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LINEAR FUNCTIONALS ON SOME NON-LOCALLY CONVEX GENERALIZED ORLICZ SPACES

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Abstract. The purpose of this paper is to provide theorems on existence and nonexistence of nonzero continuous linear functionals on non-locally generalized Orlicz spaces of functions with values in a p-normable space. We present theorems which are generalizations of the results of S. Rolewicz [19] (Theorem 0.1) and L. Drewnowski [5] (Theorem 0.2).

Key words: Orlicz space, vector valued function, linear functional, non-locally convex space.

Classification: 46E30

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0. Introduction. Orlicz spaces of vector valued functions have been developed by many authors. They can be considered as a special case of both Banach spaces - e.g. Skaff [21],[22], Kozek [12], Chen Shutao [3], Jamison and Loomis [11], and Fréchet spaces - e.g. Hernandez [7],[8]. The purpose of this paper is to establish theorems on existence and nonexistence of nonzero continuous linear functionals on non-locally convex generalized Orlicz spaces of functions with values in a p-normable space. Banach (see [1]) has given an example of a metric linear space which has no nonzero continuous linear functionals. In 1940, Day (see [3]) proved that the spaces L^p over an atomless measure with $0<p<1$ have this property as well. In the case of Orlicz spaces the most important result was obtained by Rolewicz in 1959, namely

0.1. Theorem. If Φ satisfies the condition Δ_2 and

$$\liminf_{u\to+\infty}\frac{\Phi(u)}{u}>0$$

then there are nonzero continuous linear functionals in the Orlicz space $L^{\Phi}(T,\Sigma,\mu)$.

The converse implication remains true provided the measure μ is atomless.

At the same time an analogical theorem for modular-continuous linear functionals in modular spaces was obtained by Musielak and Orlicz [15].

Similar results were presented by Cater [2] in 1962 and Gramsch [6] in 1967.

Pallaschke and Urbański [18] in 1985 studied the case of (X,ρ) being a modular space over a field with valuation $(K,|\cdot|)$. Let us recall that ρ is a (w,v)-convex modular on X if $\rho(x)=\rho(-x)$, $\rho(0)=0$, if $|ax|=0$ for every $a\in K\setminus\{0\}$, then $\rho(x)=0$ and $\rho(ax+by)\leq v(a)\rho(x)+v(b)\rho(y)$ for all $x,y\in X$, $a,b\in K$ with $w(a)+w(b)\leq 1$. They claim that there are no nonzero continuous linear functionals on the space (X,ρ) provided the modular ρ is (w,v)-convex, where

$$\liminf_{a\to+\infty}\frac{v(a)}{a}=0 \text{ and } \limsup_{a\to+\infty}\frac{w(a)}{a}<+\infty .$$

In particular, if Φ is a Φ-function with a parameter (see Definition 1.1 below) and, moreover, it is p-convex $(0<p<1)$ in the following sense

$$\Phi(ax+by,t)\leq |a|^p\Phi(x,t)+|b|^p\Phi(y,t)$$

for all $x,y\in X$, $a,b\in \mathbf{R}$, $|a|+|b|\leq 1$ and for almost every $t\in T$, then the modular

$$I_\Phi(f)=\int_T\Phi(f(t),t)d\mu$$

is $(|\cdot|^1,|\cdot|^p)$-convex. Hence there are no nonzero continuous linear functionals on the Musielak-Orlicz space L^Φ. Therefore it is worth studying $(|\cdot|^1,|\cdot|^0)$-convex modulars I_Φ only.

Some additional properties of linear functionals and linear operators in modular spaces have also been studied in [10] in 1983.

In 1986 Drewnowski proved the following (see [5])

0.2. Theorem. Let μ be a σ-finite, atomless measure and let Φ be a Musielak-Orlicz function with finite values. The space E^Φ has a topological dual zero if and only if

$$\liminf_{u\to+\infty}\frac{1}{u}\Phi(u,t)=0 \text{ for a.e. } t\in T.$$

(For detailed definitions, we refer to Section 1 below.)

Section 2 is aimed at solving the above discussed problems in the case of Musielak-Orlicz spaces of functions with values in a p-normable space X. In Section 3 we give a number of examples.

1. Preliminaries. Let (T,Σ,μ) be a measure space, where T is an abstract set, Σ is a σ-algebra of subsets of T and μ is a non-negative, complete, atomless and σ-finite measure on Σ . $(X, \|\cdot\|)$ will denote a p-normable space with a p-homogeneous norm $\|\cdot\|$. By Aoki-Rolewicz Theorem (see [20]) every locally bounded space X is locally p-convex for some $p>0$, so there is a p-homogeneous norm $\|\cdot\|$ equivalent to the original one such that $(X, \|\cdot\|)$ is a p-normed space. By $\mathcal{B}_X$ we will denote the σ-algebra of Borel subsets of X. Let $\mathcal{M}(T,X)$ be the linear space of all μ-equivalence classes of strongly measurable functions $f:T \to X$, i.e. functions for which there is a sequence of simple functions $\{f_n\}$ such that $f_n(t) \to f(t)$ as $n \to +\infty$ for almost every (a.e.) $t \in T$.

