Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 1, 49--61

Persistent URL: http://dml.cz/dmlcz/106596

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
29,1 (1988)

ON THE DECOMPOSITION OF CONTINUOUS FLOWS ON THE POLISH SPACE

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Abstract: The purpose of this paper is to prove the Borel measurability of the set of regular points of a continuous flow on the Polish space using the methods of J.C. Oxtoby [4] and S. Fomin [2]. The including into a compact dynamical system in discrete time is replaced by that into the shift-flow on the Polish space of Hilbert cube-valued continuous functions. It is shown that the set of regular points is $G_{\delta\sigma\delta}$.

Key words: Polish space, continuous flow, regular points, invariant measure.

Classification: 28D15, 60B10

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1. Definitions and the ergodic theorem. The set of all natural numbers $\{1,2,\dots\}$ will be denoted by $\mathbf{N}$.

Let G be a locally compact Abelian group with the second axiom of separability (in such a case G is a σ-compact Polish space); it will represent the time. A Haar measure on the class $\mathcal{B}_G$ of Borel sets in G will be denoted by λ , the unit element and the group operation of G by e and +, respectively.

At one with [6] we say that $\{\gamma_k\}_{k=1}^{\infty}$ is an (i)-sequence if γ_k is a measure on $\mathcal{B}_G$ of the form $\gamma_k(C)=\frac{\lambda(C\cap C_k)}{\lambda(C_k)}$, $C\in\mathcal{B}_G$, for every $k\in\mathbf{N}$, where $\{C_k\}_{k=1}^{\infty}$ is a nondecreasing sequence of Borel sets satisfying the following conditions (i1)-(i4):

(i1) $0<\lambda(C_k)<\infty$ for any $k\in\mathbf{N}$,

(i2) $\lim\limits_{k}\frac{\lambda(C_k\cap(C_k+t))}{\lambda(C_k)}=1$ for any $t\in G$,

(i3) for each $k\in\mathbf{N}$ there is a nondecreasing sequence $\{C_{k,j}\}_{j=1}^{\infty}$ of compact sets with the union C_k, (i4) $\sup_k \frac{\lambda(C_k - C_k)}{\lambda(C_k)} < \infty$.

In what follows, let $\{\gamma_k\}_{k=1}^{\infty}$ be a fixed (i)-sequence.

By the flow (on a probability space $(\Omega, \mathcal{F}, \mu)$) we mean any group $\{T_t\}_{t\in G}$ of invertible measure preserving transformations of Ω with respect to the composition which is measurable in the sense that $\varphi(\omega,t)=T_t\omega$ $(\omega\in\Omega, t\in G)$ is an $\mathcal{F}\times\mathcal{B}_G - \mathcal{B}_G$ measurable mapping. Such a flow will be denoted by $(\Omega, \mathcal{F}, \mu, \{T_t\}_{t\in G})$.

Hereafter in this section let us consider a flow $(\Omega, \mathcal{F}, \mu, \{T_t\}_{t\in G})$. A set $E\subset\Omega$ is called invariant ($\{T_t\}_{t\in G}$-invariant) if $T_tE=E$ for any $t\in G$. The σ-algebra of all invariant $\mathcal{F}$-measurable sets will be denoted by $\mathcal{J}$. The measure μ is called ergodic if there is no $E\in\mathcal{J}$ with $0<\mu(E)<1$. Symbols $L^1(\mu)$ and $L^2(\mu)$ will designate all real $\mathcal{F}$-measurable functions $f=f(\omega)$ defined on Ω μ-a.e. such that $\int_\Omega |f(\omega)|d\mu(\omega)<\infty$, and $\int_\Omega |f(\omega)|^2 d\mu(\omega)<\infty$, respectively. The usual pseudonorms will be denoted by $\| \ \|_1$ and $\| \ \|_2$.

Let f be a real $\mathcal{F}$-measurable function defined on Ω μ-a.e. For any $k\in\mathbf{N}$, $\omega\in\Omega$ let $M(f,\omega,k)=f^{(k)}(\omega)=\int_G f(T_t\omega)d\gamma_k(t)$ provided $\gamma_k(\{t\in G: T_t\omega\notin D_f\})=0$ (D_f is the domain of f), and further, $M(f,\omega)=f^*(\omega)=$ $=\lim_k M(f,\omega,k)$ provided the limit exists.

Any point $\omega\in\Omega$ and any $k\in\mathbf{N}$ also defines a probability measure m_ω^k on $(\Omega,\mathcal{F})$ by $m_\omega^k(E)=\int_G \chi_E(T_t\omega)d\gamma_k(t)$, $E\in\mathcal{F}$. Obviously, $\int f(z)dm_\omega^k(z)=f^{(k)}(\omega)$ for every real $\mathcal{F}$-measurable function f on Ω.

Proposition 1 (Ergodic theorem). For any $f\in L^p(\mu)$ $(p\in\{1,2\})$ there is an $\mathcal{J}$-measurable function $f^*\in L^p(\mu)$ such that $\lim_k f^{(k)}(\omega)=f^*(\omega)$ μ-a.e. and $\lim_k \|f^{(k)}-f^*\|_p=0$. Besides $\int_E f^*(\omega)d\mu(\omega)=\int_E f(\omega)d\mu(\omega)$, whenever $E\in\mathcal{J}$.

Proof. The assertion is a special case of Theorems 6.1, 6.2 and 6.4 in [6].

Lemma 1. Let $f\in L^2(\mu)$. Then the function $\psi_k(\omega)=M((f^{(k)}-f^*)^2,\omega)$ is well defined (μ-a.e.) for any $k\in\mathbf{N}$, and $\lim_k \int_\Omega \psi_k d\mu=0$.

Proof. By Proposition 1, for any $k\in\mathbf{N}$ it holds that $f^{(k)}-f^*\in L^2(\mu)$, i.e. $(f^{(k)}-f^*)^2\in L^1(\mu)$. Hence ψ_k's are well defined (μ-a.e.), and (again by Proposition 1) $\lim_k \int_\Omega \psi_k d\mu=0$.

