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BLOW UP ABOVE STATIONARY SOLUTIONS OF CERTAIN NONLINEAR PARABOLIC EQUATIONS

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Abstract: The instability properties of nontrivial equilibria of the Dirichlet problem for a class of nonlinear parabolic equations are studied. It is shown that every time dependent solution with initial function lying above an arbitrary nontrivial equilibrium blows up in a finite time.

Key words: Degenerate parabolic equations, blow up, stationary solutions.

Classification: 35K65, 35K55

0. Introduction

The present paper is concerned with the instability properties of nontrivial equilibria of the problem

(0.1) $$u_t = \Delta u^m + u^p - au \qquad x\in D,\ t>0,$$

(0.2) $$u(x,t) = 0 \qquad x\in \partial D,\ t>0,$$

(0.3) $$u(x,0) = u_o(x)(\geq 0) \qquad x\in D,$$

where $D\subset R^N$ is a smoothly bounded domain, $a\geq 0$, $0<m\leq p$, $1<p$ and $pm^{-1}<(N+2)(N-2)^{-1}$ if $N\geq 3$.

It is known that for $u_o\in L^\infty(D)$, $u_o\geq 0$, the unique weak solution $u(t,u_o)$ exists locally, i.e. there is a $t_{max}(u_o) = \sup\{\tau>0: u(t,u_o)$ exists on $[0,\tau]\}$, $t_{max}\leq\infty$. If $t_{max}<\infty$ then $\|u(t,u_o)\|_{L^\infty(D)}\to\infty$ as $t\to t_{max}$ and we say that u blows up in finite time.

There are various conditions in the literature which ensure that solutions of related problems do not exist globally and thus that a blow up occurs (see e.g. Ball [2], Fila and Filo [6],[7], Galaktionov [9], Payne and Sattinger [18], Nakao [15], Ni, Sacks and Tavantzis [17], Sacks [20], Tsutsumi [22], Weissler [23]). In [6],[7] Lyapunov functional is used to describe a set of initial data for which the solutions of (0.1)-(0.3) blow up.

The aim of this paper is to establish a new sufficient condition for blow up, namely to show that $t_{max}(u_0)<\infty$ if $u_0\in B^+(v) = \{u_0\in L^\infty(D): u_0\geq v,\ u_0\not\equiv v$ on the support of $v\}$, where v is an arbitrary nonnegative nontrivial equilibrium solution (equilibria with compact support in D may occur only if $1<m$, $0<a$). We prove this result in the following cases:

(i) $a=0$, $0<m<p$,

(ii) $a>0$, $0<m\leq p$, $N=1$,

(iii) $a\geq 0$, $m=1$.

For a,m as in (i),(iii), the existence of positive equilibria is well known. If (ii) is fulfilled ($D=(-L,L)$, $L>0$), nonnegative nontrivial equilibria exist when $m<p$ or $m=p$, $L>\pi/2$ (Section 3) . The present characterization of nonnegative initial functions which cause blow up is different than in [6],[7](see Remark 1) , except of the case when (i) or (iii) is satisfied and the positive equilibrium is unique. But the positive steady states are known to be not unique for example if D is an N-dimensional annulus, $a=0$, $m=1$, p is close to $(N+2)(N-2)^{-1}$ ([4]). For the interesting geometry of the set of (many) positive equilibria for $a=1$, $m=1$, p an odd integer, D an annulus, we refer to [10, Example 2.13] .

In the case (iii) we do not restrict ourselves to nonnegative equilibria and initial data. Instead of u^p-au we consider the growth term $|u|^{p-1}u-au$ and we show in addition that the solution $u(t,u_0)$ blows up if $u_0\in B^+(v)\cup B^-(v)$ ($B^-(v)=\{u_0\in L^\infty(D):u_0\leq v,\ u_0\not\equiv v\}$), where v is an arbitrary equilibrium which changes the sign. The existence of infinitely many steady states with this property was demonstrated in [21] for a nonlinearity more general than $|u|^{p-1}u$.

