Ľubica Šedová

Parabolic equations with deviating argument in advance

Commentationes Mathematicae Universitatis Carolinae, Vol. 28 (1987), No. 4, 613--628

Persistent URL: http://dml.cz/dmlcz/106575

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
28,4(1987)

PARABOLIC EQUATIONS WITH DEVIATING ARGUMENT IN ADVANCE

Ľubica ŠEDOVÁ

Abstract: The phase space for the equation $\frac{du}{dt}+Au=Lu_{t+b}$ is decomposed into the sum $Y=Y_1+Y_2$ of two $T(t)$-invariant subspaces, where $T(t)$ is the corresponding semigroup of solutions, L is a linear operator and u_{t+b} denotes the deviation of u in advance. Also a nonlinear generalization $\frac{du}{dt}+Au=f(t,u_{\omega(t)})$ of the above problem is treated.

Key words: Functional differential equations, parabolic equations with delay in advance, sectorial operator, ordered Banach space, stability.

Classification: 35R10, 34K30

---

§ 1. Introduction and results

The paper extends the results of J. Milota [Mi] to the problems with deviating argument in advance. A decomposition of the phase space $Y=Y_1+Y_2$ into two $T(t)$-invariant subspaces is established. This decomposition can be applied in investigation of the asymptotic stability. Some ideas of [Mi] have been used. The results obtained in the linear problem are similar to those by [Mi] provided the norm of L is small.

Solving the nonlinear parabolic functional differential equation, the existence and uniqueness of the solution have been proved (see Theorem 5). A result of [Š] has been extended from functional ODE to the parabolic problems.

Moreover, under the monotonicity assumptions on f, restrictions concerning the growth assumptions can be removed. In that case we apply some techniques from the theory of partially ordered Banach spaces.

In the paper some examples are given which illustrate the special behaviour of solutions to the problem with advanced argument.

The theory of semigroups, the original Banach space X, the power spaces X^{α} and the estimates from [Mi] and [He] are used.

§ 2. Assumptions and denotations

Suppose that X is a Banach space and A is a sectorial operator in X (for the definition see [He] or [Mi]). Let Re $\sigma(A) > 0$ and let $\alpha \in (0,1)$. Denote X^{α} the fractional Banach space of X, following [He] terminology.

Let $\gamma > 0$. Let $Y = \{ y \in C((-\infty,0\rangle, X^{\alpha});\ \sup_{\theta \in (-\infty,0\rangle} e^{\gamma\theta} |y(\theta)|_{\alpha} < \infty$ and $e^{\gamma\theta} y(\theta)$ is a uniformly continuous function from $(-\infty,0\rangle$ to $X^{\alpha}\}$. Clearly, Y is a Banach space with the norm $\|y\|_{Y} = \sup e^{\gamma\theta}|y(\theta)|_{\alpha}$. Let $b > 0$ and let $L: Y \longrightarrow X$ be a continuous linear operator. Further we assume that there exists a constant $C_{\alpha} > 0$ such that the following estimates hold for each $t \in (0,\infty)$:

(A1) $|A^{\alpha}e^{-At}| \leq C_{\alpha} t^{-\alpha} e^{-at}$

(A2) $\|L\| C_{\alpha} \int_0^t (t-s)^{-\alpha} e^{-a(t-s)} ds \leq q$, where $0 \leq q < 1$ and $\|L\|$ is the norm of the operator L.

The first assumption takes always place (see [He]).
Let $Z = \{ u \in C(\langle 0,\infty), X^{\alpha});\ \sup_{t \in \langle 0,\infty)} |u(t)|_{\alpha} < \infty \}$ and let $|||u||| = \sup_{t \in \langle 0,\infty)} |u(t)|_{\alpha}$ for each $u \in Z$. $(Z, |||\cdot|||)$ is a Banach space. If $u \in C((-\infty,\infty), X^{\alpha})$, we denote $u_t(\theta) = u(t+\theta)$ for each $\theta \in (-\infty,\infty)$.

Definition of a mild solution. Let $h \in Y$ be an initial function. We consider the equation

(E) $\frac{du}{dt} + Au = L(u_{t+b})$ (the equation with the deviation)

together with the initial condition

(C_0) $u_0 = h$ on the interval $(-\infty,0)$.

Any solution u in the space $C((-\infty,\infty), X^{\alpha}) \cap Z$ of the integral equation

(E1) $u(t) = e^{-At}h(0) + \int_0^t e^{-A(t-s)} L(u_{s+b}) ds$,

which satisfies the initial condition (C_0), is said to be a mild solution to the initial problem (E,C_0).

§ 3.

Existence theorem 1. Let $h \in Y$. Then under the assumptions given above there exists a unique mild solution of the problem (E,C_0) in the class of

$|\cdot|_\alpha$-bounded continuous functions.

Proof. Let $S_1 = \{x \in Z;\ x$ is a continuous extension of the function $h\}$. Clearly, S_1 is a closed subset of the Banach space Z. We shall consider the operator $T: S_1 \to S_1$ which is defined for each $t \geq 0$, by $Tx(t) = e^{-At}h(0) + \int_0^t e^{-A(t-s)} L(x_{s+b})ds$.

Since $|Tx(t)|_\alpha \leq Ce^{-at}|h(0)|_\alpha + C_\alpha \int_0^t e^{-a(t-s)}(t-s)^{-\alpha}\|L\|\,\|x_{s+b}\|ds \leq C_\alpha |h|_Y + C_\alpha \|L\| \int_0^t e^{-a(t-s)}(t-s)^{-\alpha}ds(\|h\|_Y + |||x|||)$, T maps S_1 into S_1. Further

$$|Tx_1(t) - Tx_2(t)|_\alpha = \left|\int_0^t e^{-A(t-s)}L(x_{1,b+s} - x_{2,b+s})ds\right|_\alpha \leq$$
$$\leq \|L\| C_\alpha \int_0^t e^{-a(t-s)}(t-s)^{-\alpha}ds\ |||x_1 - x_2||| \leq q\,|||x_1 - x_2|||\ .$$

Hence $|||T(x_1 - x_2)||| \leq q\,|||x_1 - x_2|||$ and by the Banach fixed point theorem there exists a unique solution of (E1) in S_1.