2. Probability measures on metric spaces. Let $V=(V, \rho_V)$ be a metric space. As usually, spherical neighbourhoods will be denoted by $U^{\varepsilon}(v)$ ($\varepsilon > 0$, $v \in V$). By $\mathcal{M}(V)$ we mean the set of all probability measures on $\mathcal{B}_V$ (on the σ-algebra of Borel sets). $\mathcal{C}(V)$ will be the space of all real bounded continuous functions on V equipped with the topology of uniform convergence on compact sets (the base of which is formed by sets of the type $U(f;B,\varepsilon)=\bigcap_{t\in B}\{g \in \mathcal{C}(V): |f(t)-g(t)| < \varepsilon\}$, $f \in \mathcal{C}(V)$, $B \in \mathcal{K}(V)$, $\varepsilon > 0$; $\mathcal{K}(V)$ denotes the class of all compact subsets of V). Let $F(V)=\bigcap_{v \in V}\{f \in \mathcal{C}(V):$ $: |f(v)| \leq 1\}$.

Further, let us recall the following probabilistic convention. A sequence $\{\mu_k\}_{k=1}^{\infty}$ in $\mathcal{M}(V)$ weakly* converges to the measure $\mu \in \mathcal{M}(V)$ ($\mu_k \to \mu$, $k \to \infty$) if $\lim_k \int f d\mu_k = \int f d\mu$ for each $f \in \mathcal{C}(V)$. A subset $\mathcal{S} \subset \mathcal{M}(V)$ is said to be relative compact if for every sequence $\{\mu_k\}_{k=1}^{\infty}$ in $\mathcal{S}$ there is a measure $\mu \in \mathcal{M}(V)$ and a subsequence $\{\mu_{k(j)}\}_{j=1}^{\infty}$ such that $\mu_{k(j)} \to \mu$, $j \to \infty$. Finally, a subset $\mathcal{S} \subset \mathcal{M}(V)$ is said to be tight if for any $\varepsilon > 0$ there is $K \in \mathcal{K}(V)$ with the property that, for each $\mu \in \mathcal{S}$, $\mu(K) > 1-\varepsilon$. By the Prochorov's theorem the conditions of relative compactness and tightness are the same in the case of Polish space V.

Proposition 2. Let V be a Polish space, $\mu \in \mathcal{M}(V)$, and let $L_\mu(f)=\int_V f d\mu$, $f \in \mathcal{C}(V)$. Then L_μ constitutes a linear functional on $\mathcal{C}(V)$ which is continuous on $F(V)$.

Proposition 3. There is a countably dense subset in $\mathcal{C}(V)$ if V is Polish.

The Proposition 2 follows from Prochorov's theorem, and for the proof of the last assertion see [3]. Consequently, if V is Polish, there is a countable dense subset in F(V), too. Let us fix it and denote by $F_0(V)$.

By the Hilbert cube we mean the compact metric space $J=\langle 0,1\rangle^{\mathbf{N}}$ equipped with the metric ρ_J defined as $\rho_J(x,y)=\sum_{n=1}^{\infty} \frac{1}{2^n} |x_{(n)}-y_{(n)}|$, $x,y \in J$, $x=\{x_{(n)}\}_{n=1}^{\infty}$, $y=\{y_{(n)}\}_{n=1}^{\infty}$. As it is well-known, every Polish space is homeomorphic to a G_δ-subset of J (Urysohn).

3. The space $\mathcal{C}_J(V)$. In this section we shall suppose that $V=(V,\rho_V)$ is a locally compact Polish space. As it is well known, $V=\bigcup_{n=1}^{\infty} B_n^o$ (the union of interiors) where $\{B_n\}_{n=1}^{\infty}$ is a sequence of compact sets. The space of all continuous mappings of V into the Hilbert cube with the topology of uniform convergence on compact sets will be denoted by $\mathscr{C}_J(V)$ (the base of this topology is formed by sets $U(\tilde{x};B,\varepsilon)=\bigcap_{t\in B}\{\tilde{y}\in\mathscr{C}_J(V):\rho_J(\tilde{x}(t),\tilde{y}(t))<\varepsilon\}$, $\tilde{x}\in\mathscr{C}_J(V)$, $B\in\mathscr{K}(V)$, $\varepsilon>0$).

Proposition 4. $\mathscr{C}_J(V)$ is Polish.

Proof. First we shall show that the above topology is metrizable. Let us set $\rho_n(\tilde{x},\tilde{y})=\sup\{\rho_J(\tilde{x}(t),\tilde{y}(t)):t\in B_n\}$, $n\in\mathbf{N}$, $\tilde{x},\tilde{y}\in\mathscr{C}_J(V)$, and further

(1) $$\rho(\tilde{x},\tilde{y})=\sum_{n=1}^{\infty}\frac{1}{2^n}\cdot\rho_n(\tilde{x},\tilde{y}),$$

$\tilde{x},\tilde{y}\in\mathscr{C}_J(V)$. Obviously, ρ is a metric on $\mathscr{C}_J(V)$, and for any neighbourhood $U^{\varepsilon}(\tilde{x})$ ($\varepsilon>0$, $\tilde{x}\in\mathscr{C}_J(V)$) there is $n_0\in\mathbf{N}$ such that $U(\tilde{x};\bigcup_{n=1}^{n_0}B_n,\varepsilon/2)\subset U^{\varepsilon}(\tilde{x})$. On the other hand, let us have a set $U(\tilde{x};B,\varepsilon)$ ($\tilde{x}\in\mathscr{C}_J(V)$, $B\in\mathscr{K}(V)$, $\varepsilon>0$). Since $V=\bigcup_{n=1}^{\infty}B_n^0$, $B\subset\bigcup_{n=1}^{n_0}B_n^0\subset\bigcup_{n=1}^{n_0}B_n$ for some $n_0\in\mathbf{N}$. Consequently, $U^{\delta}(\tilde{x})\subset$ $\subset U(\tilde{x};B,\varepsilon)$ for $\delta=\varepsilon/(2^{n_0})$, which together gives the metrizability.