The basic step of the proofs of our results consists in finding a solution w of (0.1),(0.2) such that $u(t_0,u_0)\geq w(\cdot,0)$ for some $t_0>0$, $\|w(\cdot,t)\|_{L^\infty}$ is unbounded and $w(x,t)$ is nondecreasing in t for each $x \in D$. A Moser-type estimate (Proposition 4) yields that $\|w(\cdot,t)\|_{L^{m+p}}$ is unbounded. The monotonicity in t implies the existence of limits of $\|w(\ ,t)\|_{L^q}$ as $t \to t_{max}(w(\cdot,0))$ for every $q \geq 1$. An argument based on an idea of Sattinger and Payne [18](Proposition 5) leads then to the conclusion that $t_{max}(u_o) < \infty$.

It is not difficult to see that every solution of (0.1)-(0.3) with $u_o \in B^+(v)$ is unbounded in the L^∞-norm. From [17] it follows that unbounded positive solutions of certain parabolic equations blow up. However, the problems considered in [17] cover our problem only in the particular cases a=0, m=1, $1<p<(N+2)/N$, D-convex or a=0, $1<m<p<(N+2)/N$, D-convex and u_o- decreasing near ∂D in a suitable way.

Since the behaviour of solutions to (0.1)-(0.3) depends strongly upon the structure of the set of nonnegative stationary solutions, in Section 3 we describe the dependence of this set on m when N=1. This detailed information about equilibria will be useful in the proof of our result in the case (ii).

1. Preliminaries

Throughout this paper we shall use the following hypotheses about the data D and u_o:

(H1) D is a bounded domain in R^N whose boundary ∂D, is of class C^3.

(H2) $u_o^m \in L^\infty(D) \cap H_o^1(D)$ and $u_o \geq 0$ a.e. in D.

We shall refer to these hypotheses collectively as (H). For simplicity we restrict ourselves to $u_o \in L^\infty(D) \cap H_o^1(D)$, as it is known that any weak solution $u(t,u_o)$ of (0.1)-(0.3) for $u_o \in L^\infty(D)$ only is actually in $L^\infty(D) \cap H_o^1(D)$ at any later time (see e.g.[11]).

Let us now introduce some notation: $Q_T = D\times(0,T)$, $S_T = \partial D\times(0,T)$, $|D|$- Lebesgue measure of the set D, $|u|_q = \|u\|_{L^q(D)}$, $1 \leq q \leq \infty$, $|u|_q^q = (|u|_q)^q$, $\|u\| = (\int_D |\nabla u|^2 dx)^{1/2}$, $\int_D h(t) = \int_D h(x,t)dx$, $\iint_{Q_T} h = \iint_{Q_T} h(x,t)dxdt$ and $(u(t),v(t)) = \int_D u(x,t)v(x,t)dx$.

DEFINITION 1. By a solution of Problem (0.1)-(0.3) on [0,T] we mean a nonnegative function u such that

$$u\in C([0,T];L^2(D))\cap L^\infty(Q_T),\ u^m\in L^\infty(0,T;H_0^1(D))$$

and u satisfies

$$(1.1)\qquad (u(t),\varphi(t))-\iint_{Q_t}(u\varphi_t-\nabla u^m\nabla\varphi+f(u)\varphi)=(u_0,\varphi(0))$$

for all $t\in[0,T]$ and $\varphi\in H^1(0,T;L^2(D))\cap L^\infty(0,T;H_0^1(D))$, $f(u)=u^p-au$.

A subsolution (supersolution) of Problem (0.1)-(0.3) is defined as above with equality in (1.1) replaced by $\leq$ ($\geq$) whenever $\varphi\geq 0$.

In the sequel we shall often denote the solution $u(=u(x,t))$ of Problem (0.1)-(0.3) by $u(t,u_0)$.

DEFINITION 2. By a stationary solution (equilibrium) of Problem (0.1)-(0.3) we mean a nonnegative function v such that

$$v^m\in C^2(D)\cap C^1(\bar{D}),\ v=0 \text{ on } \partial D \text{ and } \Delta v^m+f(v)=0 \text{ in } D .$$

By $E=E(D)$ we shall denote the set of all nontrivial stationary solutions of Problem (0.1)-(0.3) .

PROPOSITION 1 (Comparison principle). Suppose that D satisfies (H1) and that u_0 and v_0 both satisfy (H2). If u is a subsolution and v is a supersolution of Problem (0.1)-(0.3) on $[0,T]$ with

$$u_0\leq v_0 \quad\text{then}\quad u\leq v \quad \text{a.e. in } Q_T.$$

For the proof of this proposition for $m\geq 1$ we refer to [1] and for $0<m<1$ to [8](the comparison principle holds also for weak sub- and supersolutions with initial functions from $L^\infty(D)$ only) .