The question arises, when the mild solution is a strong one.

Theorem 2. Let $h(0)$ be an element of $X^{\alpha+\varepsilon}$ for some $\varepsilon > 0$ and let $e^{\gamma\Theta}h(\Theta)$ be Hölder continuous in $(-\infty, 0\rangle$. Then the mild solution of (E, C_0) is a strong one.

Proof. We shall apply Th. 3.2.2 in [He]. Put $f(s) = Lx_{s+b}$. Clearly, $\int_0^\rho |f(s)|_X ds < \infty$ for each $\rho > 0$, because $f: \langle 0, \infty) \to X$ is a continuous function. Further, we have to show that f is locally Hölder continuous. This follows from the following two statements:

(i) $e^{\gamma\Theta}h(\Theta)$ is a Hölder continuous function;

(ii) $x(t)$ as a mild solution is a Hölder continuous function in $\langle 0, \infty)$.

We have

$$|x(t_1+k) - x(t_1)|_\alpha = |(e^{-Ak} - I)e^{-At_1}h(0)|_\alpha + \left|\int_0^t (e^{-Ak} - I)\ .\right.$$
$$\left.\cdot e^{-A(t_1-s)}Lx_{s+b}ds\right|_\alpha + \left|\int_{t_1}^{t_1+k} e^{-A(t_1+k-s)}Lx_{s+b}ds\right|_\alpha \leq Ck^\varepsilon e^{-at_1}|h(0)|_{\alpha+\varepsilon} +$$
$$+ C(\varepsilon)k^\varepsilon C_{\alpha+\varepsilon}\int_0^t (t_1-s)^{-\alpha-\varepsilon}e^{-a(t_1-s)}ds\|L\|(\|h\|_Y + |||x|||) +$$
$$+ \|L\|(\|h\|_Y + |||x|||)C_\alpha \int_{t_1}^{t_1+k} e^{-a(t_1+k-s)}(t_1+k-s)^{-\alpha}ds \leq Kk^\varepsilon + K_1(t_1)k^{1-\alpha},$$ so that (ii) takes place.

$h(\Theta)$ is a locally Hölder continuous function on $(-\infty, 0\rangle$ because $e^{\gamma\Theta}h(\Theta)$ is Hölder continuous. This also implies that the map $s \to x_{s+b}$ as a map from

$\langle 0,\infty)$ into Y is locally Hölder continuous and thus $f(s)=Lx_{s+b}$ is locally Hölder continuous, too.

Apriori estimates for the mild solutions and some properties of those solutions

1) <u>Continuous dependence of solutions on the initial condition.</u> Let the solution x_1 correspond to φ_1 and x_2 to φ_2. Then we have $|||x_1-x_2||| \leq (C_1+q)\|\varphi_1-\varphi_2\|_Y+q|||x_1-x_2|||$, from where it follows

$$(1)\quad |||x_1-x_2||| \leq \frac{C_1+q}{1-q}\|\varphi_1-\varphi_2\|_Y$$

and

$$(2)\quad \|x_t\|_Y \leq \frac{C_1+1}{1-q}\|\varphi_Y\|.$$

Now we shall consider the operator $T(t):Y \longrightarrow Y$, which is defined for each $t \geq 0$ as follows:

If $u(t)$ is a mild solution of the equation (E) with the initial function $\varphi \in Y$, then $T(t)\varphi$, similarly as in [Mi] will mean $u_t \in Y$. On the basis of (2), it is clear that so defined $T(t)$ is a semigroup of the class C_0. The assumption on the uniform continuity of $e^{\gamma\Theta}\varphi(\Theta)$ is needed in the proof of $\lim_{t\to 0} T(t)\varphi = \varphi$. For the operator $L=0$ we denote this semigroup by $S(t)$, similarly as in [Mi].

2) The solution $x(t)$ is bounded on the interval $\langle \sigma,\infty)$ for each $\sigma > 0$ also in the space $X^{\alpha+\varepsilon}$ such that $\varepsilon > 0$ and $\alpha+\varepsilon < 1$, and locally Hölder continuous as a function of $s \in \langle \sigma,\infty)$ into X^{α}:

$$(3)\quad |u(s)|_{\alpha+\varepsilon} \leq |e^{-As}\varphi(0)|_{\alpha+\varepsilon} + \left|\int_0^s e^{-A(s-r)}Lu_{r+b}\,dr\right|_{\alpha+\varepsilon} \leq Ce^{-as}s^{-\varepsilon}|\varphi(0)|_{\alpha} +$$
$$+C\int_0^s e^{-a(s-r)}(s-r)^{-\alpha-\varepsilon}\|L\|(\|\varphi\|_Y+|||u|||)\,dr \leq C\sigma^{-\varepsilon}|\varphi(0)|_{\alpha} + C\int_0^s e^{-a(s-r)}(s-r)^{-\alpha-\varepsilon}\cdot$$
$$\cdot dr\|L\|\left(1+\frac{C_1+q}{1-q}\right)\|\varphi\|_Y.$$