1.1. Definition. A function $\Phi: X \times T \to [0,+\infty]$ is said to be a Φ-function if there is a set T_0 of measure 0 such that:

a) Φ is $\mathcal{B}_X \times \Sigma$ -measurable,

b) $\Phi(0,t)=0$ and $\Phi(x,t)=\Phi(-x,t)$ for every $x \in X$ and $t \notin T_0$,

c) $\Phi(\cdot,t)$ is not identically equal to 0 and is lower semicontinuous on X for $t \notin T_0$, i.e. for every $t \notin T_0$, $x_0 \in X$ and $a < \Phi(x_0,t)$ there exists an open neighbourhood U of x_0 such that $a < \Phi(x,t)$ for all $x \in U$.

d) $\Phi(ux+vy,t) \leq \Phi(x,t) + \Phi(y,t)$ for every $u,v \geq 0$, $u+v \leq 1$, $x,y \in X$ and $t \notin T_0$,

e) $\lim_{u \to 0} \Phi(ux,t)=0$ for all $x \in \{y \in X: \Phi(y,t) < +\infty\}$ and $t \notin T_0$.

Since X is a linear metric space, every strongly measurable function f is Borel measurable i.e. $f^{-1}(U) \in \Sigma$ for every $U \in \mathcal{B}_X$. Hence the composition $t \mapsto \Phi(f(t),t)$ is measurable. So, we can define the functional $I_\Phi: \mathcal{M}(T,X) \to [0,+\infty]$ by the formula

$$I_\Phi(f)=\int_T \Phi(f(t),t)d\mu .$$

Let us note that I_Φ is a pseudomodular on $\mathcal{M}(T,X)$ in the sense of [14], [16].

By the generalized Orlicz space $L^\Phi(\mathcal{M}(T,X))$ (or shortly L^Φ if it does not lead to misunderstanding) we mean the set of all functions $f \in \mathcal{M}(T,X)$ such that $I_\Phi(af) < +\infty$ for some $a>0$, equipped with the F-seminorm

$$|f|_\Phi = \inf\{u>0: I_\Phi(u^{-1}f) \leq u\}.$$

Let us note (cf. [13]) that $|f-f_n|_\Phi \to 0$ as $n \to +\infty$ if and only if $I_\Phi(a(f-f_n)) \to 0$ as $n \to +\infty$ for all $a>0$. The sets $\varepsilon B_\Phi(\varepsilon)$, where $\varepsilon > 0$ and

$$B_\Phi(\varepsilon) = \{f \in \mathcal{M}(T,X): I_\Phi(f) < \varepsilon\}$$

form a base of neighbourhoods of 0 in the space $(L^\Phi, |\cdot|_\Phi)$. By $E^\Phi(\mathcal{M}(T,X))$ (or shortly E^Φ) we denote a linear subspace of L^Φ defined as follows

$$E^{\Phi} = \{f \in L^{\Phi} : I_{\Phi}(af) < +\infty \text{ for all } a > 0\}.$$

Before we pass to the main part of this paper, we state some connections between Φ-functions and Musielak-Orlicz functions in the sense of the following definition:

1.2. Definition. A function $\Phi : R \times T \longrightarrow [0,+\infty]$ is said to be a Musielak-Orlicz function if

a′) $\Phi(u,\cdot)$ is measurable for each $u \in \mathbf{R}$,

b′) $\Phi(0,t)=0$, $\Phi(-u,t)=\Phi(u,t)$ for every $u \in \mathbf{R}$ and a.e. $t \in T$,

c′) $\Phi(\cdot,t)$ is not identically equal to zero and is left-continuous on $(0,+\infty)$ for a.e. $t \in T$,

d′) $\Phi(\cdot,t)$ is nondecreasing on $(0,+\infty)$ for a.e. $t \in T$,

e′) $\Phi(\cdot,t)$ is continuous at zero.

The next proposition is a simple modification of Theorem 6.1 in [9].

1.3. Proposition. Let (Z,d) be a separable metric space and $h : Z \times T \longrightarrow [0,+\infty]$ be a function such that $h(\cdot,t)$ is lower semicontinuous for every $t \in T$. If one of the following conditions is satisfied:

a) $h(\cdot,t)$ is continuous on the set $\{z \in Z : h(z,t) < +\infty\}$ (shortly: continuous) for every $t \in T$,

b) $Z=\mathbf{R}$ and $h(\cdot,t)$ is left-continuous for every $t \in T$,

then the following are equivalent:

(i) h is $\mathcal{B}_Z \times \Sigma$-measurable,

(ii) $t \longmapsto h(z,t)$ is measurable for every $z \in Z$.

Proof. (i) $\Longrightarrow$ (ii) is obvious.

