COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

ON THE BASIC REPRESENTATION OF THE AFFINE KAC-MOODY LIE ALGEBRAS $D_n^{(1)}$

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Abstract: We compute the constants needed for the principal realization given in [3] of the affine Kac-Moody Lie algebras $D_n^{(1)}$, $n\geq 4$.

Key words: Affine Kac-Moody Lie algebras, graded Lie algebras.

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1. Introduction. Let $n\geq 4$. Let $\{E_{ij}\}_{j=1,\dots,2n}$ be the standard basis of the space of $2n\times 2n$ complex matrices, so that the matrix E_{ij} is 1 in the ij-entry and 0 in all the other entries. In the case of Kac-Moody Lie algebra $\mathfrak{g}(A)$ of type $D_n^{(1)}$ $(n\geq 4)$ we have (see [4],[9])

(1) $\mathfrak{g}=o(2n,\mathbb{C})$, $(x|y)=\mathrm{tr}\, xy$.

Instead of the standard representations [2], we consider all $2n\times 2n$ complex matrices in the form given in [1], for type D_n. We can take the Chevalley generators [3] (cf. [8]) e_i,f_i,h_i given in [10] by the following relations, for $i=1,\dots,n-1$.

$$(2)\quad\begin{cases} e_0=E_{2n-1,1}-E_{2n,2},\ e_i=E_{i,i+1}-E_{2n-i,2n-i+1}, \\ e_n=E_{n-1,n+1}-E_{n,n+2}, \\ f_0=E_{1,2n-1}-E_{2,2n},\ f_i=E_{i+1,i}-E_{2n-i+1,2n-i}, \\ f_n=E_{n+1,n-1}-E_{n+2,n}, \\ h_0=E_{2n,2n}+E_{2n-1,2n-1}-E_{22}-E_{11}, \\ h_i=E_{2n-i,2n-i}+E_{ii}-E_{2n-i+1,2n-i+1}-E_{i+1,i+1}, \\ h_n=E_{nn}+E_{n-1,n-1}-E_{n+2,n+2}-E_{n+1,n+1} \end{cases}$$

Let us denote (see [10]) by $\underline{\kappa}$ the number defined in the following way:

$\underline{\kappa}=\kappa$ if $\kappa \leq n$ and $\underline{\kappa}=\kappa-1$ if $\kappa > n$ for every κ of Z. Moreover let $h=2(n-1)$ be the Coxeter number [3]. Then we have the following

Proposition 1 (see [10]) . A Lie algebra $\mathfrak{G}$ of type D_n is a graded modh where the 1-principal Z/h Z-gradation is given by setting

$$\deg E_{ij}=(\underline{j}-\underline{i}) \bmod h.$$

So we can write

$$\mathfrak{G}=\bigoplus_{i\in Z/hZ}\mathfrak{G}_i.$$

We take a normalized 1-cyclic element [6] as follows: $E=\beta_i e_i$ where $\beta_0=\beta_1=\beta_{n-1}=\beta_n=1/\sqrt{2}$, $\beta_i=1$ for $i=2,\dots,n-2$. One obtains the relations

$$(3)\quad \begin{cases} E^{h+1}=(-1)^n E,\ E^{2\kappa-1}=(-1)^n\, {}^tE^{2(n-\kappa)-1} \text{ for } \kappa\in\mathbf{N};\ \kappa\leq\frac{n}{2}, \\ \text{and } \operatorname{tr} E^h=(-1)^n h. \end{cases}$$

The centralizer S of E is a CSA of $\mathfrak{G}$ with dimension n (see [6]). A basis of S is $\{E,E^3,\dots,E^{2n-3},E_0\}$ where

$$E_0=E_{1n}-E_{1,n+1}+E_{n1}+E_{n,2n}-E_{n+1,1}-E_{n+1,2n}+E_{2n,n}-E_{2n,n+1},$$

and we have

$$(4)\quad E_0^2+(-1)^n 4E^h=4I,\ \operatorname{tr}E_0^2=8, \text{ and } E_0E^\nu=E^\nu E_0=0 \text{ for every } \nu.$$