$$(4)\quad |u(s+h)-u(s)|_{\alpha} = |(e^{-Ah}-I)e^{-As}\varphi(0)|_{\alpha} + \left|\int_0^s e^{-A(s-r)}(e^{-Ah}-I)\cdot\right.$$
$$\left.\cdot Lu_{r+b}\,dr\right|_{\alpha} + \left|\int_s^{s+h} e^{-A(s+h-r)}Lu_{r+b}\,dr\right|_{\alpha} \leq Ch^{\varepsilon}e^{-as}s^{-\varepsilon}|\varphi(0)|_{\alpha} +$$
$$+C\int_0^s e^{-a(s-r)}(s-r)^{-\alpha-\varepsilon}\,dr\,h^{\varepsilon}\|L\|\left(1+\frac{C_1+q}{1-q}\right)\|\varphi\|_Y +$$
$$+C\int_s^{s+h} e^{-(s+h-r)\alpha}(s+h-r)^{-\alpha}dr\|L\|\left(\frac{C_1+q}{1-q}\right)\|\varphi\|_Y \leq (C\sigma^{-\varepsilon}h^{\varepsilon}+C^{*}h^{\varepsilon}+C^{*}h^{1-\alpha})\|\varphi\|_Y.$$

Now we shall consider the problem, when $T(t)-S(t):Y \longrightarrow Y$, is a compact operator. A similar lemma as in [Mi] is true:

Lemma 1. If A has a compact resolvent, then the operator $T(t)-S(t):$ $:Y \to Y$ is compact for each $t>0$.

Proof. We have

(5) $((T(t)-S(t))\varphi)(\tau-t)=\int_0^{\tau} e^{-A(\tau-s)} Lu_{s+b} ds$ for each $\tau \in (0,t\rangle$

where u is a mild solution of (E) with $u_0=\varphi$ and $((T(t)-S(t))\varphi)(\tau-t)=0$ for each $\tau \in (-\infty,-0\rangle$.

Let, now, $\varphi_n \in Y$ be a sequence of bounded elements of Y. We shall show that we can extract a subsequence from this sequence such that $(T(t)-S(t))\varphi_n$ converges in Y.

By (5) it suffices to show that there exists a subsequence of the sequence of mappings $\tau \to \int_0^{\tau} e^{-A(\tau-s)} Lu_{n,s+b} ds$, which converges in the space $C(\langle 0,t),X^{\alpha})$. This will be shown as follows:

I. The sequence of mappings from $\langle 0,t\rangle$ into X^{α}

$\tau \to \int_0^{\tau} e^{-A(\tau-s)} Lu_{n,s+b} ds$ is equicontinuous.

II. For each $\tau \in \langle 0,t\rangle$ there exists a subsequence of φ_n such that $\int_0^{\tau} e^{-A(\tau-s)} Lu_{n,s+b} ds$ converges in X^{α}.

From I, II the compactness follows.

To prove I we consider

$$\left| \int_0^{\tau+h} e^{-A(\tau+h-s)} Lu_{n,s+b} ds - \int_0^{\tau} e^{-A(\tau-s)} Lu_{n,s+b} ds \right|_{\alpha} \le$$

$$\le \left| \int_0^{\tau} (e^{-Ah}-I) e^{-A(\tau-s)} Lu_{n,s+b} ds \right|_{\alpha} + \left| \int_{\tau}^{\tau+h} e^{-A(\tau+h-s)} Lu_{n,s+b} ds \right|_{\alpha},$$

from where the equicontinuity by (2) follows.

Further,

$$\left| \int_0^{\tau} e^{-A(\tau-s)} Lu_{n,s+b} ds \right|_{\alpha+\varepsilon} \le C_{\alpha+\varepsilon} \int_0^{\tau} e^{-a(\tau-s)} (\tau-s)^{-\alpha-\varepsilon} \|L\| \, \|u_{n,s+b}\|_Y ds \le$$

$$\le C_{\alpha+\varepsilon} \int_0^{\tau} e^{-a(\tau-s)} (\tau-s)^{-\alpha-\varepsilon} ds \left(\frac{C_1+q}{1-q}\right) \|\varphi_n\|_Y \le C.$$

From this inequality as well as from the compactness of the embedding $X^{\alpha+\varepsilon} \hookrightarrow X^{\alpha}$ the statement II follows.

The compactness of the operator A is a sufficient but not necessary condition (even when the right-hand side of the equation (E) is different from 0), for the operator $T(t)-S(t)$ to be compact, as the following example shows.

Example 1. Let $X=L_2(-\infty,\infty)$. Let $Au=-u_{xx}+u$ and $D(A)=W_2^2(R)$. According to [He] A is a sectorial operator, $\sigma(A)=\langle 1,\infty)$, $\alpha=1/2$ and $X^{1/2}=W_2^1(R)=X^{\alpha}$.

Hence the operator A has no compact resolvent. Let us take the equation (E), where $L:C(\langle -b+\beta,0\rangle, X^{\alpha}) \longrightarrow X$ is a linear and continuous operator, for some fixed $\beta > 0$ such that $-b+\beta < 0$. Further we suppose that there exists a sequence L_r, for $r=1,2,\ldots$, such that:

(i) each $L_r : C(\langle -b+\beta,0\rangle, W_2^{2\alpha}(-r,r)) \longrightarrow X$ is a continuous and linear operator,

(ii) $L_r - L \longrightarrow 0$ for $r \longrightarrow \infty$ in the space $[C(\langle -b+\beta,0\rangle, X^{\alpha}), X]$.

Since $W_2^{2\alpha}(-\infty,\infty)$ is continuously embedded into $W_2^{2\alpha}(-r,r)$, it is clear that $L_r : C(\langle -b+\beta,0\rangle, X^{\alpha}) \longrightarrow X$ is also linear and continuous for each $r \in \{1,2,\ldots\}$.

