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PERFECT CODES IN REGULAR GRAPHS

I. DVOŘÁKOVÁ-RULÍČOVÁ

Abstract: We prove that for any finite set M of positive integers and for any graph G with maximum degree d, G is an induced subgraph of a (d+1)-regular graph containing t-perfect codes for every $t \in M$.

Key words: Graph, regular graph, perfect code

Classification: 05C99, 94B25

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1. Introduction. Perfect codes in graphs were defined by Biggs [1] as a generalization of the classical notions of perfect Hamming- and Lee-error-correcting codes. However, Biggs and others considered only distance-regular graphs, since strong necessary conditions for the existence of perfect codes in such graphs were derived [1]. It turns out that perfect codes do not appear too often in distance-regular graphs, and new classes of distance-regular graphs with perfect codes are regarded with interest. On the other hand, one can easily construct general graphs containing perfect codes. The aim of this paper is to show that in a certain sense there exists a lot of regular graphs containing perfect codes.

All graphs considered are finite undirected and without loops and multiple edges. For a graph G with a vertex set V and edge set E, the notation $G=(V(G),E(G))$ will be used. Further we denote by $d(u,v)$ the distance function (i.e. the length of the shortest path between vertices u, v). The disjoint union of sets is denoted by $\uplus$.

Let $G=(V(G),E(G))$ be a graph and t a positive integer. Any set $C \subseteq V(G)$, $|C|>1$, is called a <u>(nontrivial) t-perfect code</u> if and only if for each vertex u of G there exists exactly one code-vertex $c \in C$ such that $d(u,c) \leq t$. Obviously, a graph contains a t-perfect code if and only if each of its connected components contains a t-perfect code. Therefore only connected graphs are considered.

Observation: For a graph G, $C\subset V(G)$, and $t>0$ the following statements are equivalent (provided $|C|>1$):

i) $C\subseteq V(G)$ is a t-perfect code in the graph G,

ii) $V(G)=\bigcup_{c\in C} S_t(c)$, where $S_t(c)$ is the closed neighbourhood of the vertex c of radius t.

iii) For any two distinct code vertices c_1, c_2 we have $d(c_1,c_2)\geq 2t+1$, and for each $u\in V(G)$ there exists $c\in C$ so that $d(u,c)\leq t$.

Obviously, every graph G is an induced subgraph of a graph G' containing a 1-perfect code (it is sufficient to put $U=\{u'|u\in V(G)\}$ and $G'=(V(G)\cup U, E(G)\cup\{\{u,u'\}|u\in V(G)\}$, then C=U is a 1-perfect code in G').

An essentially stronger result is proved in the sequel.

2. Main result

Theorem 1: Let t be a positive integer. Every graph with maximum degree d is an induced subgraph of a (d+1)-regular graph containing a t-perfect code.

Definition: Let n, d be positive integers. We define a P-path P(w,z) as the graph defined by the vertex set

$$V=\{w,z\}\cup\bigcup_{i=1}^{n} V(P_i^w)$$

and the edge set

$$E=\bigcup_{i=1}^{n} E(P_i^w)\cup\{\{y_i^w,x_{i+1}^w\}|i=1,\dots,n-1\}\cup\{\{w,x_1^w\},\{z,y_n^w\}\},$$

where P_i^w is the graph determined by the sets

$$V(P_i^w)=\{x_i^w,y_i^w\}\cup\{v_j^{w,i}|j=1,\dots,d\},$$

$$E(P_i^w)=\{\{v_j^{w,i},v_k^{w,i}\}|j\neq k,j,k=1,\dots,d\}\cup\{\{x_i^w,v_j^{w,i}\},\{y_i^w,v_j^{w,i}\}|j=1,\dots,d\}.$$

The distance of the vertices w, z is called the length of the P-path.

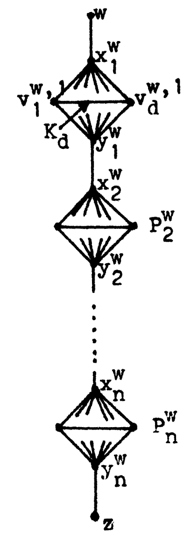
Proof of Theorem 1: Let G be a given graph with maximum degree d and t positive integer.

i) Let d be even.

At first we embed the graph G by a standard procedure into a d-regular graph G_d. Further, for each vertex $w\in V(G_d)$ we construct the graph H^w with the vertex set

$$V(H^w)=V(P(w,y))\cup\bigcup_{i=1}^{d/2} V(P(z_i^w,z_{d-i+1}^w))$$

and the edge set



$$E(H^w)=E(P(w,y)) \cup \bigcup_{i=1}^{d/2} E(P(z_i^w,z_{d-i+1}^w)) \cup \{\{z_i^w,z_j^w\} \mid i \neq j,$$