In [5] and [3] a construction of the basic representation of the affine Kac-Moody Lie algebras have been introduced which construction is a generalization of the one introduced in [7]. This construction has been called principal realization. For this realization, in the case of $D_n^{(1)}$, one needs n root vectors $A_1,\dots,A_n$, with respect to S, such that their projections on $\mathfrak{G}_0$ form a basis of $\mathfrak{G}_0$. Moreover if $\{T_s\}_{s=1,\dots,n}$ is a basis of S which is normalized such that $(T_i|T_{n+1-j})=\delta_{ij}$ for all $i,j=1,\dots,n$, then the constants λ_{rs} defined by the relation $[T_s,A_r]=\lambda_{rs}A_r$ are needed for the principal realization. If we decompose the vectors A_r with respect to the 1-principal gradation

$$A_r=\sum_\nu A_{r\nu},\ r=1,\dots,n;\ \nu\in Z/hZ,$$

then the elements $A_{r\nu},T_s$ where $r,s=1,\dots,n$; $\nu=0,\dots,n-1$, form a basis of $\mathfrak{G}$. The aim of this paper is to compute the constants λ_{rs} in the case of $D_n^{(1)}$, $n\geq 4$.

2. A basis of $\mathfrak{g}$. Let $n=2\kappa$. Then decomposing the S with respect to the 1-principal gradation we have dimension one for the degrees 1,3,...,n-3, n+1, n+3,...,2n-3, and dimension two for the degree n-1. For the dimension n-1 we have a basis: $\{E^{n-1},E_0\}$.

For the one dimensional degrees, in order that the relation $(T_i|T_{n+1-j})=\delta_{ij}$ should be valid, we can take the vectors

$$(5)\qquad \begin{cases} T_s=\frac{1}{\sqrt{n}}E^{2s-1} \text{ for } s=1,\dots,\kappa-1; \\ T_s={}^tT_{n+1-s}=\frac{1}{\sqrt{n}}E^{2s-3} \text{ for } s=\kappa+2,\dots,n. \end{cases}$$

It remains to find two vectors T_κ , $T_{\kappa+1}$ of dimension n-1, i.e. from the linear span of E^{n-1}, E, such that $(T_\kappa|T_\kappa)=0$, $(T_{\kappa+1}|T_{\kappa+1})=0$ and $(T_\kappa|T_{\kappa+1})=1$. Using the relations (3) and (4) one obtains the vectors.

$$(6)\quad T_\kappa=\frac{1}{\sqrt{2}}E^{n-1}+\frac{i}{4}E_0,\ T_{\kappa+1}=\frac{1}{\sqrt{2n}}E^{n-1}-\frac{i}{4}E_0, \text{ where } i^2=-1.$$

Therefore a normalized basis in case that $n=2\kappa$ is given by the relations (5) and (6).

Let $n=2\kappa+1$. In this case for S we have only dimension one on the degrees 1,3,...,n-2, n-1, n,...,2n-3. So in a similar way as above we have the following as a normalized basis of S:

$$(7)\qquad \begin{cases} T_s=\frac{1}{\sqrt{n}}E^{2s-1} \text{ for } s=1,\dots,\kappa;\ T_{\kappa+1}=\frac{1}{2\sqrt{2}}E_0; \\ T_s=-{}^tT_{n+1-s}=\frac{1}{\sqrt{n}}E^{2s-3} \text{ for } s=\kappa+2,\dots,n. \end{cases}$$

Proposition 2. Let $\Theta_r=\frac{r\pi}{n-1}$, and $X=\mathrm{diag}(x_1,\dots,x_n,-x_n,\dots,-x_1)$ be the elements of $\mathfrak{g}_n$. Then the $(\mathrm{ad}E)^h$ has eigenvalues

$$(8)\quad \lambda_r=(-1)^{n+r}.2^h\cos^h\frac{\Theta_r}{2} \text{ for } r=1,\dots,n-2, \text{ and } \lambda_{n-1}=\lambda_n=(-1)^n$$

with corresponding, appropriate, eigenvectors

$$(9)\quad A_{ro}=\mathrm{diag}(0,-\sin\Theta_r,\dots,(-1)^{n-2}\sin(n-2)\Theta_r,0,$$
$$0,-(-1)^{n-2}\sin(n-2)\Theta_r,\dots,\sin\Theta_r,0),$$

$$(10)\qquad \begin{cases} A_{n-1,o}=\mathrm{diag}(1,0,\dots,\nu_1,-\nu_1,0,\dots,-1) \\ A_{no}=\mathrm{diag}(1,0,\dots,\nu_2,-\nu_2,0,\dots,-1), \end{cases}$$

where ν_1, ν_2 are the square roots of $(-1)^n$.