(ii) $\Longrightarrow$ (i). Let $0 \le c < +\infty$. Then

$$h^{-1}([0,c]) = \{(z,t) : h(z,t) \le c\} =$$

$$= \begin{cases} \bigcap_{n=1}^{+\infty} \bigcup_{y \in X_0} \{t \in T : h(y,t) < c + \frac{1}{n}\} \times \{z \in Z : d(z,y) < \frac{1}{n}\} & \text{(by assumption a))} \\ \bigcap_{n=1}^{+\infty} \bigcup_{y \in Q} \{t \in T : h(y,t) < c + \frac{1}{n}\} \times \{z \in R : 0 \le z-y < \frac{1}{n}\} & \text{(by assumption b))} \end{cases}$$

where X_0 is a countable and dense subset of X and Q stands for the set of all rational numbers. We shall prove only the inclusion $\supset$ of the last equality (by assumption a)). Let $z \in Z$, $t \in T$ be such elements that there is a sequence $\{y_n\} \subset X_0$ such that

$$h(y_n,t) < c + \frac{1}{n} \text{ and } d(z,y_n) < \frac{1}{n} \text{ for every } n \in \mathbf{N}.$$

Hence $y_n \longrightarrow z$.

We claim that $h(z,t) < +\infty$. Suppose $h(z,t)=+\infty$. Then, by the lower semicontinuity of $h(\cdot,t)$, for every $m\in \mathbf{N}$ there is $\delta_m>0$ such that $d(y,z)<\delta_m$ implies $h(y,t)>m$ for every y Z. Let $m=c+2$ and n be such a number that $\frac{1}{n}<\delta_m$. Then $d(y_n,z)<\frac{1}{n}<\delta_m$, so

$$c+\frac{1}{n}>h(y_n,t)>m=c+2$$

- a contradiction.

Now, in virtue of the continuity of $h(\cdot,t)$ at the point z, $h(y_n,t)\longrightarrow$ $\longrightarrow h(z,t)$. Since

$$h(z,t)\le |h(z,t)-h(y_n,t)|+h(y_n,t)$$

$$\le |h(z,t)-h(y_n,t)|+c+\frac{1}{n}$$

we obtain $h(z,t)\le c$.

Now, the thesis is evident.

Let Φ be any Φ - [resp. Musielak-Orlicz-] function satisfying all the conditions of Definition 1.1 [resp. 1.2] with some set T_0 of measure zero but not necessarily empty. Let us consider a new measure space (S,Σ_S,μ_S) where $S=T\setminus T_0$, $\Sigma_S=\{A\cap S:A\in\Sigma\}$, $\mu_S=\mu|_S$. Then the measure μ_S is also nonnegative, atomless, σ-finite and complete. Furthermore, the spaces $L^\Phi(\mathcal{M}(T,X))$ and $L^\Phi(\mathcal{M}(S,X))$ are isometric, because $I_\Phi(f)=I_\Phi(f\chi_S)$ for every $f\in\mathcal{M}(T,X)$.

Let $X=\mathbf{R}$. Without loss of generality, we can assume now that the sets T_0 appearing in Definitions 1.1 and 1.2 are empty. Then the following implications hold: (a) $\Rightarrow$ (a´), [(a´) and (c´)] $\Rightarrow$ (a), (b) $\Leftrightarrow$ (b´), (c) $\Leftrightarrow$ (c´), (d) $\Leftrightarrow$ (d´), (e) $\Leftrightarrow$ (e´), so the conceptions of Φ-functions and Musielak-Orlicz functions are equivalent. The space $E^\Phi(\mathcal{M}(T,\mathbf{R}))$ is a closed subspace of $L^\Phi(\mathcal{M}(T,\mathbf{R}))$ and the following conditions are equivalent:

(i) $f\in E^\Phi(\mathcal{M}(T,\mathbf{R}))$,

(ii) f is μ-continuous i.e. $|f\chi_{A_n}|_\Phi\longrightarrow 0$ for every nonincreasing sequence $\{A_n\}$ of measurable subsets of T such that

$$\mu\Big(\bigcap_{m=1}^{+\infty} A_n\Big)=0,$$

(iii) $\lim\limits_{\mu(A)\to 0}|f\chi_A|_\Phi=0$ and $\forall_{\varepsilon>0}\ \exists_{A\in\Sigma}\ \mu(A)<+\infty$ and $|f\chi_{T\setminus A}|_\Phi<\varepsilon$.

2. Main results. If $X=\mathbf{R}$ then the space $E^\Phi(\mathcal{M}(T,\mathbf{R}))$ is equal to

$$E^\Phi_S(\mathcal{M}(T,\mathbf{R}))=\mathrm{cl}\,\{g\in S(T,\mathbf{R}):I_\Phi(g)<+\infty\},$$

where $S(T,\mathbf{R})$ denotes the space of all simple functions with support of finite measure and the closure is taken with respect to the norm $|\cdot|_\Phi$. The problem of the structure of the space $E^\Phi_S(\mathcal{M}(T,X))$ is more complicated in the case of vector valued functions. Sometimes, the fact that $E^{\Phi}\subset E^{\Phi}_s$ is very useful. Unfortunately, the above inclusion does not hold in general. Moreover, there are known conditions which ensure that $E^{\Phi}_s \neq \emptyset$. One of them is the following: (cf. [12],[25]).

