by saying that such an X is an S(1)-set of itself but not an H-set; hence the converse of Prop. 2.3 does not hold.

2.6. Proposition. Let X be S(n)-θ-closed, $n>1$, and let $M\subset X$. M is an S(n-1)-set of X whenever it is θ^{n-1}-closed in X.

Proof. Let $\{U_i\}_I$ be an S(n-1)-cover w.r. to M. Then $\{X-M\}\cup\{U_i\}_I$ is an S(n-1)-cover w.r. to X. The proposition follows by the remark above.

2.7. Theorem. Let X be an S(2)-θ-closed space and let $M\subset X$. Consider the following statements:

(a) M is an H-set of X.

(b) M is an S(1)-set of X.

(c) $M=cl_\theta M$.

Then (a) $\longrightarrow$ (b) $\longleftrightarrow$ (c) always. In case X is an S(2)-space which is H-closed, (a), (b) and (c) are all equivalent.

Proof. The implication (a) $\longrightarrow$ (b) is contained in Prop. 2.3. The equivalence of (b) and (c) follows from Prop. 2.5 and 2.6, for n=2. The last assertion is motivated by (b) of the introduction.

The following example is a modification of [HE], Beisp. 5 and [DG], Ex. 4.3. Let $I=I_1 \cup I_2 \cup I_3$ be a partition of the unit real interval, where each I_i, i=1,2,3, is dense in I and $0 \in I_1$. Let us denote by σ the coarsest topology on I, containing the usual compact topology τ and I_2, I_3 as open sets. (I, σ) is an S(2)-space; it is actually an S(2)-closed space which is not H-closed.

Let (x_n) be any sequence, contained in I_3, converging to 0 in (I, τ). $D = \{x_n : n \in N\}$ is θ-closed in (I, σ), that is $D \in \sigma_\theta$, hence $\sigma_\theta > \tau$ and D is not compact in (I, σ_θ). By Prop. 2.4, D cannot be an S(1)-set of (I, σ).

2.8. Definition. Let X be an S(n)-space. A subset M of X is said to be S(n)-<u>embedded</u> in X if, for every open set V of X, one has

$$M \cap \mathrm{int}_{\theta^n} V = \mathrm{int}^M_{\theta^n} (M \cap V),$$

where $\mathrm{int}^M_{\theta^n}$ denotes the θ^n-interior in the subspace M of X.

2.9. Theorem. Let X be an S(n)-θ-closed space, $n > 1$, and let $M \subset X$ be S(n-1)-embedded in X.

Consider the following statements:

(a) M is an S(n)-θ-closed space.

(b) M is an S(n-1)-set of X.

Then (a) $\longrightarrow$ (b) always and (a) $\longleftrightarrow$ (b) for n=2.

Proof. Suppose M is an S(n)-θ-closed space in the induced topology from X. Then the implication (a) $\longrightarrow$ (b) follows from the remark preceding Prop. 2.6 and from the easy observation that, for every space X and $M \subset Y \subset X$, M is and S(m)-set of X whenever it is an S(m)-set of Y, $m \in N$.

Let now n=2 and assume that M is an S(1)-set of X. Let $\{U_i\}_I$ be an S(1)-cover w.r. to M; then $U_i = M \cap V_i$, V_i open in X, for every $i \in I$. The family $\{M \cap \mathrm{int}_{\theta^{n-1}} V_i\} = \{\mathrm{int}^M_{\theta^{n-1}} U_i\}$ is a cover of M. Now $\{X-M\} \cup \{V_i\}_I$ is an S(1)-cover w.r. to X by Th. 2.7, hence it admits a finite subcover $\{X-M, V_{i_1}, \dots, V_{i_m}\}$.

It follows that $\{U_{i_1},\dots,U_{i_m}\}$ is a finite subcover for M, so that M is S(2)-θ-closed.

Let X be the subset $\{(0,0)\} \cup \bigcup_{n=1}^{\infty} \{1/n\} \times [0,1/n]$ of the Euclidean plane. Let τ be the weakest topology on X, finer than the subspace topology σ and containing as closed subsets all subsets of $F=\{(1/n,0):n \in N\}$. Then (X,τ) is an H-closed Urysohn space while $M=cl_\theta F=F \cup \{(0,0)\}$ is an H-set and an S(1)-set of X which is not S(1)-embedded in X (cf. [DG], Ex. 4.2).

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