PROPOSITION 2 (Existence). Suppose that (H) holds. Then there exists a number t_{max}, $0<t_{max}\leq\infty$(which depends on the data D,m,f and u_0) such that Problem (0.1)-(0.3) possesses a unique solution u on $[0,T]$ for any $T\in(0,t_{max})$. If $t_{max}<\infty$ then

$$(1.2)\qquad \lim_{t\to t_{max}^-}|u(t,u_0)|_\infty=\infty.$$

Moreover for $0\leq s<t<t_{max}$ u satisfies

$$(1.3)\qquad \frac{4m}{(m+1)^2}\int_s^t|(u^{(m+1)/2})_t|_2^2+J(u^m(t,u_0))\leq J(u^m(s,u_0)) ,$$

where

$$(1.4) \qquad J(w) = \frac{1}{2} \| w \|^2 - \int_D \int_0^w f(r^{1/m}) dr \ .$$

For the proof of Proposition 2 for $m \geq 1$ we refer to [12] and for $0<m<1$ to [8].

To study the asymptotic behaviour of solutions to Problem (0.1)-(0.3) we introduce the notion of ω-limit set:

$$\omega(u_0) = \{ w \in C(\overline{D}) : \text{there exists } t_n \to \infty \text{ such that } u^m(t_n, u_0) \to w^m \text{ uniformly as } n \to \infty \} \ .$$

Some basic observations are collected in the next proposition.

PROPOSITION 3. Suppose that (H) holds and that $u(\cdot, u_0)$ is a bounded (in L^∞ norm) solution of Problem (0.1)-(0.3) on $[0,\infty)$. Then

(i) $\{ u^m(t,u_0) : t \geq \varepsilon \}$ is relatively compact in $C(\overline{D})$ for any $\varepsilon > 0$.

(ii) $\omega(u_0)$ is nonempty .

(iii) $\omega(u_0) \subset E \cup \{0\}$.

For the proof of the assertions (i) and (iii) we refer to [11] and (ii) follows from (i).

PROPOSITION 4. Suppose that (H) holds.

(i) Assume further that

$$(1.5) \qquad \begin{array}{ll} 0<m<p, \ 1<p & \text{for } N=1,2 \ , \\ 0<m<p<(N+2)m/(N-2), \ 1<p & \text{for } N \geq 3. \end{array}$$

Let $|u(t,u_0)|_{m+p}$ be bounded on $[0,t_{max})$. Then $t_{max} = \infty$ and

$$|u(t,u_0)|_\infty \leq C(|u_0|_\infty \ , \sup_{0 \leq t < \infty} |u(t,u_0)|_{m+p})$$

for $0 \leq t < \infty$.

(ii) Let $m=p$ and let $|u(t,u_0)|_{m+1}$ be bounded on $[0,t_{max})$. Then $t_{max} = \infty$ and

$$|u(t,u_0)|_\infty \leq C(|u_0|_\infty \ , \sup_{0 \leq t < \infty} |u(t,u_0)|_{m+1})$$

for $0 \leq t < \infty$.

For the proof we refer to [7](for $m\geq 1$, $m<p$ see [16]).

PROPOSITION 5. Suppose that (H) holds.

(i) If $0<m<p$ and $|u(t,u_o)|_{m+p}\to\infty$ as $t\to t_{max}$, then $t_{max}<\infty$.

(ii) If $m=p$, $a>0$ and $|u(t,u_o)|_{m+1}\to\infty$ as $t\to t_{max}$, then $t_{max}<\infty$.

Proof. We shall prove the assertion (i). For the proof of (ii) (which is analogous and more simple) we refer to [7]. Following an idea from [18] we proceed by contradiction. Suppose that $t_{max}=\infty$ and denote $M(t) = \int_0^t |u|_{m+1}^{m+1}$.