Condition (B). There are an increasing sequence $\{T_i\}$ of sets of finite measure, $\bigcup_{i=1}^{+\infty} T_i=T$, and a sequence $\{\xi_n\}$ of measurable functions from T into $[0,+\infty]$ such that

$$\forall_{n\in N} \sup_{\|x\|<n} \Phi(x,t) \leq \xi_n(t) \text{ for a.e. } t\in T$$

and

$$\forall_{i\in N} \forall_{n\in N} \int_{T_i} \xi_n(t)d\mu<+\infty.$$

2.1. Lemma. If Φ satisfies Condition (B) then $E^{\Phi}\subset E^{\Phi}_s$.

Proof. Let $f\in E^{\Phi}$ and $\{T_i\}$ be a sequence taken from (B). Denote

$$A_n=\{t\in T_n: \|f(t)\|\leq n\} \text{ and } f_n=f\chi_{A_n},$$

$n=1,2,\dots$. Then each f_n is a bounded function vanishing outside a set of finite measure and $|f_n-f|_{\Phi}\longrightarrow 0$ as $n\longrightarrow +\infty$. In virtue of Proposition 3.2 from [12] (or Theorem 21(a) in [25]) $f_n\in E^{\Phi}_s$ for every natural number n. The rest of the proof is obvious.

Let us note that Condition (B) is not necessary for the inclusion $E^{\Phi}\subset E^{\Phi}_s$ (cf. Example 3.2). Moreover, Condition (B) is always satisfied provided X is a finite dimensional normed space and Φ is a continuous Φ-function with finite values. If X is separable, then Φ satisfies Condition (B) if and only if there is a set T_0 of measure 0 such that

$$\forall_{n>0} \forall_{t\notin T_0} \sup_{\|x\|<n} \Phi(x,t)<+\infty$$

(cf. [25]).

We shall say that the elements $\{e_1,e_2,\dots\}\subset X$ form a basis of the space X, if for each $x\in X$ there is exactly one sequence $\{a_n\}$ of numbers such that

$$\|x-\textstyle\sum_{k=1}^{n} a_k e_k\| \longrightarrow 0 \text{ whereas } n\longrightarrow +\infty .$$

In the sequel, we will denote by Φ_z $(z\in X)$ a Φ-function defined as follows:

$$\Phi_z:\operatorname{lin}\{z\}\times T\longrightarrow [0,+\infty] , \quad \Phi_z(uz,t)=\Phi(uz,t).$$

2.2. Theorem. Let us assume that

(+) there exist $z\in X\setminus\{0\}$ and a set A of a positive measure such that $\Phi(uz,t)<+\infty$ and $\liminf_{u\to+\infty} \frac{1}{u}\Phi(uz,t)>0$ for all $u>0$ and $t\in A$.

Then the following are equivalent:

(a) $(E^{\Phi})^{*} \neq \{0\}$.

(b) For every measurable set $B \subset A$ such that $z\chi_B \in E^{\Phi}$ there is a continuous linear operator $P_B : E^{\Phi}(\mathcal{M}(T,X)) \longrightarrow E^{\Phi_z}(\mathcal{M}(T,\text{lin}\{z\}))$ such that $\text{lin}\{z\chi_B\} \subset P_B(E^{\Phi})$.

Proof. (b) $\Longrightarrow$ (a). Let us define

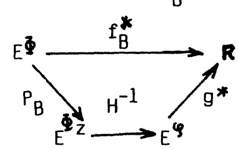
$$\varphi : R \times A \longrightarrow [0,+\infty],\ \ \varphi(u,t) = \Phi_z(uz,t).$$

It is obvious that φ is a Musielak-Orlicz function. Let us consider the operator $H : E^{\varphi} \longrightarrow E^{\Phi_z}$ defined by $H(g)=gz$ for all $g \in E^{\varphi}$. Then $|H(g)|_{\Phi_z} = |gz|_{\Phi_z} = |g|_{\varphi}$, so H is an isometry.