Hence, we consider the problem

$$\frac{du}{dt} - u_{xx} = Lu_{t+b} - u(t);\ u(0)=u_0, \text{ where } u_0 \in X^{1/2} = W_2^1(R).$$

We can extend the initial function to the interval $(-\infty,0\rangle$ as a constant function. We shall show that under (i) and (ii) $T(t)-S(t):Y \longrightarrow Y$ is a compact operator for each $t>0$.

We have to prove the statements I and II from the previous proof under the assumption that the sequence φ_n in Y is bounded.

The statement I has been already shown in the previous proof.

Now we prove the statement II: First we shall show that for each $r \in \{1,2,\ldots,\}$ there exists a subsequence of φ_n such that the corresponding $u_n(s)$ converge in the space $W_2^1(-r,r)$ for each $s \in \langle \beta, t+b\rangle$, uniformly with respect to s. Choose an arbitrary, but fixed $r>0$. On the basis (4) we have:

(6) u_n are equicontinuous as the mappings from $\langle \beta, t+b\rangle$ into $X^{\alpha} \hookrightarrow W_2^1(-r,r)$.

Further, (3) gives us that $|u_n(s)|_{\alpha+\varepsilon} \le K \|\varphi_n\|_Y$ and so $|u_n(s)|_{W_2^{2\alpha+2\varepsilon}} \le C$ for each $s \in \langle \beta, t+b\rangle$. Hence

(7) $|u_n(s)|_{W_2^{2\alpha+2\varepsilon}(-r,r)} \le C$ for each $s \in \langle \beta, t+b\rangle$.

As $W_2^{2\alpha+2\varepsilon}(-r,r) \hookrightarrow\hookrightarrow W_2^1(-r,r)$, from (6) and (7) the existence of a subsequence u_n uniformly converging in the space $W_2^1(-r,r)$ with respect to s follows.

Now, step by step, we put $r=1,2,\ldots$ and we construct a subsequence of φ_n such that the corresponding u_n will converge uniformly with respect to $s \in \langle \beta, t+b\rangle$ in the space $W_2^1(-r,r)$ for every $r>0$. (This does not imply the convergence in $W_2^1(-\infty,\infty)$.)

Let $\tau > 0$, $0 < \tau < t$, be arbitrary and let φ_n^* be such a subsequence constructed above that the corresponding subsequence u_n converges in $W_2^1(-r,r)$ for each $r\in\{1,2,\dots\}$. Then

$$|(S(t)-T(t))(\varphi_n-\varphi_m)(\tau-t)|_\alpha \leq |\int_0^\tau e^{-A(\tau-s)}L_r(u_{n,s+b}-u_{m,s+b})ds|_\alpha +$$

$$+|\int_0^\tau e^{-A(\tau-s)}(L_r-L)(u_{n,s+b}-u_{m,s+b})ds|_\alpha \leq$$

$$\leq \sup \|L_r\| \int_0^\tau C_\alpha e^{-a(\tau-s)}(\tau-s)^{-\alpha}ds \|u_{n,s+b}-u_{m,s+b}\|_{C(\langle -b+\beta,0\rangle, W_2^{2\alpha}(-r,r))} +$$

$$+ \|L_r-L\| C_\alpha \int_0^\tau e^{-a(\tau-s)}(\tau-s)^{-\alpha}(\|u_{n,s+b}\|+\|u_{m,s+b}\|) \leq$$

$$\leq C_1 \sup\|L_r\| \|u_n-u_m\|_{C(\langle\beta,t+b\rangle, W_2^{2\alpha}(-r,r))} + \|L_r-L\|C_2,$$

where we have used (2).

Then to each $\varepsilon>0$ there exists an r_0 such that $\|L_{r_0}-L\| < \frac{\varepsilon}{2C_2}$. To this r_0 there is an n_0 such that for each $n,m>n_0$:

$|u_n-u_m|_{(C(\langle\beta,t+b\rangle, W_2^{2\alpha}(-r_0,r_0))} < \frac{\varepsilon}{C_1 \sup\|L_r\|.2}$ so that

$|(S(t)-T(t))(\varphi_n-\varphi_m)(\tau-t)|_\alpha < \varepsilon$ for each $n,m>n_0$. Hence $S(t)-T(t): Y \to Y$ is certainly a compact operator.

An example of an operator L satisfying the conditions (i), (ii) above, is the operator $[Lu](x)=a(x)u_x(0,x)$, where $a(x)$ is a continuous function defined on $(-\infty,\infty)$ such that $\lim\limits_{\substack{x\to\infty\\ x\to-\infty}} a(x)=0$. Then we can take $(L_r u)(x)= =a_r(x)u_x(0,x)$, where a_r is a sequence of continuous functions such that $\operatorname{supp} a_r \subset (-r,r)$ and $a_r \rightrightarrows a$ on $(-\infty,\infty)$.

An estimate for the essential spectrum of the operator L under the assumption that $T(t)-S(t)$ is compact, is the following one:

(8) $\quad \sigma_{es}(T(t)) \leq Ce^{-\min(a,\gamma)t}$.

For the proof see [Mi], who estimated $|\sigma(S(t))| \leq Ce^{-\min(a,\gamma)t}$ and since $S(t)$ here and in [Mi] have the same meaning, we have that $\sigma_{es}(T(t))= = \sigma_{es}(S(t)+T(t)-S(t))= \sigma_{es}(S(t))$, by [4.1, Mi].

Now, we shall deal with the relation between $\sigma(B)-\sigma_{es}(B)$ and $\sigma(T(t))-\sigma_{es}(T(t))$, where B is the infinitesimal generator of the semigroup $T(t)$.

We shall extend the result of [Mi, Lemma 2] to our case.

Lemma 2.