Since the completeness in the metric ρ is easy to see, it remains to verify the separability, i.e. to find a countable set which intersects any open set $U(\tilde{x};B,\varepsilon)$. To this end let us define, for each $n\in\mathbf{N}$, an auxiliary continuous mapping φ_n of J into itself by the rule $\varphi_n(t)=(t_{(1)},t_{(2)},\ldots$ $\ldots,t_{(n)},0,0,\ldots)$, $t=(t_{(1)},t_{(2)},\ldots)\in J$. As it follows from the separability of the space of all real continuous functions on $\langle 0,1\rangle^n$ (with the supreme norm), there is a countable subset $A_n\subset\mathscr{C}_J(J)$ with the property that for each $\tilde{z}\in\mathscr{C}_J(J)$ and $\varepsilon>0$ there exists $\tilde{y}\in A_n$ satisfying

(2) $$\rho_J(\tilde{y}(t),\varphi_n(\tilde{z}(\varphi_n(t))))<\varepsilon/2$$

for any $t\in J$. Let us suppose that $\tilde{x}\in\mathscr{C}_J(V)$, $B\in\mathscr{K}(V)$, and $\varepsilon>0$. According to Urysohn's assertion, we may assume that V itself is a subset of J. Further, there is a continuous mapping $\tilde{z}$ of J into itself such that $\tilde{z}=\tilde{x}$ on B (from the normality of J). Since J is compact (and $\tilde{z}$ is uniformly continuous), we can find $n\in\mathbf{N}$ such that, for any $t\in J$,

(3) $$\rho_J(\varphi_n(\tilde{z}(\varphi_n(t))),\tilde{z}(t))<\varepsilon/2.$$

Summarizing (2) and (3), $\rho_J(\tilde{y}(t),\tilde{x}(t))<\varepsilon$ for any $t\in B$, i.e. $\tilde{y}\in U(\tilde{x};B,\varepsilon)$. The desired countable set corresponds to $\bigcup_{n=1}^{\infty}A_n$.

Let us define the module of continuity (of $\tilde{x}$ on B) for any $\tilde{x}\in\mathscr{C}_J(V)$, $B\in\mathscr{K}(V)$ and $\delta>0$ by $w_{\tilde{x}}(B,\delta)=\sup\{\rho_J(\tilde{x}(s),\tilde{x}(t)):s,t\in B,\rho_V(s,t)<\delta\}$.

Lemma 2. Let $E \subset \mathcal{C}_J(V)$. Its closure $\overline{E}$ is compact if and only if, for any $n_0 \in \mathbf{N}$,

$$\lim_{\delta\to 0} \sup_{\tilde{x}\in E} w_{\tilde{x}}\Big(\bigcup_{n=1}^{n_0} B_n, \delta\Big)=0. \tag{4}$$

Proof. Let $\overline{E}$ be compact. Clearly, the functions $w_{\tilde{x}}(B,\frac{1}{k})$ of $\tilde{x}$ (k=1,2,...) are continuous and (in k) nonincreasing, for each $B \in \mathcal{K}(V)$, which implies the uniform convergence $\lim_{k} w_{\tilde{x}}(B, \frac{1}{k})$ on E, and hence (4). Conversely, (4) allows us to construct, for any $\varepsilon > 0$, a finite ε-net of E in the space J^V (with the metric defined also by (1)) making use of the step-J-valued functions vanishing outside a sufficiently large compact $\bigcap_{n=1}^{n_0} B_n$. Thus the compactness of $\overline{E}$ follows from the fact that $\mathcal{C}_J(V)$ is closed.

Lemma 3. A sequence $\{\mu_k\}_{k=1}^{\infty}$ in $\mathcal{M}(\mathcal{C}_J(V))$ is relative compact if and only if, for any $n_0, j, \ell \in \mathbf{N}$, there is $p \in \mathbf{N}$ such that, for every $k \in \mathbf{N}$,

$$\mu_k\Big(\{\tilde{x} \in \mathcal{C}_J(V): w_{\tilde{x}}\Big(\bigcup_{n=1}^{n_0} B_n, \frac{1}{p}\Big) \geq \frac{1}{j}\}\Big) \leq \frac{1}{\ell}. \tag{5}$$

Proof. If $\{\mu_k\}_{k=1}^{\infty}$ is relative compact, then, according to Prochorov, it is tight. Thus for any $\ell \in \mathbf{N}$ there is $K \in \mathcal{K}(\mathcal{C}_J(V))$ such that, for every $k \in \mathbf{N}$, $\mu_k(K) > 1-\frac{1}{\ell}$. As it follows from Lemma 2, for any n_0, $j \in \mathbf{N}$, there is $p \in \mathbf{N}$ such that $K \subset \{\tilde{x} \in \mathcal{C}_J(V): w_{\tilde{x}}(\bigcup_{n=1}^{n_0} B_n, \frac{1}{p}) < \frac{1}{j}\}$, which implies (5).

Conversely, for any $n_0, j, \ell \in \mathbf{N}$, let $p=p(n_0,j,\ell) \in \mathbf{N}$ be such that, for every $k \in \mathbf{N}$, $\mu_k(E_{j,n_0,\ell}) > 1-\frac{1}{\ell}\cdot\frac{1}{2^j}\cdot\frac{1}{2^{n_0}}$, where $E_{j,n_0,\ell}=$ $= \{\tilde{x} \in \mathcal{C}_J(V): w_{\tilde{x}}(\bigcup_{n=1}^{n_0} B_n, \frac{1}{p}) < \frac{1}{j}\}$. Setting $E_\ell = \bigcap_{n_0=1}^{\infty} \bigcap_{j=1}^{\infty} E_{j,n_0,\ell}$ we obtain (following Lemma 2 again) a compact set $\overline{E_\ell}$: for any $n_0 \in \mathbf{N}$

$$\lim_{\delta\to 0} \sup \{w_{\tilde{x}}(\bigcup_{n=1}^{n_0} B_n, \delta): \tilde{x} \in E_\ell\} \leq \lim_{\delta\to 0} \sup \{w_{\tilde{x}}(\bigcup_{n=1}^{n_0} B_n, \delta): \tilde{x} \in \bigcap_{j=1}^{\infty} E_{j,n_0,\ell}\} =0.$$

Moreover, since $\mu_k(E_\ell) > 1-\frac{1}{\ell}$ for any $k \in \mathbf{N}$, we have proved the tightness, which is sufficient.

4. Continuous flow on a metric space. Let Ω be a metric space and $\{T_t\}_{t\in G}$ a group of homeomorphisms of Ω (with the composition as the group operation) which is continuous as a mapping of $\Omega \times G$ into Ω with respect to the product topology. The triad $(\Omega, \mathcal{B}_\Omega, \{T_t\}_{t\in G})$ will be called a continuous flow (on a metric space). The set of all $\{T_t\}_{t\in G}$-invariant probability measures on $(\Omega, \mathcal{B}_\Omega)$ will be denoted by $\mathcal{M}_{st}(\Omega, \{T_t\}_{t\in G})$. Recall that for any $\mu \in \mathcal{M}_{st}(\Omega, \{T_t\}_{t\in G})$ it holds that $\mu \circ T_t = \mu$, whenever $t \in G$, i.e. $(\Omega, \mathcal{B}_\Omega, \mu, \{T_t\}_{t\in G})$ is a flow on a probability space. Thus we shall use the notions introduced in Section 1 in the case of a continuous flow, too (the σ-algebra $\mathcal{F}$ will be replaced by $\mathcal{B}_\Omega$). A set $E \in \mathcal{B}_\Omega$ is said to have invariant measure one if, for any $\mu \in \mathcal{M}_{st}(\Omega, \{T_t\}_{t\in G})$, $\mu(E)=1$. Let us note that it may happen $\mathcal{M}_{st}(\Omega, \{T_t\}_{t\in G})=\emptyset$ (see [5]); the theory below is vacuous for such flows.