The set $\{A_{10},\dots,A_{n0}\}$ is a basis of $\mathfrak{g}_0$.

Proof. The non-degenerate operator adE shifts the gradation by 1, see [3], so the vectors

$$(\mathrm{ad}E)^h X = \sum_{s=0}^{h} (-1)^s \binom{h}{s} E^{h-s} X E^s$$

are of degree zero. Since $E^{h-s}=(-1)^n\, {}^tE^s$, $0<s<h$, we obtain

$$(11)\quad (\mathrm{ad}E)^h X = (-1)^n \sum_{s=1}^{h-1} \binom{h}{s} {}^tE^s\, X\, E^s + E^h X + X E^h,$$

Our problem is to solve the matrix equation $(\mathrm{ad}E)^h X = \lambda X$ or from relation (11) to solve the equation

$$(12)\quad \sum_{s=1}^{h-1} \binom{h}{s} {}^tE^s\, X\, E^s + (-1)^n (E^h X + X E^h) = (-1)^n \lambda X.$$

We have two cases:

Case A. Let $X=\mathrm{diag}(0,x_2,\dots,x_{n-1},0,0,-x_{n-1},\dots,-x_2,0)$. In this case depending on the skew symmetry with respect to the second diagonal of $x_2,\dots$ $\dots,x_{n-1}$ in (12) we have to solve the homogeneous system on $x_2,\dots,x_{n-1}$ with coefficient matrix the $(n-2)\times(n-2)$ matrix which has on the ν-th row, $\nu=1,\dots,n-2$, the following entries

$$(-1)^{\nu+1}\left[\binom{h}{\nu-1}-\binom{h}{\nu+1}\right],\ (-1)^{\nu+2}\left[\binom{h}{\nu-2}-\binom{h}{\nu+2}\right],\dots,$$

$$-\left[\binom{h}{1}-\binom{h}{2\nu-1}\right],\ 2-\binom{h}{2\nu}-(-1)^n\lambda,\ -\left[\binom{h}{1}-\binom{h}{2\nu+1}\right],\dots,$$

$$(-1)^{n-\nu-3}\left[\binom{h}{n-3-\nu}-\binom{h}{n-3+\nu}\right],\ (-1)^{n-\nu-2}\left[\binom{h}{n-2-\nu}-\binom{h}{n-2+\nu}\right].$$

Solving the above system we finally obtain that the n-2 eigenvalues of the $(\mathrm{ad}E)^h$ are given by the relation (8) for $r=1,\dots,n-2$ and the corresponding eigenvectors are given by the relation (9). For the above computation we use the well known identity

$$\cos^h\theta = \frac{1}{2^h-1}\sum_{\kappa=0}^{h/2-1}\binom{h}{\kappa}\cos(h-2\kappa)\theta + \frac{1}{2^h}\binom{h}{h/2}$$

Case B. Let $X=\mathrm{diag}(x_1,0,\dots,0,x_n,-x_n,0,\dots,0,-x_1)=A_{r0}$ where $r=n-1,n$. In this case all matrices $(\mathrm{ad}E)^\nu A_{r0}$ have non-zero entries only on $1,n,n+1,2n$ rows and columns. Moreover $A_r=\sum_\nu A_{r\nu}$ must be an eigenvector of the $\mathrm{ad}E_{0*}$ i.e. $(\mathrm{ad}E_0)A_r=\mu A_r$. Solving the above system we obtain the eigenvalue $(-1)^n$ for $(\mathrm{ad}E)^h$ and a corresponding basis of X's is the one given by (10). From the cases A and B we obtain that $\{A_{10},\dots,A_{n0}\}$ is a basis of $\mathfrak{g}_0$.

Remark. Let ε_r be a h-primitive root of $(-1)^{n+r}$ for $r=1,\dots,n-2$, and $\varepsilon_{n-1}=\varepsilon_n=\varepsilon$ be a h-primitive root of $(-1)^n$. We set

$$(13)\quad A_r=\sum_{\nu=1}^{h}\mu_r^{-\nu}(\mathrm{ad}E)^{\nu}A_{ro}$$

where $\mu_r=2\varepsilon_r\cos\frac{\theta_r}{2}$, and $\mu_{n-1}=\mu_n=\varepsilon$. Then T_s for $s=1,\dots,n$ together with the homogeneous components

$$(14)\quad A_{r\nu}=\mu_r^{-\nu}(\mathrm{ad}E)^{\nu}A_{ro},\quad r=1,\dots,n;\ \nu=0,\dots,h-1$$

of A_r, form a basis of $\mathcal{G}$.