(i) If $B\varphi = \lambda\varphi$, then $0 \geq \mathrm{Re}\,\lambda \geq -\gamma$ and $T(t)\varphi = e^{\lambda t}$ where $\varphi(\Theta) = e^{\lambda\Theta}d$ for $\Theta \leq 0$ and $d \in D(A)$ and d solves the characteristic equation

(9) $\lambda\varphi(0) + A\varphi(0) = e^{\lambda b}L(e^{\lambda\Theta}\varphi(0))$.

(ii) If $0 \geq \mathrm{Re}\,\lambda \geq -\gamma$ and (9) has a nontrivial solution, then $\lambda \in P_\sigma(B)$ (point spectrum of the operator B).

(iii) If $\mu \in P_\sigma(T,t))$ and $\mu \neq 0$, then there exists at least one λ such that $\lambda \in P_\sigma(B)$ and $e^{\lambda t} = \mu$ and there are at most finitely many such λ.

Proof.

(i) In the same way as in [Mi] we can show that $\varphi(\Theta) = e^{\lambda\Theta}\varphi(0)$ and hence $\mathrm{Re}\,\lambda \geq -\gamma$ and $[T(t)\varphi](0) = e^{\lambda t}\varphi(0)$. By the boundedness of $T(t)\varphi$ this implies that $\mathrm{Re}\,\lambda \leq 0$. Further the function $f(s) = Lu_{s+b} = L(e^{\lambda(s+b+\Theta)}\varphi(0)) = e^{\lambda(s+b)}L(e^{\lambda\Theta}\varphi(0))$ is locally Hölder continuous on the interval $\langle 0, \infty)$ into X and so $u(t) = [T(t)\varphi](0)$ is a strong solution from where $e^{\lambda t}\varphi(0) \in D(A)$ for each $t > 0$. This implies $\varphi(0) \in D(A)$ and for each $t > 0$ it is true that $\frac{du}{dz} + Au = Lu_{t+b}$.

From this it follows that $\lambda e^{\lambda t}\varphi(0) + e^{\lambda t}A\varphi(0) = L(e^{\lambda(t+b+\Theta)}\varphi(0))$. Hence $\lambda\varphi(0) + A\varphi(0) = e^{\lambda b}L(e^{\lambda\Theta}\varphi(0))$.

(ii) Analogously as in [Mi].

(iii) By [Hi, Phi], similarly as in [Mi], as the semigroups of C_0 are the semigroups of the class A, the existence of such a λ already follows. We shall show by contradiction that to each $\mu \neq 0$ there exist only finitely many $\lambda \in P_\sigma(B)$ such that $e^{\lambda t} = \mu$.

Let there exist to some $\mu \neq 0$ infinitely many such $\lambda_n \in P_\sigma(B)$ so that $e^{\lambda_n t} = \mu$. Then $\lambda_n = t^{-1}\log\mu + \frac{i2\pi k_n}{t}$ where k_n is an integer. Hence $|\lambda_n|$ converge to ∞, whereby $\mathrm{Re}\,\lambda_{n_1} = \mathrm{Re}\,\lambda_{n_2}$ for each n_1, n_2. Since A is a sectorial operator, from certain n_0 all λ_n belong to the resolvent set of the operator $-A$. Thus we can take

$$d_n = (\lambda_n I + A)^{-1}e^{\lambda_n b}L(e^{\lambda_n\Theta}d_n)$$

by (9), where d_n are the eigenvectors. We can normalize them in such a way that $\|d_n\|_\alpha = \|e^{\lambda_n\Theta}d_n\|_Y = 1$ and then

$$1 = \|d_n\|_\alpha = |A^\alpha(\lambda_n I + A)^{-1}e^{\lambda_n b}L(e^{\lambda_n\Theta}d_n)| \leq \frac{|e^{\lambda_n b}|\,\|L\|C}{|\lambda_n + a|^{1-\alpha}} = \frac{|e^{\lambda_1 b}|\,\|L\|C}{|\lambda_n + a|^{1-\alpha}},$$

where the estimate from [Mi] was used. The right-hand side goes to 0, and this gives the contradiction.

Corollary 1. Clearly, the space $N_k(\lambda,B)=\{x;(\lambda I-B)^k x=0\}$ is $T(t)$-invariant and $N_k(\lambda,B)\subset N_k(e^{\lambda t},T(t))$.

Theorem 3. Let $T(t)-S(t):Y\to Y$ be a compact operator. Then to each $\varepsilon>0$ the set $P_\sigma(B)\cap G$, where $G=\{\lambda\in C;\mathrm{Re}\,\lambda\geq -\min(a,\gamma)+\varepsilon\}$, contains finitely many points only. Moreover, all these points are of the finite multiplicity.

Proof. First we show by contradiction that the set $M=P_\sigma(B)\cap G$ is isolated. Let there exist a sequence $\lambda_n\in M$ such that λ_n converges to λ, $\lambda_n\neq\lambda_m$ for $n\neq m$. Then $e^{\lambda_n t}\in P_\sigma(T(t))$ according to Lemma 2. Thus $e^{\lambda_n t}$ converges to $e^{\lambda t}\in \sigma(T(t))$. Moreover, $e^{\lambda t}\in \sigma_{es}(T(t))$ for each $t>0$. At the same time $\mathrm{Re}\,\lambda\geq -\min(a,\gamma)+\varepsilon$ which implies that $|e^{\lambda t}|\geq e^{-\min(a,\gamma)t+\varepsilon t}$ and hence $Ce^{-\min(a,\gamma)t}\geq e^{\mathrm{re}\,\lambda t}\geq e^{-\min(a,\gamma)t+\varepsilon t}$

But this does not hold for sufficiently great t. Thus, the set M is certainly isolated in G ant it has no point of accumulation.