Definition 1. Let $(\Omega, \mathcal{B}_\Omega, \{T_t\}_{t\in G})$ be a continuous flow on a metric space. A point $\omega \in \Omega$ is called <u>quasi-regular</u> ($\omega \in Q=Q(\Omega, \{T_t\}_{t\in G})$) if

(q1) the limit $M(f,\omega)$ exists for any $f \in F(\Omega)$,

(q2) the sequence $\{m_\omega^k\}_{k=1}^{\infty}$ (in $\mathcal{M}(\Omega)$) is relative compact.

Lemma 4. For every $\omega \in Q(\Omega, \{T_t\}_{t\in G})$ there is the unique measure $m_\omega \in \mathcal{M}_{st}(\Omega, \{T_t\}_{t\in G})$ such that $m_\omega^k \longrightarrow m_\omega$, $k \longrightarrow \infty$.

Proof. By (q2), $m_\omega^{k(j)} \longrightarrow \mu$, $j \longrightarrow \infty$, for some $\mu \in \mathcal{M}(\Omega)$ and some increasing sequence $\{k(j)\}_{j=1}^{\infty}$. By (q1) we have, for any $f \in F(\Omega)$, $\int_\Omega f d\mu = \lim_j \int_\Omega f dm_\omega^{k(j)} = \lim_j M(f,\omega,k(j)) = M(f,\omega) = \lim_k \int_\Omega f dm_\omega^k$; hence $m_\omega^k \longrightarrow \mu$, $k \longrightarrow \infty$. As such a measure μ is unique, let us write $\mu = m_\omega$. Further, since for any $f \in \mathcal{C}(\Omega)$, $k \in \mathbf{N}$, $s \in G$, $|\int f(z)dm_\omega^k(z) - \int f(T_s z)dm_\omega^k(z)| =$
$= \frac{1}{\lambda(C_k)} |\int_{C_k} f(T_t\omega)d\lambda(t) - \int_{C_k+s} f(T_t\omega)d\lambda(t)| \leq \frac{\|f\|}{\lambda(C_k)} \cdot 2\lambda(C_k \Delta (C_k+s))$
($\|f\| = \sup_{t\in G} |f(t)|$, Δ means the symmetrical difference), it follows that $m_\omega \in \mathcal{M}_{st}(\Omega, \{T_t\}_{t\in G})$ by (i2).

Lemma 5. Let Ω be a Polish space. Then $\omega \in Q(\Omega, \{T_t\}_{t\in G})$ if and only if ω satisfies the conditions (q1´) and (q2), where (q1´) means that the limit $M(f,\omega)$ exists for any $f \in F_0(\Omega)$.

Proof. Let $\omega \in \Omega$ satisfy (q2), i.e. the sequence $\{m_\omega^k\}_{k=1}^{\infty}$ is tight. Hence for any $g \in F(\Omega)$ and $\varepsilon > 0$ there is $f \in F_0(\Omega)$ such that, for every $k \in \mathbf{N}$, $|M(f,\omega,k) - M(g,\omega,k)| < \varepsilon/3$ (we find $B \in \mathcal{K}(\Omega)$ with the property $m_\omega^k(B) > 1 - \varepsilon/12$ for every $k \in \mathbf{N}$ and $f \in F_0(\Omega)$ such that $|f(z)-g(z)| < \varepsilon/6$ whenever $z \in B$). Since by (q1´) $|M(f,\omega,k) - M(f,\omega,k')| < \varepsilon/3$ for sufficiently large k, $k' \in \mathbf{N}$, the limit $M(g,\omega)$ exists.

5. The shift-flow on $\mathcal{C}_J(G)$. We can write $G=\bigcup_{n=1}^{\infty} B_n^o$ (for each $n\in\mathbf{N}$ let $B_n\in\mathcal{K}(G)$). According to Proposition 4, $\mathcal{C}_J(G)$ is a Polish space (with the metric ρ defined by (1)). Let us write in short $\mathcal{B}_{\mathcal{C}}=\mathcal{B}_{\mathcal{C}_J(G)}$. For any $\tilde{x}\in\mathcal{C}_J(G)$ and $s,t\in G$ let $(S_t\tilde{x})(s)=\tilde{x}(s+t)$. Evidently, S_t is a 1:1 mapping of $\mathcal{C}_J(G)$ onto itself for any $t\in G$.

Proposition 5. $(\mathcal{C}_J(G),\mathcal{B}_{\mathcal{C}},\{S_t\}_{t\in G})$ is a continuous flow on the Polish space.

Proof. We have to prove that $\{S_t\}_{t\in G}$ is a continuous group of homeomorphisms of $\mathcal{C}_J(G)$. Let $\{\tilde{x}_j\}_{j=1}^{\infty}$ ($\lim_j \tilde{x}_j=\tilde{x}$) and $\{t_j\}_{j=1}^{\infty}$ ($\lim_j t_j=s'$) be convergent sequences in $\mathcal{C}_J(G)$ and G, respectively. We have to show that $\lim_j S_{t_j}\tilde{x}_j=S_{s'}\tilde{x}$. As clearly $S_s\circ S_t=S_{s+t}$ for any $s,t\in G$, we may assume that $s'=e$. Let $B\in\mathcal{K}(G)$ and $\varepsilon>0$. We ask whether it holds, for any sufficiently large j and any $t\in B$, that $\rho_J((S_{t_j}\tilde{x}_j)(t),\tilde{x}(t))<\varepsilon$. Let U be a neighbourhood of e with the compact closure $\bar{U}$. Since $B'=\bigcup_{t\in\bar{U}}(B+t)$ is compact, there is $\delta>0$ such that $\rho_J(\tilde{x}(s),\tilde{x}(t))<\varepsilon/2$ whenever $s,t\in B'$ with $\rho_G(s,t)<\delta$. Further, from the continuity of the group operation there is $\Delta>0$ such that $\rho_G(t,s+t)<\delta$ whenever $t\in B$ and $\rho_G(e,s)<\Delta$. Finally, for all sufficiently large j and $t\in B$, it follows $\rho_J(\tilde{x}_j(t_j+t),\tilde{x}(t_j+t))<\varepsilon/2$ from the convergence $\lim_j \tilde{x}_j=\tilde{x}$, and $\rho_J(\tilde{x}(t_j+t),\tilde{x}(t))<\varepsilon/2$ from the uniform continuity of $\tilde{x}$ on B'.