Moreover,

$$\liminf_{u \to +\infty} \frac{1}{u}\varphi(u,t) > 0 \text{ for all } t \in A,$$

so we can apply Theorem 0.2 and we obtain $(E^{\varphi})^{*} \neq \{0\}$. Let $0 \neq g^{*} \in (E^{\varphi})^{*}$ and let us define $f_B^{*} : E^{\Phi} \longrightarrow R$ such that f_B^{*} factors as follows



i.e. $f_B^{*} = g^{*} \circ H^{-1} \circ P_B$. Then f_B^{*} is linear and continuous. Since $g^{*} \neq 0$, there is a set $B_0 \subset A$, $\mu(B_0) > 0$ such that $\chi_{B_0} \in E^{\varphi}$ and $g^{*}(\chi_{B_0}) \neq 0$. Hence

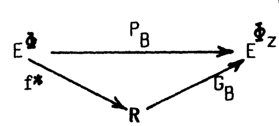
$$\int_T \Phi(cz\chi_{B_0}(t),t)d\mu = \int_T \varphi(c\chi_{B_0}(t),t)d\mu < +\infty$$

for all $c > 0$, i.e. $z\chi_{B_0} \in E^{\Phi}$. Therefore, there is a function $f \in E^{\Phi}$ such that $P_{B_0}(f) = z\chi_{B_0}$. Finally,

$$f_{B_0}^{*}(f) = g^{*}(H^{-1}(P_{B_0}(f))) = g^{*}(H^{-1}(z\chi_{B_0})) = g^{*}(\chi_{B_0}) \neq 0,$$

i.e. $f_{B_0}^{*}$ is nontrivial.

(a) $\Longrightarrow$ (b). Let f^{*} be a nontrivial continuous and linear functional on E^{Φ}. Let $B \subset A$ be a measurable set such that $z\chi_B \in E^{\Phi}$. Define



i.e. $P_B = G_B \circ f^{*}$, where $G_B(u) = uz\chi_B$. Obviously, the operator P_B is linear and continuous. Moreover,

$$P_B(E^\Phi)=\{f^*(f)z\chi_B : f\in E^\Phi\}=\{cz\chi_B : c\in \mathbf{R}\}.$$

2.3. Corollary. Let X be a p-Banach space with a Schauder basis $\{e_n\}$ and let Condition (+) of Theorem 2.2 be satisfied. If for every $\varepsilon>0$ there are $c,K>0$ and the function $h:T\longrightarrow [0,+\infty]$ such that $\int_T h(t)d\mu<\varepsilon/2$ and if the following inequality

$$\Phi(ca_ie_i,t)\le K\Phi(\textstyle\sum_{k=1}^{+\infty}a_ke_k,t)+h(t)$$

holds for all $t\in A$ and some fixed i, then there exists a nontrivial continuous linear functional on the space E^Φ.

Proof. By Theorem 2.2, it is sufficient to verify that the condition (b) of Theorem 2.2 is satisfied with $z=e_i$. The projection $P:X\longrightarrow \operatorname{lin}\{e_i\}=X_i$ defined by

$$P(\textstyle\sum_{k=1}^{+\infty}a_ke_k)=a_ie_i$$

is linear and continuous (cf. Theorem 26.1 in [20]). Every function $f:T\longrightarrow X$ can be uniquely represented as the sum of series $f(t)=\sum_{k=1}^{+\infty}f_k(t)e_k$, where $f_k:T\longrightarrow \mathbf{R}$ for k=1,2,... . Define

$$P_B:E^\Phi\longrightarrow E^{\Phi_{e_i}},\ P_B(\textstyle\sum_{k=1}^{+\infty}f_k(t)e_k)=f_i(t)\chi_B(t)e_i,$$

where B is a measurable subset of A. Then

$$\{t\in T:f_i(t)\chi_B(t)\le c\}=\{t\in B:f(t)\in P^{-1}[\{ue_i:u\le c\}]\}\in\Sigma,$$

because P is continuous and f is strongly measurable. This means that $f_i\chi_B$ is measurable. Further, by the assumption, we have

$$\Phi(cP_B[f(t)],t)\le K\Phi(f(t)\chi_B(t),t)+h(t),$$

so P_B is continuous. Finally, let $e_i\chi_B\in E^\Phi$. Then $\{ce_i\chi_B:c\in\mathbf{R}\}\subset P_B(E^\Phi)$ since $P_B(e_i\chi_B)=e_i\chi_B$.

2.4. Theorem. If $E^\Phi\subset E_s^\Phi$ and for all $z\in X\setminus\{0\}$

$$\liminf_{u\to+\infty}\frac{1}{u}\Phi(uz,t)=0 \text{ and } \lim_{u\to+\infty}\Phi(uz,t)>0,$$

for a.e. $t\in T$, then there are no nontrivial continuous linear functionals on the space E^Φ.

Proof. Let us suppose that there is a nonzero continuous linear functional F on the space E^Φ. Then $F(f)\neq 0$ for some $f\in E^\Phi$. By the assumption $E^\Phi\subset E_s^\Phi$, we can find a sequence $\{f_n\}$ of simple functions such that

$I_{\Phi}(f_n) < +\infty$ for $n=1,2,\ldots,$ and $|f_n - f|_{\Phi} \longrightarrow 0$ as $n \longrightarrow +\infty$.

Thus $F(f_n) \neq 0$ for some n. Taking into account the form of the function f_n, we infer that there are an element $z \in X$ and a set A of a positive measure such that $F(z\chi_A) \neq 0$.