In order to write explicitly the matrices A_r for $r=1,\dots,n-2$ which are skew symmetric with respect to the second diagonal lets consider the matrix

$$A_r=\begin{pmatrix} A_{r1} & A_{r2}\\ A_{r3} & A_{r4}\end{pmatrix}$$

where A_{r1}, A_{r2}, A_{r3}, A_{r4} are $n\times n$ matrices.

We set ε and $\langle\nu,\mu\rangle$ instead of ε_r and $(-1)^{\nu}\sin\mu\frac{\theta_r}{2}$ respectively, and we write simply the upper parts of the skew symmetric, with respect to the second diagonal, matrices A_{r2}, A_{r3}. Then A_{r1}, A_{r2}, A_{r3} are given respectively as follows:

$$\begin{pmatrix}
0 & \frac{\langle 1,1\rangle}{\varepsilon\sqrt{2}} & \frac{\langle 2,2\rangle}{\varepsilon^{2}\sqrt{2}} & \cdots & \frac{\langle n-3,n-3\rangle}{\varepsilon^{n-3}\sqrt{2}} & \frac{\langle n-2,n-2\rangle}{\varepsilon^{n-2}\sqrt{2}} & \frac{\langle n+r,n-1\rangle}{\varepsilon^{n-1}2}\\
\frac{\langle n+r,1\rangle}{\varepsilon^{2n-3}\sqrt{2}} & \langle 1,2\rangle & \frac{\langle 2,3\rangle}{\varepsilon} & \cdots & \frac{\langle n-3,n-2\rangle}{\varepsilon^{n-4}} & \frac{\langle n-2,n-1\rangle}{\varepsilon^{n-3}} & \frac{\langle n+r,n-2\rangle}{\varepsilon^{n-2}\sqrt{2}}\\
\frac{\langle n+r,2\rangle}{\varepsilon^{2n-4}\sqrt{2}} & \frac{\langle n+r-1,3\rangle}{\varepsilon^{2n-3}} & \langle 2,4\rangle & \cdots & \frac{\langle n-3,n-1\rangle}{\varepsilon^{n-5}} & \frac{\langle n+r-1,n-2\rangle}{\varepsilon^{n-4}} & \frac{n+r,n-3}{\varepsilon^{n-3}\sqrt{2}}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\frac{\langle n+r,n-3\rangle}{\varepsilon^{n+1}\sqrt{2}} & \frac{\langle n+r-1,n-2\rangle}{\varepsilon^{n+2}} & \frac{\langle n+r,n-1\rangle}{\varepsilon^{n+3}} & \cdots & \langle n+r,4\rangle & \frac{\langle n+r-1,3\rangle}{\varepsilon} & \frac{\langle n+r,2\rangle}{\varepsilon^{2}\sqrt{2}}\\
\frac{\langle n+r,n-2\rangle}{\varepsilon^{n}\sqrt{2}} & \frac{\langle n+r-1,n-1\rangle}{\varepsilon^{n+1}} & \frac{\langle n-1,n-2\rangle}{\varepsilon^{n+2}} & \cdots & \frac{\langle 2,3\rangle}{\varepsilon^{2n-3}} & \langle n+r-1,2\rangle & \frac{\langle n+r,1\rangle}{\varepsilon\sqrt{2}}\\
\frac{\langle n-1,n-1\rangle}{\varepsilon^{n-1}2} & \frac{\langle n-2,n-2\rangle}{\varepsilon^{n}\sqrt{2}} & \frac{\langle n-3,n-3\rangle}{\varepsilon^{n+1}\sqrt{2}} & \cdots & \frac{\langle 2,2\rangle}{\varepsilon^{2n-4}\sqrt{2}} & \frac{\langle 1,1\rangle}{\varepsilon^{2n-3}\sqrt{2}} & 0
\end{pmatrix}$$