Lemma 6. $Q(\mathcal{C}_J(G),\{S_t\}_{t\in G})$ is $G_{\delta\sigma\delta}$.

Proof. Let us write in short F_0 for $F_0(\mathcal{C}_J(G))$ and Q for $Q(\mathcal{C}_J(G),\{S_t\}_{t\in G})$. According to Lemma 5, $Q=Q_1\cap Q_2$, where

$$Q_1=\bigcap_{f\in F_0}\{\tilde{x}\in\mathcal{C}_J(G):\liminf_k M(f,\tilde{x},k)=\limsup_k M(f,\tilde{x},k)\},$$

$$Q_2=\{\tilde{x}\in\mathcal{C}_J(G):\{m_{\tilde{x}}^k\}_{k=1}^{\infty}\text{ is relative compact}\}.$$

Since for any $f\in\mathcal{C}(\mathcal{C}_J(G))$ and $k\in\mathbf{N}$ also the function $M(f,\tilde{x},k)$ belongs in $\mathcal{C}(\mathcal{C}_J(G))$ ($\lim_m \int_G f(S_t\tilde{x}_n)d\gamma_k(t)=\int_G f(S_t\tilde{x})d\gamma_k(t)$ whenever $\lim_m \tilde{x}_n=\tilde{x}$), the set Q_1 is $F_{\sigma\delta}$ ([3], p. 274). Further, making use of Lemma 3,

$$Q_2=\bigcap_{n_0=1}^{\infty}\bigcap_{j=1}^{\infty}\bigcap_{\ell=1}^{\infty}\bigcup_{p=1}^{\infty}\bigcap_{k=1}^{\infty}\{\tilde{x}\in\mathcal{C}_J(G):m_{\tilde{x}}^k(\{\tilde{y}\in\mathcal{C}_J(G):w_{\tilde{y}}(\bigcup_{n=1}^{n_0}B_n,\tfrac{1}{p})\geq\tfrac{1}{j}\})\leq\tfrac{1}{\ell}\}.$$

Considering the continuity of the function (of $\tilde{y}$) $w_{\tilde{y}}(\bigcup_{n=1}^{n_0}B_n,\frac{1}{p})$, the set

$\{\tilde{y} \in \mathscr{C}_J(G): w_{\tilde{y}}(\bigcup_{m=1}^{m_0} B_n, \frac{1}{p}) \geq \frac{1}{j}\}$ is closed, hence the function (of $\tilde{x}$) $m_{\tilde{x}}^k(\{\tilde{y} \in \mathscr{C}_J(G): w_{\tilde{y}}(\bigcup_{m=1}^{m_0} B_n, \frac{1}{p}) \geq \frac{1}{j}\})$ is of the first Baire class. Thus Q_2 (and also Q) is $G_{\delta\sigma\delta}$.

Lemma 7. $Q(\mathscr{C}_J(G), \{S_t\}_{t\in G})$ has invariant measure one.

Proof. We shall show that both sets Q_1 and Q_2 (cf. the proof of Lemma 6) have invariant measure one. For Q_1 this assertion follows immediately from Proposition 1. Concerning Q_2, it holds by Prochorov's theorem that $Q_2 = \bigcap_{j=1}^{\infty} \bigcup_{K\in\mathcal{K}} \bigcap_{k=1}^{\infty} \{\tilde{x} \in \mathscr{C}_J(G): m_{\tilde{x}}^k(K) > 1 - \frac{1}{j}\}$, where $\mathcal{K} = \mathcal{K}(\mathscr{C}_J(G))$. Let $\mu \in \mathcal{M}_{st}(\mathscr{C}_J(G), \{S_t\}_{t\in G})$ and let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers which tend to 0. There is a sequence $\{K_n\}_{n=1}^{\infty}$ in $\mathcal{K}$ such that, for any $n \in \mathbf{N}$, $\mu(K_n) > 1 - \varepsilon_n^2$ and, consequently, by Proposition 1, $\mu(A_n) \geq 1 - \varepsilon_n$ where $A_n = \{\tilde{x} \in \mathscr{C}_J(G): \chi_{K_n}^*(\tilde{x}) > 1 - \varepsilon_n\}$. Together, $\mu(\bigcap_{m_0=1}^{\infty} \bigcup_{m=m_0}^{\infty} A_n) = 1$. For any fixed $j \in \mathbf{N}$ and $\tilde{y} \in \bigcap_{m_0=1}^{\infty} \bigcup_{m=m_0}^{\infty} A_n$ there is $n \in \mathbf{N}$ such that $\chi_{K_n}^*(\tilde{y}) > 1 - \frac{1}{j}$. Therefore $\chi_{K_n}^{(k)}(\tilde{y}) > 1 - \frac{1}{j}$ for all sufficiently large $k \in \mathbf{N}$, and hence $m_{\tilde{y}}^k(K) = \chi_K^{(k)}(\tilde{y}) > 1 - \frac{1}{j}$ for all $k \in \mathbf{N}$ for some $K \in \mathcal{K}$. Thus $\bigcap_{m_0=1}^{\infty} \bigcup_{m=m_0}^{\infty} A_n \subset Q_2$.

Lemma 8. For any Borel measurable real function f on $\mathscr{C}_J(G)$, $\int f dm_{\tilde{x}}$ is a Borel measurable function of $\tilde{x}$ on $Q = Q(\mathscr{C}_J(G), \{S_t\}_{t\in G})$, and

(6) $$\int f d\mu = \int_Q (\int f dm_{\tilde{x}}) d\mu(\tilde{x})$$

for every $\mu \in \mathcal{M}_{st}(\mathscr{C}_J(G), \{S_t\}_{t\in G})$.

Proof. According to Definition 1 and Lemma 4, $\int f dm_{\tilde{x}} = f^*(\tilde{x})$ for any $\tilde{x} \in Q$ and $f \in \mathscr{C}(\mathscr{C}_J(G))$. Further, $\int f d\mu = \int f^* d\mu = \int_Q f^* d\mu$ for any $\mu \in \mathcal{M}_{st}(\mathscr{C}_J(G), \{S_t\}_{t\in G})$, i.e. such a function f fulfils (6). By the theorem on monotonous convergence, (6) is true also for characteristic functions of closed sets, which enables us to deduce the desired assertion for all characteristic functions of Borel sets, and hence for all Borel measurable functions.