Define

$$\varphi_z(u,t) = \Phi(uz,t) \text{ for } u \in \mathbf{R} \text{ and } t \in A.$$

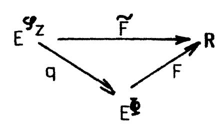
Then, in virtue of the assumption, $\varphi_z : \mathbf{R} \times A \longrightarrow [0,+\infty]$ is a (non-identically equal to 0 for $t \in A$) Musielak-Orlicz function.

Let $q : E^{\varphi_z} \longrightarrow E^{\Phi}$ be a linear operator defined by $q(g) = g\chi_A z$ for $g \in E^{\varphi_z}$. Then q is continuous. Indeed, if $|g_n - g|_{\varphi_z} \longrightarrow 0$ as $n \longrightarrow +\infty$, then

$$I_{\Phi}(a[q(g_n) - q(g)]) = \int_A \Phi(a[g_n(t) - g(t)]z, t)\,d\mu =$$

$$= I_{\varphi_z}(a(g_n - g)) \longrightarrow 0 \text{ whereas } n \longrightarrow +\infty$$

for all $a > 0$. Thus $|q(g_n) - q(g)|_{\Phi} \longrightarrow 0$ as $n \longrightarrow +\infty$. Therefore, the functional $\tilde{F} : E^{\varphi_z} \longrightarrow \mathbf{R}$ defined in the following manner



(i.e. $\tilde{F} = F \circ q$) is linear and continuous. Moreover, it is nonzero as well, since

$$\tilde{F}(\chi_A) = F \circ q(\chi_A) = F(z\chi_A) \neq 0.$$

Hence $(E^{\varphi_z})^* \neq \{0\}$.

On the other hand

$$\liminf_{u \to +\infty} \frac{1}{u} \varphi_z(u,t) = \liminf_{u \to +\infty} \frac{1}{u} \Phi(uz,t) = 0$$

for $t \in A$, so φ_z takes finite values and by Theorem 0.2, the functional $\tilde{F}$ must be identically equal to zero. The obtained contradiction ends the proof.

2.5. Corollary. Let X be a one-dimensional p-Banach space. The following are equivalent:

(a) $(E^{\Phi})^* \neq \{0\}$.

(b) There are $z \neq 0$ and a set A of a positive measure such that

$$\liminf_{u \to +\infty} \frac{1}{u} \Phi(uz,t) > 0 \text{ for every } t \in A.$$

Proof. (b) $\Longrightarrow$ (a). Let $B \subset A$ and $z\chi_B \in E^{\Phi}$. Define $P_B : E^{\Phi} \longrightarrow E^{\Phi_z}$ by the formula $P_B(f)=f$. Then P_B is linear and continuous because $X_z=X$ and $\Phi_z=\Phi$. Further, $\operatorname{lin}\{z\chi_B\}\subset E^{\Phi}=P_B(E^{\Phi})$. Hence, by Theorem 2.2, $(E^{\Phi})^*\neq\{0\}$.

(a) $\Longrightarrow$ (b). Suppose that the implication is not true. Assume

$$\liminf_{u\to+\infty}\frac{1}{u}\Phi(uz,t)=0$$

for all $z\in X\setminus\{0\}$ and a.e. $t\in T$. Since $\Phi(\cdot,t)\not\equiv 0$, so $\lim_{u\to+\infty}\Phi(uz,t)>0$ for all $z\in X\setminus\{0\}$ and a.e. $t\in T$.

Let us fix $z\in X\setminus\{0\}$. Defining $\varphi_z:R\times T\longrightarrow[0,+\infty]$ by the formula $\varphi_z(u,t)=\Phi(uz,t)$, it is easy to verify that spaces E^{φ_z} and E^{Φ} as well as $E_s^{\varphi_z}$ and E_s^{Φ} are isomorphic. Since $E^{\varphi_z}=E_s^{\varphi_z}$, then $E^{\Phi}=E_s^{\Phi}$. Now, applying Theorem 2.4, we obtain $(E^{\Phi})^*=\{0\}$. Contradiction.

3. Examples and corollaries. We say that a Φ-function Φ satisfies Condition Δ_2 if there are a set T_0 of measure zero, a number $K>0$ and an integrable function $h:T\longrightarrow[0,+\infty]$ such that

$$\Phi(2x,t)\leq K\,\Phi(x,t)+h(t)$$

for all $x\in X$ and $t\in T\setminus T_0$.

It is easy to verify that the spaces E^{Φ} and L^{Φ} are equal provided Φ satisfies Condition Δ_2. Thus, in this case, Theorems 2.2 and 2.4 can be considered as theorems on existence and nonexistence of a nontrivial continuous linear functional on the space L^{Φ}.