If there were infinitely many p s $\lambda_n\in M$, the sequence λ_n should be unbounded. Thus, since $\mathrm{Re}\,\lambda$ is bounded for $\lambda_n\in M$, the set M should have λ_n with unbounded imaginary part. Then there would exist a sequenmce d_n such that

$$1=\|d_n\|_\alpha=|e^{\lambda_n b}|\,|A^\alpha(\lambda_n I+A)^{-1}L(e^{\lambda\Theta}d_n)|\leq\frac{e^{\mathrm{Re}\,\lambda_n b}\|L\|}{|\lambda_n+a|^{1-\alpha}},$$

where the right-hand side again tends to zero. That is a contradiction.

Now, we shall show that all points of the point spectrum B in the set G are of the finite multiplicity. This follows from Corollary 1 and from the fact that the value $e^{\lambda t}$ is for sufficiently great t a normal point of the operator $T(t)$.

Corollary 2. If $S(t)-T(t):Y\to Y$ is under the assumptions above a compact operator and $\mathrm{Re}\,\lambda<0$ for each solution of the characteristic equation (9), then 0 is an asymptotically stable solution to (E) in this class of mild, $|\cdot|_\alpha$-bounded solutions of the equation (E), and

$$|T(t)|\leq Ce^{-\delta t}\text{ for some }\delta>0.$$

Proof. The proof is based on the estimate for the essential spectrum and on the fact that

$\sup\{|\lambda|;\lambda\in P_\sigma(T(t))\}=e^{\sup \operatorname{Re} P_\sigma(B)\cdot t}=e^{\Delta t}$, where $\Delta<0$.
Thus $|T(t)|\leq C(\sigma_1)e^{-\sigma_1 t}$, where $0<\sigma_1<\min\{-\Delta,\min(a,\gamma)\}$.

Corollary 3. Let $T(t)-S(t):Y\longrightarrow Y$ be a compact operator. Then the following statements are true:

Y can be decomposed into the sum $Y=Y_1+Y_2$ such that:

(i) The spectrum of $B|_{Y_1}$ contains the finitely many eigenvalues of the finite multiplicity, whereby $\operatorname{Re}\lambda=0$ for $\lambda\in\sigma(B|_{Y_1})$,

(ii) Y_1, Y_2 are T(t)-invariant.

(iii) The zero solution is asymptotically stable for $T(t)|_{Y_2}$.

(iv) $Y_1\subset D(B)$ and $B|_{Y_1}$ is a continuous linear operator generating a group which is an extension of $T(t)|_{Y_1}$ in [Mi, Cor. 2].

Proof. The proof is similar to that one of the corresponding theorem in [Mi].

Example 2. Let us take $X=L_2(0,\pi)$, $X^{1/2}=\overset{\circ}{W}{}^1_2$, $Au=-u''$, $D(A)=\overset{\circ}{W}{}^1_2\cap W^2_2$ and the problem $\frac{du}{dt}+Au=Lu_{t+b}$; $u_0=h_0$.

We shall consider the assumption A_2, that means, let us calculate $C_{1/2}$ of this operator.

Let us take $u\in\overset{\circ}{W}{}^1_2\cap W^2_2$, $u=\sum_{n=1}^{\infty}b_n\sin nx$. Then
$Ae^{-At}u=\sum_{n=1}^{\infty}n^2e^{-n^2t}b_n\sin nx$, from where we have

$$|A^{1/2}e^{-At}u|^2_{L_2}=\sum_{n=1}^{\infty}n^2e^{-2n^2t}|b_n\sin nx|^2_{L_2}.$$

Now, $\operatorname{Re}\sigma(A)>\sigma$, where $\sigma<1$. After some calculations we get that $|A^{1/2}e^{-At}u|_{L_2}\leq\left(\frac{1}{2e(1-\sigma)}\right)^{1/2}t^{-1/2}e^{-\sigma t}|u|_{L_2}$ for each $u\in D(A)$ and by the density of D(A) in X, also for all $u\in X$. Thus,

$$|L|<\frac{(2e)^{1/2}[(1-\sigma)\sigma]^{1/2}}{\Gamma(1/2)}\quad\text{for some } \sigma\in(0,1).$$

Hence,

$$|L|<\frac{(2e)^{1/2}}{\Gamma(1/2)}\cdot\frac{1}{2}=\left(\frac{e}{2\pi}\right)^{1/2}.$$

When dealing with equations with advancing argument, we meet many difficulties. Consider the following example:

Example 3.

$$\frac{du}{dt} - u'' = u(t+1),$$
$$u(t,0) = u(t,\pi) = 0,$$
$$u(0,x) = 0 \text{ for each } x \in (0,\pi).$$

Clearly, this problem has for example these two solutions:

(1) $u_1 = 0$,

(2) $u_2(t,x) = kt \sin x$.

However, under certain assumptions and in some classes, the uniqueness takes place.

§ 4.

In the present section we study a nonlinear problem with a rather general nonlinear deviation. Of course, to prove the existence of a solution it is necessary to put stronger assumptions. We shall use the following assumptions and denotations:

A is a sectorial operator with $\mathrm{Re}\,\sigma(A) > 0$.

$\omega : \langle 0,\infty) \longrightarrow R$ is a continuous function.

$h \in C((-\infty,0\rangle, X^{\alpha})$ is a uniformly continuous function.

$\psi : \langle 0,\infty) \longrightarrow (0,\infty)$ is a nondecreasing continuous function.