6. About the relation $\int f dm_\omega = M(f, \omega)$ more generally. In this section let $(\Omega, \mathcal{B}_\Omega, \{T_t\}_{t\in G})$ be a continuous flow on a Polish space. For any $\omega \in \Omega$ let $O^+(\omega) = \bigcup_{k=1}^{\infty} \{T_t\omega : t \in C_k\}$ (cf. § 1).

Lemma 9. Let $K_0 \in \mathcal{K}(\Omega)$ and $\omega \in Q(\Omega, \{T_t\}_{t\in G})$. Then there is $K \in \mathcal{K}(\Omega)$ such that $K_0 \subset K \subset K_0 \cup 0^+(\omega)$ and

(7) $$\liminf_{k} M(\chi_K, \omega, k) \geq m_\omega(K_0).$$

Proof. For any $n \in \mathbf{N}$ let us define $U_n = \{z \in \Omega : \mathrm{dist}(z, K_0) < \frac{1}{n}\}$, and take $f_n \in \mathcal{C}(\Omega)$ such that $\chi_{K_0} \leq f_n \leq \chi_{U_n}$. As $\lim_n M(f_n, \omega) = m_\omega(K_0)$, for an increasing sequence $\{k_n\}_{n=1}^{\infty}$ it holds that, for any n and $k \geq k_n$, $M(f_n, \omega, k) > m_\omega(K_0) - \frac{1}{n}$. For the sake of simplicity let us suppose that all the sets C_k are compact (the general case will be considered below). Using a continuous mapping $\varphi_\omega(t) = T_t\omega$, $t \in G$, of G into Ω, let us set

(8) $$K = K_0 \cup \bigcup_{n=1}^{\infty} (\overline{U}_n \cap \varphi_\omega C_{k_{n+1}}).$$

Every open cover $\{V_\alpha\}$ of K allows us to find sets $V_{\alpha_1}, \dots, V_{\alpha_i}$ such that, for some $n \in \mathbf{N}$, $U_n \subset \bigcup_{j=1}^{i} V_{\alpha_j}$. Taking into account that $K \setminus U_n = \bigcup_{j=1}^{n} (\overline{U}_j \cap \varphi_\omega C_{k_{j+1}} \cap U_n^c)$ and the sets $\varphi_\omega C_{k_{j+1}}$ being compact, we can see that $K \setminus U_n$ is compact, too. Hence there is a finite subcover $\{V_{\alpha_1}, \dots, V_{\alpha_i}, V_{\alpha_{i+1}}, \dots, V_{\alpha_p}\}$ of K, i.e. K is compact. It follows from (8) that, for any n, $n' \geq n$ and $k \in \{k_{n'}, k_{n'}+1, \dots, k_{n'+1}\}$, $m_\omega(K_0) - \frac{1}{n} \leq m_\omega(K_0) - \frac{1}{n'} < M(f_{n'}, \omega, k) \leq \frac{1}{\lambda(C_k)} \int_{C_k} \chi_K(T_t\omega)\, d\lambda(t)$. So $m_\omega(K_0) - \frac{1}{n} < M(\chi_K, \omega, k)$ for each $k \geq k_n$, which implies (7). In general case we shall take (according to (i3)) some compact set $C_{k,j(k)}$ ($k \in \mathbf{N}$) satisfying $\frac{\lambda(C_k \setminus C_{k,j(k)})}{\lambda(C_k)} < 1/k$, instead of C_k.

Lemma 10. Let f be Borel measurable bounded real function on Ω and $\omega \in Q(\Omega, \{T_t\}_{t\in G})$. If for each $\varepsilon > 0$ there is a set $E_\varepsilon \in \mathcal{B}_\Omega$ such that the contraction of f to E_ε is continuous, $0^+(\omega) \subset E_\varepsilon$ and $m_\omega(E_\varepsilon) > 1 - \varepsilon$, then $\int f\, dm_\omega = M(f, \omega)$.

Proof. Let $\sup_{z\in\Omega} |f(z)| = \alpha$ and let $\varepsilon > 0$. From the regularity of the measure it results that there is a compact set $K_0 \subset E_\varepsilon$ such that $m_\omega(K_0) > 1 - \varepsilon$. According to Lemma 9 we can take a compact set K, $K_0 \subset K \subset K_0 \cup 0^+(\omega)$, for which (7) holds. Since the contraction of f to K is continuous, there is $g \in \mathcal{C}(\Omega)$ such that $\sup_{z\in\Omega} |g(z)| = \alpha$ and f=g on K. Now, for every $k \in \mathbf{N}$ we have $|\int f\, dm_\omega - M(f, \omega, k)| \leq |\int f\, dm_\omega - \int g\, dm_\omega| + |\int g\, dm_\omega - M(g, \omega, k)| + |M(g, \omega, k) - M(f, \omega, k)| \leq$

$\leq 2\alpha m_\omega(\Omega\setminus K)+|M(g,\omega)-M(g,\omega,k)|+2\alpha(1-M(\chi_k,\omega,k))$, thus
$\limsup_k |\int f dm_\omega - M(f,\omega,k)| < 4\alpha\varepsilon$.

Lemma 11. Let f be Borel measurable bounded real function on Ω and $\omega\in Q(\Omega,\{T_t\}_{t\in\Omega})$. If there is a $\{T_t\}_{t\in G}$-invariant set $E\in\mathcal{B}_\Omega$ containing ω such that the contraction of f to E is continuous and $m_\omega(E)=1$, then $\int f dm_\omega = M(f,\omega)$.

Proof. This lemma is a consequence of the previous assertion.

7. Relations between flows. Two continuous flows $(\Omega,\mathcal{B}_\Omega,\{T_t\}_{t\in G})$ and $(\Omega',\mathcal{B}_{\Omega'},\{T_t'\}_{t\in G})$ on metric spaces are called homeomorphic if there is a homeomorphism Φ of Ω onto Ω' such that $T_t=\Phi^{-1}T_t'\Phi$ for each $t\in G$.

Lemma 12. The sets $Q=Q(\Omega,\{T_t\}_{t\in G})$ and $Q'=Q(\Omega',\{T_t'\}_{t\in G})$ of quasi-regular points of two homeomorphic (under $\Phi:\Omega\to\Omega'$) continuous flows satisfy the relation $Q=\Phi^{-1}Q'$.