3.1. Example. Let $\Lambda:R\times T\longrightarrow[0,+\infty]$ be a Musielak-Orlicz function. Then the function $\Phi:X\times T\longrightarrow[0,+\infty]$ defined by

$$\Phi(x,t)=\Lambda(\|x\|,t)$$

for $x\in X$, $t\in T$ is a Φ-function. We will prove only the $\mathcal{B}_X\times\Sigma$-measurability of Φ. Let $c\in R$. Then

$$\{(x,t):\Phi(x,t)>c\}=\{(x,t):\Lambda(\|x\|,t)>c\}=\bigcup_{u\in Q}\{(x,t):\|x\|>u \text{ and } \Lambda(u,t)>c\}=$$

$$=\bigcup_{u\in Q}(\{x:\|x\|>u\}\times T\cap\{(x,t):\Lambda(u,t)>c\})\in\mathcal{B}_X\times\Sigma,$$

where Q denotes the set of all rational numbers. Thus the space E^{Φ} has the topological dual zero, provided E^{Λ} has the same property.

The generalized Orlicz spaces generated by Φ-functions defined in the same manner as in Example 3.1 are solid function spaces.

3.2. Example (of nonsolid generalized Orlicz space). Let $X=l^0$ be the space of all sequences $\{x_n\}$ of real numbers, possessing a finite number of nonzero elements, with the norm

$$\|x\| = \max_{n\in\mathbf{N}} |x_n|.$$

Moreover, let $r_n:T\longrightarrow(0,+\infty)$ be measurable functions ($n\in\mathbf{N}$) such that

$$\inf_{n\in\mathbf{N}} r_n(t)>0 \text{ for a.e. } t\in T.$$

Define

$$\Phi(x,t)=\sum_{n=1}^{+\infty}|x_n|^{r_n(t)},$$

where $x=(x_1,x_2,\dots,x_n,\dots)\in l^0$. Then Φ is a Φ-function with finite values. Indeed, the properties b), d) of Definition 1.1 are obvious. Moreover, $\Phi(x,t)\neq 0$ for all $x\neq 0$ and for a.e. $t\in T$. Let $x\in l^0$, $x\neq 0$ and let ε be an arbitrary positive number. Then $x_n=0$ for sufficiently large n, say for $n>n_0$. Further, the (finite) family of functions $u\longrightarrow|u|^{r_n(t)}$, $n=1,2,\dots,n_0$, is equicontinuous, so there is $\delta>0$ such that

$$|u-x_n|<\delta \Longrightarrow \left||u|^{r_n(t)}-|x_n|^{r_n(t)}\right|<\frac{\varepsilon}{n_0}$$

for all $n=1,2,\dots,n_0$, Hence, for every $y\in l^0$ such that $\|y-x\|<\delta$ we have $|y_n-x_n|<\delta$ for $n=1,2,\dots,n_0$, so

$$\Phi(x,t)-\Phi(y,t)=\sum_{n=1}^{n_0}(|x_n|^{r_n(t)}-|y_n|^{r_n(t)})-\sum_{n=n_0+1}^{+\infty}|y_n|^{r_n(t)}\leq$$

$$\leq\sum_{n=1}^{n_0}\left||x_n|^{r_n(t)}-|y_n|^{r_n(t)}\right|<\varepsilon.$$

Thus, the function $\Phi(\cdot,t)$ is lower-semicontinuous. Moreover, Φ is $\mathcal{B}_{l^0}\times\Sigma$-measurable. Indeed, for arbitrary i and $c>0$ we have

$$\{(x,t)\in l^0\times T:|x_i|^{r_i(t)}>c\}=\bigcup_{q\in Q_+}\{(x,t)\in l^0\times T:|x_i|>q>c^{1/r_i(t)}\}=$$

$$=[\bigcup_{q\in Q_1}(\{x\in l^0:|x_i|>q\}\times T)\cap(l^0\times\{t\in T:r_i(t)<\log_q c\})]\cup$$

$$\cup[\bigcup_{q\in Q_2}(\{x\in l^0:|x_i|>q\}\times T)\cap(l^0\times\{t\in T:r_i(t)>\log_q c\})]\in\mathcal{B}_{l^0}\times\Sigma,$$

where Q_+ is a set of positive rational numbers and $Q_1=Q_+\cap(0,1)$, $Q_2=Q_+\cap(1,+\infty)$. Now, $\mathcal{B}_{l^0}\times\Sigma$-measurability of Φ is obvious. Finally, let $0<u\leq 1$ and $r_t=\inf_{n\in\mathbf{N}} r_n(t)$. Then

$$\Phi(ux,t)=\sum_{n=1}^{+\infty}|u|^{r_n(t)}|x_n|^{r_n(t)}\le|u|^{r_t}\sum_{n=1}^{+\infty}|x_n|^{r_n(t)},$$

so $\lim_{u\to 0}\Phi(ux,t)=0$.