Then we have

$$M'(t) = |u_o|_{m+1}^{m+1} + \int_0^t\int_D (u^{m+1})_t = |u_o|_{m+1}^{m+1} + \int_0^t (m+1)(-\|u^m\|^2 + |u|_{m+p}^{m+p} - a|u|_{m+1}^{m+1})$$

and further

$$M''(t) = (m+1)(-(m+p-\vartheta)m^{-1}J(u^m(t)) + (p-m-\vartheta)(2m)^{-1}\|u^m(t)\|^2 + a(p-1-\vartheta)(m+1)^{-1}|u(t)|_{m+1}^{m+1} + \vartheta(m+p)^{-1}|u(t)|_{m+p}^{m+p}),$$

where we choose $0<\vartheta<\min(p-1,p-m)$. Now (1.3) leads to the inequality

$$\begin{aligned} MM'' - (m+p-\vartheta)(m+1)^{-1}M'^2 &> \\ &> 4(m+1)^{-1}(m+p-\vartheta)\Big(\int_0^t\int_D u^{m+1}\int_0^t\int_D (u^{(m+1)/2})_t^2 - \\ &- \Big(\int_0^t\int_D u^{(m+1)/2}(u^{(m+1)/2})_t\Big)^2\Big) + (m+p)^{-1}\vartheta M|u|_{m+p}^{m+p} - \\ &- (1+m^{-1})(m+p-\vartheta)J(u_o^m)M - 2(m+p-\vartheta)(m+1)^{-1}|u_o|_{m+1}^{m+1}M' . \end{aligned}$$

The first term on the right hand side is nonnegative according to the Schwarz inequality and the last term may by estimated as follows

$$-2(m+p-\vartheta)(m+1)^{-1}|u_o|_{m+1}^{m+1}M' \geq -\varepsilon(M')^{m+p} - C(\varepsilon,m,p,\vartheta,|u_o|_{m+1}^{m+1}),$$ where

we put $0<\varepsilon<(m+p)^{-1}\vartheta 2^{-1}|D|^{(1-p)/(m+1)}\int_0^1 |u|_{m+1}^{m+1}$. Hence we get

$$MM'' - (m+p-\vartheta)(m+1)^{-1}(M')^2 >$$

$$(1.6)\quad > M\left(2^{-1}(m+p)^{-1}\vartheta |u|_{m+p}^{m+p} - (1+m^{-1})(m+p-\vartheta)J(u_0^m) \right) +$$
$$+ |u|_{m+1}^{m+p}(2^{-1}(m+p)^{-1}\vartheta |D|^{(1-p)/(m+1)}M - \varepsilon) - C$$

and it is easy to see that there exists a $t_0 \geq 1$ such that the right hand side of (1.6) is positive for $t \geq t_0$, therefore

$$(M^{-\lambda})'' < 0 \quad \text{for } t \geq t_0 \text{ where } \lambda = (p-1-\vartheta)/(m+1) .$$

Since $M^{-\lambda}$ is decreasing, it must have a root $t_1 > 0$ what is a contradiction.

2. Main results

THEOREM 1. Suppose that (H) holds, and let v be an arbitrary nonnegative nontrivial stationary solution of Problem (0.1)-(0.3). Then any solution $u(t,u_0)$ with $u_0 \in B^+(v)$ blows up in a finite time if

(i) $a=0$ and (1.5) holds

or

(ii) $a>0$, $N=1$, $0<m\leq p$ and $1<p$.

THEOREM 2. Suppose that $m=1$, the reaction term in (0.1) is replaced by $|u|^{p-1}u - au$, $1<p$, $p<(N+2)/(N-2)$ if $N\geq 3$, $a\geq 0$, D satisfies (H1) and $u_0 \in L^\infty(D) \cap H_0^1(D)$. Let v be a nontrivial equilibrium. Then $u(t,u_0)$ blows up in a finite time if

(i) v is positive or it changes the sign and $u_0 \in B^+(v)$,

or

(ii) v is negative or it changes the sign and $u_0 \in B^-(v)$.

Proof of Theorem 1(i). Take an arbitrary nontrivial stationary solution v. According to the Hopf maximum principle for elliptic equations ([18]) we have $v>0$ in D, $\partial(v^m)/\partial\nu < 0$ on ∂D and we can apply the theory of Bertch and Rostamian [3] to show that v is unstable in the linearized sense. It is shown in [3] that in the case $m>1$ the behaviour of u near v depends on the spectrum of the eigenvalue problem

$$-\Delta w = g'(v^m)w + \lambda\beta'(v^m)w \quad \text{in } D ,$$
$$w = 0 \quad \text{on } \partial D,$$