Proof. Clearly, $m_\omega^k\Phi^{-1}=m_{\Phi\omega}^k$ for any $\omega\in\Omega$, $k\in\mathbf{N}$. Since Φ is continuous, the weak* convergence of an arbitrary sequence $\{\mu_k\}_{k=1}^\infty$ in $\mathcal{M}(\Omega)$ implies the weak* convergence of $\{\mu_k\Phi^{-1}\}_{k=1}^\infty$ in $\mathcal{M}(\Omega')$, hence $\Phi Q\subset Q'$. The converse follows from the symmetry.

We can see that a continuous flow $(\tilde\Omega,\mathcal{B}_{\tilde\Omega},\{\tilde T_t\}_{t\in G})$ (on a metric space) induces a new one on every nonempty $\{\tilde T_t\}_{t\in G}$-invariant Borel subset $\Omega\subset\tilde\Omega$. This flow $(\Omega,\mathcal{B}_\Omega,\{T_t\}_{t\in G})$ ($\mathcal{B}_\Omega=\Omega\cap\mathcal{B}_{\tilde\Omega}$, $T_t=\tilde T_t\wedge\Omega$, $t\in G$) will be called a subsystem of $(\tilde\Omega,\mathcal{B}_{\tilde\Omega},\{\tilde T_t\}_{t\in G})$.

Lemma 13. Let $(\Omega,\mathcal{B}_\Omega,\{T_t\}_{t\in G})$ be a subsystem of a continuous flow $(\tilde\Omega,\mathcal{B}_{\tilde\Omega},\{\tilde T_t\}_{t\in G})$, Then $Q(\Omega,\{T_t\}_{t\in G})=\Omega\cap\{\omega\in Q(\tilde\Omega,\{\tilde T_t\}_{t\in G}):m_\omega(\Omega)=1\}$.

Proof. Let us write $\tilde\mu$ instead of μ whenever $\mu\in\mathcal{M}(\tilde\Omega)$, and let $Q=Q(\Omega,\{T_t\}_{t\in G})$, $\tilde Q=Q(\tilde\Omega,\{\tilde T_t\}_{t\in G})$. If $\omega\in Q$ then the limit $M(f,\omega)$ exists for any $f\in F(\tilde\Omega)$ because $f\wedge\Omega\in F(\Omega)$. Further, the convergence in $\mathcal{M}(\Omega)$ $m_\omega^k\to m_\omega$, $k\to\infty$, implies that $(m_\omega^k)^\sim\to\tilde m_\omega$, $k\to\infty$ (in $\mathcal{M}(\tilde\Omega)$), where we set $(m_\omega^k)^\sim(E)=m_\omega^k(\Omega\cap E)$ and $\tilde m_\omega(E)=m_\omega(\Omega\cap E)$ for $k\in\mathbf{N}$, $E\in\mathcal{B}_{\tilde\Omega}$. Thus $\omega\in\Omega\cap\tilde Q$ and $\tilde m_\omega(\Omega)=1$.

On the other hand, let us suppose that $\omega\in\Omega\cap\tilde Q$ and $\tilde m_\omega(\Omega)=1$. If $f\in F(\Omega)$, let us define a real function g on $\tilde\Omega$ in such a way that g=f on Ω

and g=0 on $\tilde{\Omega}\setminus\Omega$. According to Lemma 11, $\int g d\tilde{m}_\omega = M(g,\omega) = M(f,\omega)$, i.e. the limit $M(f,\omega)$ exists. As $m_\omega^k \to \mu$, $k\to\infty$ (in $\mathcal{M}(\Omega)$), where $\mu = \tilde{m}_\omega \restriction \mathcal{B}_\Omega$, we conclude that $\omega\in Q$.

Lemma 14. Every continuous flow $(\Omega, \mathcal{B}_\Omega, \{T_t\}_{t\in G})$ on a Polish space is homeomorphic to a subsystem of the shift-flow $(\mathcal{C}_J(G), \mathcal{B}_\mathcal{C}, \{S_t\}_{t\in G})$.

Proof. By Urysohn's theorem there is a homeomorphism Ψ of Ω onto a G_δ-subset $\Psi\Omega$ of the Hilbert cube J. Let us define $(\Phi\omega)(t) = \Psi(T_t\omega)$ for any $t\in G$, $\omega\in\Omega$; Φ is a 1:1 mapping of Ω into $\mathcal{C}_J(G)$. Obviously, its range $\Phi\Omega$ is $\{S_t\}_{t\in G}$-invariant and $T_t = \Phi^{-1}S_t\Phi$ for any $t\in G$. We have to show that Φ is bicontinuous.

In order to prove the continuity of Φ let us suppose that $\omega\in\Omega$, $B\in\mathcal{K}(G)$ and $\varepsilon>0$; we have to find $\delta>0$ such that $\Phi(U^\delta(\omega))\subset U(\Phi\omega;B,\varepsilon)$. To this end let us define a real function Δ on G as follows:

$$\Delta(t) = \sup\{\delta>0: \forall t'\in G\ \ \forall z\in\Omega\ (\rho_G(t',e)<\delta \ \&\ \rho_\Omega(T_t\omega,z)<\delta) \Rightarrow$$
$$\Rightarrow \rho_\Omega(T_t\omega, T_{t'}z)<\varepsilon/2\}.$$

According to the definition of a flow, Δ is continuous and positive on G and has a minimum $d>0$ on B. Making use of the group operation continuity, we find a finite set $\{t_1,\dots,t_j\}\subset B$ such that $B\subset\bigcup_{i=1}^{j} a_i(U^d(e))$, where we set $a_i(t)=t+t_i$, $t\in G$. Further, $U^\delta(\omega)\subset\bigcap_{i=1}^{j} T_{-t_i}(U^d(T_{t_i}\omega))$ for some $\delta>0$. If now $t\in B$, then $t\in a_i(U^d(e))$ for some i and, as prescribed by Δ,
$\rho_\Omega(T_{t_i}\omega, T_tz) = \rho_\Omega(T_{t_i}\omega, T_{t-t_i}(T_{t_i}z)) < \varepsilon/2$ for each $z\in U^\delta(\omega)$. Thus
$\rho_\Omega(T_t\omega, T_tz) \le \rho_\Omega(T_t\omega, T_{t_i}\omega) + \rho_\Omega(T_{t_i}\omega, T_tz) < \varepsilon$, i.e. $\Phi(U^\delta(\omega))\subset U(\Phi\omega;B,\varepsilon)$. Let us note that $\Phi\Omega\in\mathcal{B}_\mathcal{C}$ since it is a 1:1 continuous image of a Polish space (see [3], p. 397).