F will mean the space $C(\langle 0,\infty), X^{\alpha})$ with the norm $|||x||| = \sup \frac{|x(t)|_{\alpha}}{\psi(t)}$

$C = \{y \in C((-\infty,0\rangle, X^{\alpha}), y$ be a bounded function in $X^{\alpha}\}$ is a Banach space with the norm $\|y\| = \sup_{t \in (-\infty,0\rangle} |y(t)|_{\alpha}$;

$f : \langle 0,\infty) \times C \longrightarrow X$ is a continuous function.

$u_z(\Theta) = u(z+\Theta)$ for each $\Theta \leq 0$.

We shall consider a mild solution of the problem (E_2, C_0).

(E2) $\frac{du}{dt} + Au = f(t, u_{\omega(t)})$,

(C_0) $u_0 = h$,

that means, a continuous solution of the integral equation

$$u(t) = e^{-At}h(0) + \int_0^t e^{-A(t-s)} f(s, u_{\omega(s)})\,ds \text{ for each } t > 0$$

which satisfies the initial condition $u_0 = h$.

Solving (E2, C_0) we shall assume (B1 - B5) where

(B1) $\int_0^t e^{-a(t-s)}(t-s)^{-\alpha} |f(s,0)|_X\,ds \leq K\psi(t)$ for each $t \geq 0$,

(B2) $|f(t,z_1) - f(t,z_2)| \leq n(t)\,\|z_1 - z_2\|$ for each $t \in \langle 0,\infty)$, where $n(t) : \langle 0,\infty) \longrightarrow R$ is a continuous function.

(B3) $\int_0^t (t-s)^{-\alpha} e^{-a(t-s)} n(s)\,ds$ is a ψ-bounded function on the interval $\langle 0,\infty)$.

(B4) $C_\alpha \int_0^t (t-s)^{-\alpha} e^{-a(t-s)} n(s)\ \operatorname{sgn} \omega^+(s)\ \psi(\omega^+(s))ds \leq q\,\psi(t)$ for each $t \geq 0$ where $0 \leq q < 1$, $\omega^+(s) = \max\{0, \omega(s)\}$ and C_α means the constant from (A1).

Theorem 5. If B1 -B4 are satisfied then there exists a unique mild ψ-bounded solution of (E2, C_0) on the interval $\langle 0, \infty)$.

Proof. The idea of the proof is due to [Š], whereby the conditions here are analogous to those in that paper. The proof is based on the Banach fixed point theorem.

Put $S_1 = \{y \in F; y(0) = h(0)\}$, whereby we extend $y \in S_1$ for $\Theta \leq 0$ by $y(\Theta) = h(\Theta)$. We define the operator $T: S_1 \longrightarrow S_1$ as follows:

$(Ty)(\Theta) = h(\Theta)$ for $\Theta \leq 0$,

$(Ty)(t) = e^{-At} h(0) + \int_0^t e^{-A(t-s)} f(s, y_{\omega(s)})ds$ for $t \geq 0$.

Now we show that $T: S_1 \longrightarrow S_1$. We have that

$|Tx(t)|_\alpha \leq Ce^{-at}|h(0)|_\alpha + C_\alpha \int_0^t e^{-a(t-s)}(t-s)^{-\alpha}|f(s, x_{\omega(s)})|ds \leq Ce^{-at}|h(0)|_\alpha +$
$+ C_\alpha \int_0^t e^{-a(t-s)}(t-s)^{-\alpha}|f(s,0)|ds + C_\alpha \int_0^t e^{-a(t-s)}(t-s)^{-\alpha} n(s) \|x_{\omega(s)}\| ds \leq$
$\leq Ce^{-at}|h(0)|_\alpha + C_\alpha \int_0^t e^{-a(t-s)}(t-s)^{-\alpha}|f(s,0)|ds + C_\alpha \int_0^t e^{-a(t-s)}(t-s)^{-\alpha} n(s)(\|h\| +$
$+ |||x|||\ \operatorname{sgn}\ \omega^+(s)\psi(\omega^+(s))ds$.

Thus there is a $C > 0$ such that

$|Tx(t)|_\alpha \leq C\,\psi(t)$ for $0 \leq t < \infty$.

T is a contraction. In fact,

$$\frac{|Tx_1 - Tx_2(t)|_\alpha}{\psi(t)} = \frac{1}{\psi(t)} \int_0^t |e^{-A(t-s)}(f(s, x_{1,\omega(s)}) - f(s, x_{2,\omega(s)})|_\alpha\ ds \leq$$
$$\leq \frac{1}{\psi(t)} \int_0^t C_\alpha (t-s)^{-\alpha} e^{-a(t-s)} n(s)\ \|x_{1,\omega(s)} - x_{2,\omega(s)}\|\ ds \leq$$
$$\leq \frac{1}{\psi(t)} \int_0^t C_\alpha (t-s)^{-\alpha} e^{-a(t-s)} n(s)\ |||x_1 - x_2|||\ \operatorname{sgn}\ \omega^+(s)\psi(\omega^+(s))ds \leq q|||x_1 - x_2|||$$

for $0 \leq t < \infty$.

Thus $|||Tx_1 - Tx_2||| \leq q\,|||x_1 - x_2|||$.

Now we shall prove an existence theorem in the partially ordered Banach space X^α. The Tarski fixed point theorem is here of no use, because X^α in general is no conditionally complete lattice, as e.g. W_2^1 does not. Instead of the assumptions B1 - B5 from the previous theorem, an assumption on the monotonicity of f as well as some special growth assumption of f are used.

Suppose that $(X, \leqq)$ is an ordered real Banach space (for the definition see [He].

Let $A:X \to X$ be a sectorial operator with the compact resolvent and let $(\lambda I+A)^{-1}$ be increasing for all λ such that Re $\lambda \geqq \lambda_0$. According to Exercise 6 in [He, p.60], $e^{-At} \geqq 0$ for each $t \geqq 0$.