where $\beta(r)=r^{1/m}$ and $g(r)=f(\beta(r))$. More precisely, if $\lambda_1<0$ then there exist $\varepsilon>0$ and for $u_0 \geq v$ ($u_0 \leq v$), $u_0 \not\equiv v$ a $T=T(u_0)\geq 0$ such that $u^m(t,u_0) > v^m+\varepsilon e$ ($u^m(t,u_0) < v^m-\varepsilon e$) for $t\geq T$, where $\Delta e=-1$ in D, $e=0$ on ∂D ([3,Theorem 4.8]). To prove an analogous statement for $m<1$ is not difficult and we omit it here. The negativity of λ_1 for g convex, $g(0)=0$ and $m=1$ is well known (see e.g. [13]) and this result can be generalized to our situation by an obvious way.

The linearized unstability of v yields the existence of a constant $K>1$ such that $u(t,u_0)>Kv$ for $t\geq T$. But Kv is a subsolution of Problem (0.1)-(0.3), hence $u(t+T,u_0) \geq u(t,Kv)$ for $t\geq 0$ by the comparison principle (Proposition 1). The comparison principle implies also that $u(\cdot,Kv)$ is nondecreasing, i.e. $u(t,Kv)\leq u(s,Kv)$ for $0\leq t\leq s$, thus $u(t,Kv)$ can not tend to a stationary solution. This follows e.g. from the linearized unstability of any nontrivial stationary solution. Hence $|u(t,Kv)|_\infty$ is unbounded by Proposition 3. By Proposition 4 and monotonicity of $u(\cdot,Kv)$ we get that $|u(t,Kv)|_{m+p}\to\infty$ as $t\to t_{max}^-$, so we can apply Proposition 5 and conclude that $t_{max}<\infty$.

REMARK 1. If the positive equilibrium is unique and a=0 or a>0,m=1 , then $Kv^m\in B$ for $K>1$, where $B=\{ w\in H_0^1(D),\ w\geq 0 \text{ a.e. in } D, w\not\equiv 0: J(\lambda w)< d \text{ for } 1\leq\lambda<\infty,\ d=J(v^m)\}$. It follows from [6] that $u(t,u_0)$ blows up if $u_0\in B$, therefore the blow up result in [6] is stronger in this particular case.

Proof of Theorem 1(ii). The principle of linearized unstability seems to be not applicable because equilibria with compact support in $D=(-L,L)$ may occur and the eigenfunctions of corresponding singular eigenvalue problems are not suitable to construct comparison functions (cf.[3,Section 5]). Therefore a result from the next section will be useful.

The classical strong maximum principle implies that $u(t_1,u_0)>v$ on supp v for arbitrary $t_1>0$. According to Lemma 2(v) it is possible to choose a subsolution $\underline{v}$ (of the stationary problem) which satisfies the inequalities $u(t_1,u_0)>\underline{v}$ in $(-L,L)$, $\max \underline{v} > \max v$ (e.g. $\underline{v} = v(\cdot,L-\varepsilon)$ on $(-L+\varepsilon,L-\varepsilon)$ for some small $\varepsilon>0$ extended by zero to [-L,L]). It follows then that $u(t,\underline{v})$ is not uniformly bounded (otherwise by Proposition 3 $\omega(\underline{v})\neq\emptyset$ and $\omega(\underline{v})\subset E$, but this is impossible because by comparison $u(t+t_1,u_0)\geq u(t,\underline{v})\geq \underline{v}$ for $t\geq 0$). Another consequence of the comparison principle is the monotonicity of $u(\cdot,\underline{v})$ as we have already mentioned. Propositions 4,5 yield then that $t_{max}<\infty$.

REMARK 2. It is not difficult to see that if $1<m\leq p$, $L>T(m,\alpha(m))$ (see Section 3), we can choose for every stationary solution v an initial function u_0 for which it holds that $u_0\geq v$, $u_0\not\equiv v$ in $(-L,L)$, $u_0\equiv v$ on supp v and $u(t,u_0)\to v$ as $t\to\infty$.