Moreover, it is easy to verify that Φ satisfies Condition Δ_2 and the inequality from Corollary 2.3 for all $i\in\mathbf{N}$ and a.e. $t\in T$. In general, Φ does not satisfy Condition (B). However, $E^{\Phi}\subset E_s^{\Phi}$ for arbitrary family $\{r_n(\cdot)\}$. Indeed, let $f\in E^{\Phi}$. Then $f(t)=(x_1(t),x_2(t),\ldots)$. Define

$$f_k(t)=(x_1(t),\ldots,x_k(t),0,\ldots)\text{ for }k=1,2,\ldots .$$

Since the sets $X_k=\{x\in l^0:x_n=0\text{ for }n\ge k\}$ are closed and

$$f_k^{-1}(U)=f^{-1}(U\cap X_k)\text{ for }U\in\mathcal{B}_{l^0},$$

the functions f_k $(k=1,2,\ldots)$ are measurable, $|f_n-f|_{\Phi}\longrightarrow 0$ as $n\longrightarrow+\infty$ and $f_n\in E^{\Phi}$ for n sufficiently large. Now, let $\{T_i\}$ be an increasing sequence of sets of finite measure such that $\bigcup_{i=1}^{+\infty}T_i=T$. Denote

$$T_m'=\{t\in T_m:\max_{1\le m\le k}r_n(t)\le m\}$$

and

$$f_{k,m}(t)=\begin{cases}f_k(t)\text{ if }\|f_k(t)\|\le m\text{ and }t\in T_m',\\ 0\quad\text{otherwise.}\end{cases}$$

Then $|f_{k,m}-f_k|_{\Phi}\longrightarrow 0$ as $m\longrightarrow+\infty$ for sufficiently large k, so $f_{k,m}\in E^{\Phi}$ for sufficiently large k and m. Finally, let $\{f_{k,m,r}\}$ be a sequence of simple functions such that $\|f_{k,m,r}(t)\|\le\|f_{k,m}(t)\|$ and $f_{k,m,r}(t)\longrightarrow f_{k,m}(t)$ as $r\longrightarrow+\infty$ for a.e. $t\in T$. Then

$$\Phi(a(f_{k,m,r}(t)-f_{k,m}(t)),t)\le\sup\{\Phi(x,t):\|x\|\le 2am,\ x\in X_k\}\le$$
$$\le\max\{(2am)^{r_n(t)}:1\le n\le k\}\le(2am)^m$$

for all $a>0$, $m\ge\frac{1}{2a}$ and $t\in T_m'$. Thus, by the Lebesgue dominated convergence theorem,

$$|f_{k,m,r}-f_{k,m}|_{\Phi}\longrightarrow 0\text{ as }r\longrightarrow+\infty$$

for sufficiently large k, m. Hence $f\in E_s^{\Phi}$, i.e $E^{\Phi}\subset E_s^{\Phi}$. The above considerations lead to the following

3.3. Corollary. There is a nonzero continuous linear functional on the space E^{Φ} if and only if the set

$$D=\bigcup_{n=1}^{+\infty}\{t\in T:r_n(t)\ge 1\}$$

is of positive measure.

Proof. If $\mu(D)>0$, then the set $D_k=\{t\in T: r_k(t)\geq 1\}$ is of positive measure for some $k\in\mathbf{N}$. Thus

$$\liminf_{u\to+\infty}\frac{1}{u}\Phi(ue_k,t)=\liminf_{u\to+\infty}\frac{1}{u}u^{r_k(t)}>0 \text{ for } t\in D_k,$$

so $(E^{\Phi})^*\neq\{0\}$ by Corollary 2.3.

On the other hand, if $\mu(D)=0$ and $z\in 1^0\setminus\{0\}$, $z_n=0$ for $n\geq k$, then

$$\liminf_{u\to+\infty}\frac{1}{u}\Phi(uz,t)=\liminf_{u\to+\infty}\frac{1}{u}\sum_{n=1}^{k}u^{r_n(t)}|z_n|^{r_n(t)}=\liminf_{u\to+\infty}u^{r_k^t-1}\Phi(z,t)=0$$

for a.e. $t\in T\setminus D$, where $r_k^t=\max_{1\leq n\leq k}r_n(t)$. Thus, $(E^{\Phi})^*=\{0\}$ by Theorem 2.4.

3.4. Example. Let $C[0,1]$ be the space of all continuous functions with the norm $\|x\|=\sup_{t\in[0,1]}|x(t)|$. Let $T=[0,1]$, μ be the Lebesgue measure on T. Moreover, let λ be an Orlicz function, i.e. $\lambda:R\to[0,+\infty)$ is even, continuous, nondecreasing on $(0,+\infty)$, and vanishes only at zero. Define

$$\Phi(x,t)=\lambda\left(\int_0^t x(s)ds\right) \text{ for } x\in C[0,1],\ t\in[0,1].$$

Then Φ is a continuous Φ-function with finite values satisfying Condition (B) (cf. Proposition 1.3). Thus we conclude:

The space E^{Φ} has the zero topological dual provided $(E^{\lambda})^*=\{0\}$.

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