To show that the mapping Φ^{-1} is continuous, too, it suffices to apprehend that every set $\Phi(U^\varepsilon(\omega)) = \Phi\Omega\cap\{\tilde{x}\in\mathcal{C}_J(G): \tilde{x}(e)\in\Psi(U^\varepsilon(\omega))\}$ ($\varepsilon>0$, $\omega\in\Omega$) is open relative to $\Phi\Omega$.

8. The Borel measurability

Theorem 1. Let $(\Omega, \mathcal{B}_\Omega, \{T_t\}_{t\in G})$ be a continuous flow on a Polish space. Then the set of quasi-regular points is Borel $G_{\delta\sigma\delta}$ of invariant measure one.

Proof. Following Lemma 14, $(\Omega, \mathcal{B}_\Omega, \{T_t\}_{t\in G})$ is homeomorphic to a subsystem of $(\mathcal{C}_J(G), \mathcal{B}_\mathcal{C}, \{S_t\}_{t\in G})$, the corresponding homeomorphism being denoted by Φ ($\Phi: \Omega \longrightarrow \mathscr{C}_J(G)$). As it is known, $\Phi\Omega$ is G_δ ([3], p. 337). Let us write $Q=Q(\Omega, \{T_t\}_{t\in G})$ and $Q_{\mathscr{C}}=Q(\mathscr{C}_J(G), \{S_t\}_{t\in G})$. According to Lemma 12 and Lemma 13, $Q=\Phi^{-1}(\Phi\Omega \cap \{\tilde{x} \in Q_{\mathscr{C}}: m_{\tilde{x}}(\Phi\Omega)=1\})$. The Borel measurability of Q has already been a consequence of Lemma 8. But, in another way, the function $m_{\tilde{x}}(E)$ is of the 3^{rd} Baire class for any Borel G_δ subset $E \subset \mathscr{C}_J(G)$. Hence $\Phi\Omega \cap \{\tilde{x} \in Q_{\mathscr{C}}: m_{\tilde{x}}(\Phi\Omega)=1\}$ is $G_{\delta\sigma\delta}$ relative to $\Phi\Omega \cap Q_{\mathscr{C}}$, and so it is $G_{\delta\sigma\delta}$ by Lemma 6. Making use of the continuity of Φ^{-1} we show that Q is $G_{\delta\sigma\delta}$, too.

Provided $\mu \in \mathcal{M}_{st}(\Omega, \{T_t\}_{t\in G})$, we apply Lemma 7 to $\mu\Phi^{-1}$ to conclude that Q has invariant measure one.

We always assume $(\Omega, \mathcal{B}_\Omega, \{T_t\}_{t\in G})$ to be a continuous flow on a Polish space in the remaining part.

Lemma 15. $\int f dm_\omega$ is Borel measurable function of ω on $Q=Q(\Omega, \{T_t\}_{t\in G})$ for any real Borel measurable function f on Ω, and $\int f d\mu = \int_Q (\int f dm_\omega) d\mu(\omega)$ for any $\mu \in \mathcal{M}_{st}(\Omega, \{T_t\}_{t\in G})$.

Proof is the same as in Lemma 8.

Definition 2. A quasi-regular point $\omega \in \Omega$ is called a point of density ($\omega \in Q_D=Q_D(\Omega, \{T_t\}_{t\in G})$) if $m_\omega(U)>0$ for every open set U containing ω. A quasi-regular point ω is called transitive ($\omega \in Q_T=Q_T(\Omega, \{T_t\}_{t\in G})$) if m_ω is an ergodic measure. A point ω is called <u>regular</u> if $\omega \in R=R(\Omega, \{T_t\}_{t\in G}) = Q_D \cap Q_T$.

Lemma 16. All the sets Q_D, Q_T and R are Borel $G_{\delta\sigma\delta}$ of invariant measure one.

Proof. These facts follow in the same way as in [4]. We show that concerning Q_T only. A measure m_ω ($\omega \in Q$) is ergodic iff $m_\omega(\{z \in \Omega: M(f,z) = \int f dm_\omega\})=1$ for any $f \in F_0(\Omega)$, i.e. $Q_T = \bigcap_{f\in F_0(\Omega)} \{\omega \in Q: g_f(\omega)=0\}$, where we set $g_f(\omega) = \int (f^*(z) - \int f dm_\omega)^2 dm_\omega(z)$, $f \in F(\Omega)$. It holds that, for any $f \in F(\Omega)$,

$$g_f(\omega) = \lim_k \int_Q (f^{(k)}(z) - f^*(\omega))^2 dm_\omega(z) = \lim_k \lim_j \int_Q (f^{(k)}(z) - f^*(\omega))^2 dm_\omega^j(z) =$$

$$= \lim_k \lim_j \int_G (f^{(k)}(T_t\omega) - f^*(\omega))^2 d\gamma_j(t) = \lim_k \lim_j \lim_i \int_G (f^{(k)}(T_t\omega) - f^{(i)}(\omega))^2 d\gamma_j(t),$$

so that the function $g_f(\omega)$ is of the 3^{rd} Baire class. Hence, applying Theorem 1, Q_T is $G_{\delta\sigma\delta}$. Moreover, it has invariant measure one because

$g_f(\omega)=\lim_k \lim_j \int_G (f^{(k)}(T_t\omega)-f^*(T_t\omega))^2 d\gamma_j(t)=\lim_k \lim_j M((f^{(k)}-f^*)^2, \omega, j)=0$ μ-a.e. for any $\mu \in \mathcal{M}_{st}(\Omega, \{T_t\}_{t\in G})$ by Lemma 1, whenever $f\in F(\Omega)$.

Theorem 2. Let $(\Omega, \mathcal{B}_\Omega, \{T_t\}_{t\in G})$ be a continuous flow on a Polish space. Then the set R of regular points is Borel $G_{\delta\sigma\delta}$ of invariant measure one. Further, $m_\omega(E)$ is Borel measurable function of ω on R for any set $E\in \mathcal{B}_\Omega$, and

$$\mu(E)=\int_R m_\omega(E)d\mu(\omega)$$

for any $\mu \in \mathcal{M}_{st}(\Omega, \{T_t\}_{t\in G})$.

Proof. The assertion is a consequence of Lemma 15 and Lemma 16.

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(Oblatum 1.9. 1987)