Proof of Theorem 2(i). Since v is linearly unstable, it follows from [14, Theorem 1] that there exists a function $w(x,t)$ satisfying (0.1),(0.2) defined on $\overline{D}\times(-\infty,t_{max}(w(\cdot,0)))$, $0<t_{max}(w(\cdot,0))\leq\infty$, such that $w(\cdot,t)\to v$ in $C^2(\overline{D})$ as $t\to-\infty$, w is strictly monotone increasing in t. We show that $t_{max}(w(\cdot,0))<\infty$. If v is positive, Propositions 4,5 yield the desired result. If v changes the sign, then $w(\cdot,t)$ may also change the sign for $t\geq 0$, but $\lim_{t\to t_{max}^-}|w(\cdot,t)|_{p+1}$ exists, because

$$|w(\cdot,t)|_{p+1}^{p+1} = \int_{D^+(t)} w^{p+1}(t) + \int_{D^-(t)} (-w(t))^{p+1} = I_1(t)+I_2(t),$$

$D^+(t)=\{x\in D:\ w(x,t)>0\}$ ($D^-(t)$ is defined analogously) and I_1,I_2 are obviously monotone. The assumption of nonnegativity of u_0 in Proposition 4,5 is not needed in this case, hence the conclusion. The proof of the assertion (ii) is almost the same.

3. Structure of the set of stationary solutions in one space dimension

In this section we shall discuss the problem

$$(3.1)\qquad \begin{aligned} &(v^m)_{xx} + f(v) = 0 \qquad x\in D=(-L,L),\ L>0,\\ &v(-L) = v(L) = 0, \end{aligned}$$

where $f(v)=v^p-av$, $p>1$, $a\geq 0$ and $0<m<\infty$.

We consider first $a>0$. Set $F(m,r)=\int_0^r u^{m-1}f(u)du$, $\alpha(m) = (a(p+m)/(m+1))^{1/(p-1)}$ ($\alpha(m)$ is the positive root of $F(m,\cdot)$),

$$M=\{(m,\mu):\ 0<m<\infty,\ \mu>\alpha(m)\}\cup\{(m,\alpha(m)):\ m>1\} .$$

Let us define the time-map $T(m,\mu)$ for $(m,\mu)\in M$ by the formula

$$(3.2)\qquad T(m,\mu)=\sqrt{\frac{m}{2}}\int_0^{\mu}\frac{r^{m-1}}{\sqrt{F(m,\mu)-F(m,r)}}dr \quad \left(=\sqrt{\frac{m}{2}}\mu^m\int_0^1\frac{y^{m-1}}{\sqrt{F(m,\mu)-F(m,\mu y)}}dy\right).$$

It can be shown by direct computations that the singularities in the integrand are integrable. Analogously as in [1],[5] it can be seen that the following holds.

LEMMA 1. v is a positive stationary solution to (3.1) if and only if

$$\sqrt{\frac{m}{2}}\int_{v(x)}^{\mu}\frac{r^{m-1}}{\sqrt{F(m,\mu)-F(m,r)}}dr = |x| , \quad |x|\leq L ,$$

where $(m,\mu)\in M$ and $L>0$ are related by the equation

$$(3.3)\qquad T(m,\mu) = L .$$

This lemma yields that the number of positive stationary solutions is determined by the number of roots of the equation (3.3). Obviously, $v=0$ is a stationary solution for every L, $m>0$. The next result is the description of the function T given by (3.2).

LEMMA 2.

(i) $T\in C(M)$, $T_\mu\in C(\text{int } M)$.

(ii) $\lim_{\mu\to\alpha(m)} T(m,\mu) = \infty$ for $0<m\leq 1$.

(iii) $\lim_{\mu\to\infty} T(m,\mu) = \begin{cases} 0 & \text{if } 0<m<p, \\ \pi/2 & \text{if } m=p , \\ \infty & \text{if } m>p . \end{cases}$

(iv) $\lim_{m\to 1^+} T(m,\alpha(m))=\infty$, $\lim_{m\to\infty} T(m,\alpha(m))=\begin{cases} 0 & \text{if } 0<a<1 , \\ \infty & \text{if } 1\leq a<\infty. \end{cases}$

(v) $T_\mu(m,\mu)<0$ for $0<m\leq p$, $\alpha(m)<\mu<\infty$.

(vi) There exists a function $\varphi\in C((p,\infty))$ such that for $m>p$ the following holds

$$T_\mu(m,\mu)=0 \quad \text{if and only if } \mu=\varphi(m),$$
$$T_\mu(m,\mu)<0 \quad \text{if } \alpha(m)<\mu<\varphi(m),$$
$$T_\mu(m,\mu)>0 \quad \text{if } \mu>\varphi(m)\ .$$

(vii) $\lim\limits_{m\to\infty} \varphi(m) = a^{1/(p-1)}, \quad \lim\limits_{m\to p^+} \varphi(m) = \infty$.