$$
\begin{pmatrix}
\frac{\langle n+r,n-1\rangle}{\varepsilon^{n-1}\,2} & \frac{\langle n+r,n-2\rangle}{\varepsilon^{n}\sqrt{2}} & \frac{\langle n+r,n-3\rangle}{\varepsilon^{n+1}\sqrt{2}} & \cdots & \frac{\langle n+r,2\rangle}{\varepsilon^{2n-4}\sqrt{2}} & \frac{\langle n+r,1\rangle}{\varepsilon^{2n-3}\sqrt{2}} & 0 \\
\frac{\langle n+r,n-2\rangle}{\varepsilon^{n-2}\sqrt{2}} & \frac{\langle n+r,n-3\rangle}{\varepsilon^{n-1}} & \frac{\langle n+r,n-4\rangle}{\varepsilon^{n}} & \cdots & \frac{\langle n+r,1\rangle}{\varepsilon^{2n-5}} & 0 & \\
\frac{\langle n+r,n-3\rangle}{\varepsilon^{n-3}\sqrt{2}} & \frac{\langle n+r,n-4\rangle}{\varepsilon^{n-2}} & \frac{\langle n+r,n-5\rangle}{\varepsilon^{n-1}} & \cdots & 0 & & \\
\cdots & \cdots & \cdots & \cdots & & & \\
\frac{\langle n+r,2\rangle}{\varepsilon^{2}\sqrt{2}} & \frac{\langle n+1,1\rangle}{\varepsilon^{3}} & 0 & & & & \\
\frac{\langle n+r,1\rangle}{\varepsilon\sqrt{2}} & 0 & & & & & \\
0 & & & & & &
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{\langle n-1,n-1\rangle}{\varepsilon^{n-1}\,2} & \frac{\langle n-2,n-2\rangle}{\varepsilon^{n}\sqrt{2}} & \frac{\langle n-3,n-3\rangle}{\varepsilon^{n+1}\sqrt{2}} & \cdots & \frac{\langle 2,2\rangle}{\varepsilon^{2n-4}\sqrt{2}} & \frac{\langle 1,1\rangle}{\varepsilon^{2n-3}\sqrt{2}} & 0 \\
\frac{\langle n-2,n-2\rangle}{\varepsilon^{n-2}\sqrt{2}} & \frac{\langle n-3,n-3\rangle}{\varepsilon^{n-1}} & \frac{\langle n-4,n-4\rangle}{\varepsilon^{n}} & \cdots & \frac{\langle 1,1\rangle}{\varepsilon^{2n-5}} & 0 & \\
\frac{\langle n-3,n-3\rangle}{\varepsilon^{n-3}\sqrt{2}} & \frac{\langle n-4,n-4\rangle}{\varepsilon^{n-2}} & \frac{\langle n-5,n-5\rangle}{\varepsilon^{n-1}} & \cdots & 0 & & \\
\cdots & \cdots & \cdots & \cdots & & & \\
\frac{\langle 2,2\rangle}{\varepsilon^{2}\sqrt{2}} & \frac{\langle 1,1\rangle}{\varepsilon^{3}} & 0 & & & & \\
\frac{\langle 1,1\rangle}{\varepsilon\sqrt{2}} & 0 & & & & & \\
0 & & & & & &
\end{pmatrix}
$$

For the skew symmetric with respect to the second diagonal matrices A_{n-1}, A_n we have analogously for $\nu=\nu_1, \nu_2$ the following upper part:

$$\begin{pmatrix}
1 & \frac{-1}{\varepsilon\sqrt{2}} & \cdots & \frac{(-1)^{n-2}}{\varepsilon^{n-2}\sqrt{2}} & \frac{\nu-(-1)^n}{\varepsilon^{n-1}\,2} & \frac{-\nu-(-1)^n}{\varepsilon^{n-1}\,2} & \frac{-(-1)^n}{\varepsilon^{n}\sqrt{2}} & \cdots & \frac{-(-1)^n}{\varepsilon^{2n-3}\sqrt{2}} \\
\frac{(-1)^n}{\varepsilon^{2n-3}\sqrt{2}} & 0 & \cdots & 0 & \frac{\nu}{\varepsilon^{n-2}\sqrt{2}} & \frac{-\nu}{\varepsilon^{n-2}\sqrt{2}} & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\frac{(-1)^n}{\varepsilon^{n}\sqrt{2}} & 0 & \cdots & 0 & \frac{\nu}{\varepsilon^{2}\sqrt{2}} & \frac{-\nu}{\varepsilon^{2}\sqrt{2}} & 0 & & \\
\frac{-\nu+(-1)^n}{\varepsilon^{n-1}\,2} & \frac{\nu}{\varepsilon^{n}\sqrt{2}} & \cdots & \frac{-(-1)^n\nu}{\varepsilon^{2n-3}\sqrt{2}} & \nu & 0 & & & \\
\frac{\nu+(-1)^n}{\varepsilon^{n-1}\,2} & \frac{-\nu}{\varepsilon^{n}\sqrt{2}} & \cdots & \frac{(-1)^n\nu}{\varepsilon^{2n-3}\sqrt{2}} & 0 & & & & \\
\frac{-(-1)^n}{\varepsilon^{n-2}\sqrt{2}} & 0 & \cdots & 0 & & & & & \\
\cdots & \cdots & \cdots & & & & & & \\
\frac{1}{\varepsilon\sqrt{2}} & 0 & & & & & & & \\
0 & & & & & & & &
\end{pmatrix}$$