$$i,j=1,\ldots,d\} \cup \{\{y,z_i^w\} \mid i=1,\ldots,d\},$$

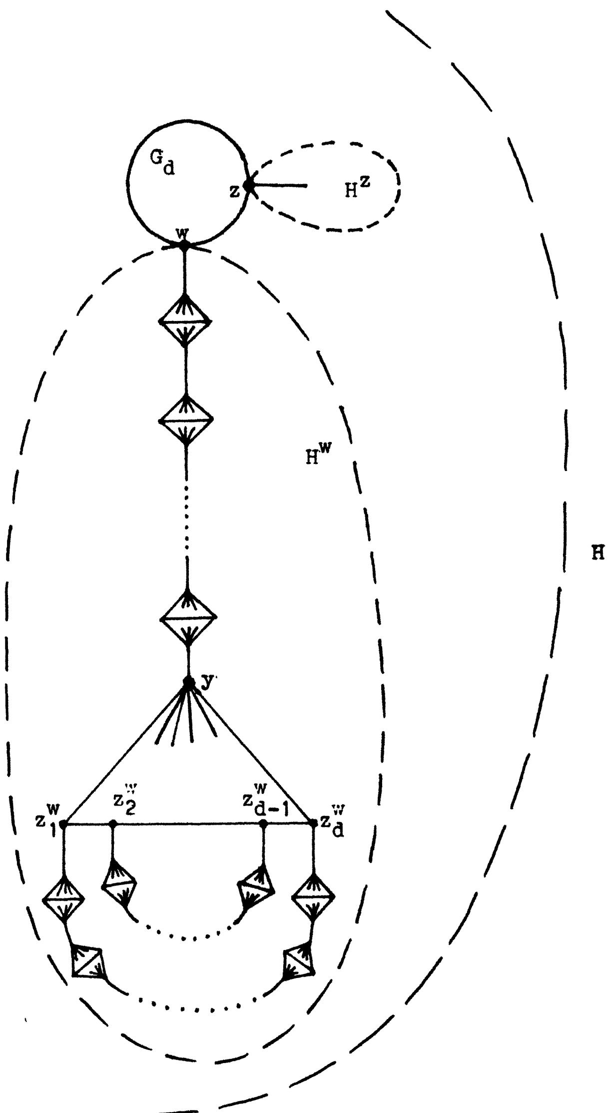
where $P(w,y)$ is a P-path of the length

$$s=\begin{cases} 2t-1 & \text{if } (2t+1) \equiv 0 \bmod 3 \\ 4t+1 & \text{if } (2t+1) \equiv 1 \bmod 3 \\ 2t & \text{if } (2t+1) \equiv 2 \bmod 3 \end{cases}$$

and each $P(z_i^w,z_{d-i+1}^w)$ is a P-path of the length

$$\bar{s}=\begin{cases} 2(t+1) & \text{if } (2t+1) \equiv 0 \bmod 3 \\ 4t+1 & \text{if } (2t+1) \equiv 1 \bmod 3 \\ 2t & \text{if } (2t+1) \equiv 2 \bmod 3 \end{cases}$$

Thus the graph H^w arises from the complete graph K_d^w on d vertices $z_1^w,\ldots,z_d^w$, the P-path $P(w,y)$ and d/2 P-paths $P(z_1^w,z_{d-i+1}^w)$ by connecting the vertex y by an edge with each of the vertices z_i^w $(i=1,\ldots,d)$, as well as $K_d^w \cap P(w,y)=\emptyset$ and $K_d^w \cap P(z_i^w,z_{d-i+1}^w)=\{z_i^w,z_{d-i+1}^w\}$. Note that all vertices of H^w have degrees


G_d
z
H^z
w
H^w
H
y
z_1^w
z_2^w
z_{d-1}^w
z_d^w

(d+1) except of the vertex w which has degree 1. Moreover, there exists a t-perfect code C^w in H^w, defined by the following:

$$C^w = \bigcup_{i=1}^{d/2} \{v_1^{z_i^w}\} \cup \{y\} \text{ for } t=1$$

$$C^w = \bigcup_{i=1}^{d/2} \{v_1^{z_i^w,2(t+2)/6}\} \cup \{x^w_{2(t-1)/6+1}\} \text{ for } t>1 \text{ such that } 2t+1 \equiv 0 \bmod 3$$

$$C^w = \bigcup_{i=1}^{d/2} \{y_{t/3}^{z_i^w}, x_{t+1}^{z_i^w}\} \cup \{y^w_{t/3}, x^w_{t+1}\} \text{ if } 2t+1 \equiv 1 \bmod 3$$

$$C^w = \bigcup_{i=1}^{d/2} \{v_1^{z_i^w,(t+1)/3}\} \cup \{v_1^{w,(t+1)/3}\} \text{ if } 2t+1 \equiv 2 \bmod 3.$$

For making the proof complete, it is now sufficient to consider the graph H= =(V,E) defined by the sets $V= \bigcup_{w \in V(G_d)} V(H^w)$ and $E= \bigcup_{w \in V(G_d)} E(H^w) \cup E(G_d)$.

From the construction of the graph H we can see that H is (d+1)-regular and contains the graph G as an induced subgraph. In addition, $C= \bigcup_{w \in V(G_d)} C^w$ is a t-perfect code in H. Thus for d even the theorem is proved.

ii) For d odd, the proof may be performed by an easy modification of the case i). It is sufficient to realize that any graph contains an even number of vertices of odd degree.

The following more general statement can be proved in a similar way:

Theorem 2: Let M be a finite set of positive integers and G a graph with maximum degree d. Then there exists a graph H with the following properties:

i) G is an induced subgraph of H,

ii) H is (d+1)-regular,

iii) H contains t-perfect codes for all $t \in M$.

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