(viii) $\lim\limits_{m\to p^+} T(m,\varphi(m))=\pi/2, \quad \lim\limits_{m\to\infty} T(m,\varphi(m)) = \begin{cases} 0 & \text{if } 0<a<1, \\ \infty & \text{if } 1\le a<\infty. \end{cases}$

The next two remarks which precede the proof explain the meaning of the observations collected in the lemma.

REMARK 3. It turns out that $(0,\infty)$ can be decomposed into four subsets: $(0,1]$, $(1,p)$, $\{p\}$, (p,∞) in such a way that for values of m belonging to different sets we get different forms of $T(m,\cdot)$ (and different numbers of solutions of (3.3)). The four types of graphs of $T(m,\cdot)$ are sketched in Figure 1 .

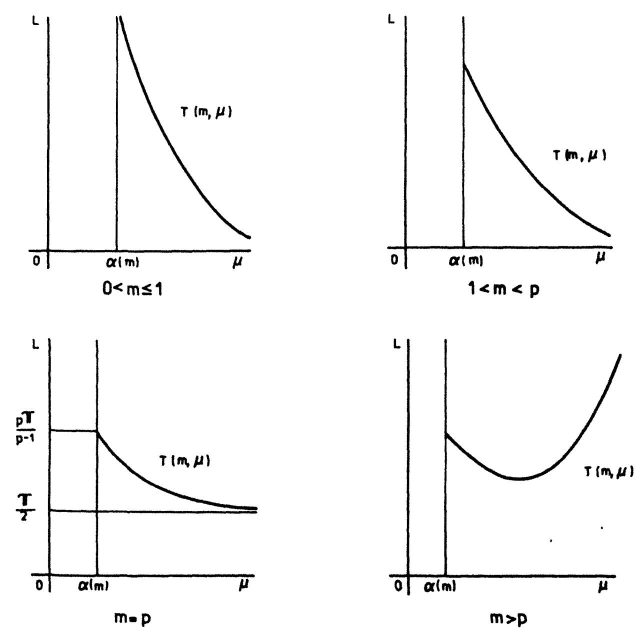

Figure 1.

REMARK 4. Similarly as in [1],[5] we can see that for $m>1$ there are continua of solutions (which are not strictly positive) on intervals larger than $2T(m,\alpha(m))$. Thus we obtain two bifurcation diagrams which are outlined in Figure 2 .

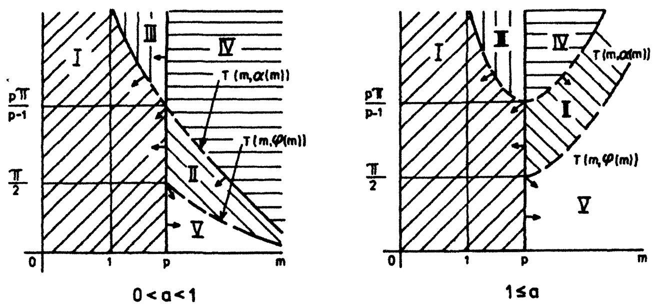

Figure 2.

The arrows on the borderlines indicate to which region particular pieces of the borderlines belong. For (m,L) lying in the region denoted by I there is one positive nontrivial solution, in the region II - two positive solutions, III - continua of nonnegative solutions, IV - continua of nonnegative solutions and one positive solution and V - no nontrivial solution. On the curve which separates regions II and V (it is the graph of the function $T(m,\varphi(m))$) there is one positive solution.

Proof of Lemma 2.

(i) Take an arbitrary point $(m_0,\mu_0)\in \operatorname{int} M$. Then for (m,μ) near (m_0,μ_0) and for y near 0 we have $F(m,\mu)-F(m,\mu y) \geq F(m,\mu)\geq C_1$, for y near 1 it holds $F(m,\mu)-F(m,\mu y) \geq C_2(1-y)$, C_1, C_2 are some positive constants depending on (m_0,μ_0). This implies the uniform existence of the integral in (3.2) for (m,μ) near (m_0,μ_0), hence we obtain the continuity of T in (m_0,μ_0). To prove the continuity in $(m_0,\alpha(m_0))$ for $m_0>1$ we use the inequality $F(m,\mu)-F(m,\mu y)\geq$ $\geq (m+1)^{-1}a(\mu y)^{m+1}(1-y^{p-1})$, which ensures that

$y^{m-1}(F(m,\mu)-F(m,\mu y))^{-1/2}$ has an integrable majorant depending only on m_0.