Now using the formulas (13) or (14) it is a simple calculation to obtain all Lie brackets of A_r with the powers of the element E as well as with E_0. Note that only the odd powers of E are elements of $\mathfrak{g}$. So the following proposition is obtained.

Proposition 3. If $j\in\{1,3,\dots,h-1\}$, then we have

$$(\mathrm{ad}E^j)A_r=2\,\varepsilon_r^j \cos\frac{j\theta_r}{2}\,A_r \text{ for } r=1,\dots,n-2$$

and

$$(\mathrm{ad}E^j)A_r=\varepsilon^j\,A_r \text{ for } r=n-1,n.$$

Moreover

$$(\mathrm{ad}E_0)A_r=0 \text{ for } r=1,\dots,n-2, \text{ and}$$

$$(\mathrm{ad}E_0)A_r=\frac{-2\nu}{\varepsilon^{n-1}}\,A_r \text{ for } r=n-1,n \text{ where } \nu=\nu_1,\nu_2.$$

3. The results. Using the above notation we can compute the constants λ_{rs} from the relation $[T_s,A_r]=\lambda_{rs}A_r$. From Proposition 3 we obtain that all constants λ_{rs}, where $r,s=1,\dots,n$, are the following:

3.1. Let $r=1,\dots,n-2$ then

$$\lambda_{rs} = \frac{2}{\sqrt{h}} \varepsilon_r^{2s-1} \cos \frac{(2s-1)\theta_r}{2} \text{ for } \begin{cases} s=1,\dots,\kappa-1 \text{ if } n=2\kappa \text{ and} \\ s=1,\dots,\kappa \quad \text{if } n=2\kappa+1. \end{cases}$$

$$\lambda_{rs} = \frac{2}{\sqrt{h}} \varepsilon_r^{2s-3} \cos \frac{(2s-3)\theta_r}{2} \text{ for } s=\kappa+2,\dots,n \text{ if } n=2\kappa \text{ or } n=2\kappa+1.$$

Moreover

$$\lambda_{r\kappa} = \lambda_{r,\kappa+1} = \frac{1}{\sqrt{2h}} \varepsilon_r^{n-1} \cos \frac{(n-1)\theta_r}{2} \text{ if } n=2\kappa \text{ and}$$

$$\lambda_{r,\kappa+1} = 0 \text{ if } n=2\kappa+1.$$

3.2. Let $r=n-1,n$ then for $\nu=\nu_1,\nu_2$ respectively we have

$$\lambda_{rs} = \frac{\varepsilon^{2s-1}}{\sqrt{h}} \text{ for } s=1,\dots,\kappa-1 \text{ if } n=2\kappa, \text{ and } s=1,\dots,\kappa \text{ if } n=2\kappa+1.$$

$$\lambda_{rs} = \frac{\varepsilon^{2s-3}}{\sqrt{h}} \text{ for } s=\kappa+2,\dots,n \text{ if } n=2\kappa \text{ or } n=2\kappa+1.$$

Moreover

$$\lambda_{r\kappa} = \left(\frac{1}{\sqrt{2h}} - \frac{\nu i}{2}\right) \varepsilon^{n-1}, \quad \lambda_{r,\kappa+1} = \left(\frac{1}{\sqrt{2h}} - \frac{\nu i}{2}\right) \varepsilon^{n-1} \text{ if } n=2\kappa$$

where $i^2 = -1$,

and

$$\lambda_{r,\kappa+1} = \frac{\nu \varepsilon^{n-1}}{\sqrt{2}} \text{ if } n=2\kappa+1.$$

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