If we set $\xi(m,s)=2mF(m,s)-s^m f(s)$, we get exactly as in [1]

$$(3.4) \qquad T_\mu(m,\mu)=\sqrt{\frac{m}{2}}\frac{1}{2\mu}\int_0^\mu \frac{\xi(m,\mu)-\xi(m,s)}{(F(m,\mu)-F(m,s))^{3/2}}s^{m-1}ds \ .$$

The uniform existence of the integral in (3.4) can be again easily verified using the mean value theorem.

(ii) As $\mu\to\alpha(m)$, $y^{m-1}(F(m,\mu)-F(m,\mu y))^{-1/2}$ tends to $C(m)y^{(m-3)/2}(1-y^{p-1})^{-1/2}$ which is not integrable if $m\leq 1$, thus the assertion follows from Fatou's lemma.

(iii) We can write

$$T(m,\mu) = (\mu^{m-p}\frac{m\mu^{m+p}}{2F(m,\mu)})^{1/2}\int_0^1 (1-\frac{F(m,\mu y)}{F(m,\mu)})^{-1/2}y^{m-1}dy \ .$$

$F(m,\mu y)/F(m,\mu)\leq y^{m+p}$ for $y\in[0,1]$, hence $y^{m-1}(1-y^{m+p})^{-1/2}$ is an integrable majorant and the integrand tends to this function pointwise as $\mu\to\infty$.

(iv) $T(m,\alpha(m))=(2m(m+1)\alpha^{m-1}(m)/a)^{1/2}\int_0^{\pi/2}(\sin x)^{\frac{m-p}{p-1}}dx/(p-1)$
$\int_0^{\pi/2}(\sin x)^\gamma dx = O(\gamma^{-1/2})$ as $\gamma\to\infty$.

(v) For $0<m\leq p$, $\mu>\alpha(m)$ the function $\xi(m,\cdot)$ from (3.4) attains its minimum on $[0,\mu]$ in the point μ.

(vi) Follows from [5,Lemma 1.2].

(vii) It is clear from (3.4) that $\varphi(m)$ lies between the point where $\xi(m,\cdot)$ attains its minimum and the positive root of $\xi(m,\cdot)$, i.e.

$$(3.5) \quad (a(m-1)/(m-p))^{1/(p-1)}<\varphi(m)<(a(m-1)(m+p)/(m+1)(m-p))^{1/(p-1)}.$$

(viii) The first assertion follows from (vi), from the fact that $T(p,\mu)\to\pi/2$ as $\mu\to\infty$ and from the continuity of T. $T(m,\varphi(m))\to 0$ as $m\to\infty$ if $0<a<1$ because $T(m,\varphi(m))<T(m,\alpha(m))$, so it remains to prove that $T(m,\varphi(m))\to\infty$ as $m\to\infty$ if $a\geq 1$. The second inequality in (3.5) yields

$$F(m,\varphi(m))-F(m,\varphi(m)y)<a\varphi^{m+1}(m)(\frac{m-1}{m-p}(1-y^{m+p})-(1-y^{m+1}))/(m+1) \ .$$

According to the mean value theorem $1-y^{m+p}\leq(m+p)(1-y^{m+1})/(m+1)$ for $y\in[0,1]$ and we obtain

$$T(m,\varphi(m)) \geq (\varphi^{m-1}(m)(m-p)/a(p-1))^{1/2} \int_0^{\pi/2} (\sin x)^{(m-1)/(m+1)} dx .$$

In the case a=0 we set $M=(0,\infty)\times(0,\infty)$ and for $(m,\mu)\in M$ we have

$$T(m,\mu) = (2m\mu^{m-p}/(m+p))^{1/2} \int_0^{\pi/2} (\sin x)^{(m-p)/(m+p)} dx .$$

We see that $T(p,\mu) = \pi/2$ for every $\mu>0$. For fixed $m>0$, $m\neq p$, the equation (3.3) has precisely one root for arbitrary $L>0$.

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