Let $f=f(x,y):\langle 0,\infty) \times C \to X$ be a continuous function, where C and the norm $\|\cdot\|$ have the same meaning as above.

The space X^{α} and hence $C((-\infty,0\rangle, X^{\alpha})$ is again an ordered Banach space with natural ordering in C.

Let two following assumptions hold:

(i) For each $z_1, z_2 \in C$, $z_1 \leqq z_2$ implies $f(s,z_1) \leqq f(s,z_2)$.

(ii) There is a continuous function $f_1:\langle 0,\infty) \times C \to X$ such that

$\|f(s,z)\|_X \leqq \max\{|f_1(s,z_1)|_X, |f_1(s,z_2)|_X\}$ for all $z \in C$, $z_1 \leqq z \leqq z_2$.

Definition. A continuous function $u_1:R \to X^{\alpha}$ will be called a lower solution (an upper solution) of the equation (E2) if it satisfies

$$u_1(t) \leqq e^{-At}u(0)+ \int_0^t e^{-A(t-s)} f(s,u_{1,\omega(s)})ds,$$

$$(u_1(t) \geqq e^{-At}u(0)+ \int_0^t e^{-A(t-s)} f(s,u_{1,\omega(s)})ds) \text{ for each } t \geqq 0.$$

Theorem 6. Let u_1 and u_2, respectively, be a lower and an upper solution, respectively, of the equation (E2) such that $u_1(\Theta)=h_1(\Theta)$ and $u_2(\Theta)=h_2(\Theta)$ for each $\Theta<0$, where $h_1,h_2 \in C$, $h_1(\Theta) \leqq h_2(\Theta)$ for each $\Theta \leqq 0$, $h_1(0)=h_2(0) \in X^{\alpha+\varepsilon}$ for some $\varepsilon>0$, for which $\alpha+\varepsilon<1$.

Let $h' \in C$ be a function such that $h_1(\Theta) \leqq h'(\Theta) \leqq h_2(\Theta)$ for each $\Theta \leqq 0$. Then there exists at least one mild solution of the problem

$$\frac{du}{dt} + Au = f(t,u_{\omega(t)}),$$

$$u_0 = h'$$

and such that $u_1(t) \leqq u(t) \leqq u_2(t)$ for each $t \geqq 0$.

Proof. We shall define the mapping $T:S_1 \to S_1$, where $S_1=\{y \in C(\langle 0,\infty), X^{\alpha}); y(0)=h'(0)\}$ such that:

1) y is an extension of the function h',

2) $(Ty)(t)=e^{-At}h'(0)+ \int_0^t e^{-A(t-s)} f(s,y_{\omega(s)})ds$ for each $t \geqq 0$,
$(Ty)(\Theta)=h'(\Theta)$ for each $\Theta \leqq 0$.

Because $e^{-At} \geqq 0$ and f is increasing, the operator T is increasing on S_1.

Since $u_1 \leq u_2$, we have $Tu_1 \leq Tu_2$. By $h_1 \leq h' \leq h_2$ we have $Tu_2 \leq u_2$ and $u_1 \leq Tu_1$. Thus $u_1 \leq Tu_1 \leq T^2u_1 \leq \ldots \leq T^2u_2 \leq Tu_2 \leq u_2$.

If we denote $v_n = T^n u_1$, then there exist $\bar{u} \in C(\langle 0,\infty),X^{\alpha})$ and a subsequence of v_n (without loss of generality we denote this subsequence again as $\{v_n\}$) such that v_n converges to u in the space $C(\langle 0,T\rangle,X^{\alpha})$ for each $T \in (0,\infty)$. To prove this we show two facts:

(i) To each $T>0$ there exists a $C(T)>0$ such that

$$|v_n(t)|_{\alpha+\varepsilon} \leq C(T) \text{ for each } t \in \langle 0,T\rangle .$$

(ii) $\{v_n\}$ is on the interval $\langle 0,T\rangle$ equicontinuous in X^{α}.

Then from (i), (ii) and from the compact embedding $X^{\alpha+\varepsilon}$ into X^{α} the existence of a $u \in C(\langle 0,T\rangle,X^{\alpha})$ and of the subsequence v_n such that v_n converges to u in the space $C(\langle 0,T\rangle,X^{\alpha})$ follows.

If we take, step by step, $T=1,2,\ldots$ and by the Cantor diagonalization process we get the existence of such a subsequence that v_n converges to u on $C(\langle 0,T\rangle,X^{\alpha})$ for each $T \in (0,\infty)$. Then for this sequence the following is true:

$$v_{n+1}(t)=e^{-At}h'(0)+\int_0^t e^{-A(t-s)}f(s,v_{n,\omega(s)})ds \text{ for each } t \geq 0.$$

We can take a limit, because

1) $v_{n,\omega(s)}$ converges to $u_{\omega(s)}$ in C for each $s \in \langle 0,\infty)$ and hence $f(s,v_{n,\omega(s)})$ converges to $f(s,u_{\omega(s)})$ for each $s \in \langle 0,\infty)$.

2) $|f(s,v_{n,\omega(s)})|_X \leq \max\{|f_1(s,u_{1,\omega(s)})|_X, |f_1(s,u_{2,\omega(s)})|_X\}$, from where $|f(s,v_{n,\omega(s)})|_X$ is a bounded function for $s \in \langle 0,t\rangle$.

This implies that

$$\left| \int_0^t e^{-A(t-s)}f(s,v_{n,\omega(s)})ds \right|_{\alpha} \leq \int_0^t e^{-a(t-s)}(t-s)^{-\alpha}C_1(s)ds.$$

By the Lebesgue theorem we have

$$u(t)=e^{-At}h(0)+\int_0^t e^{-A(t-s)}f(s,u_{\omega(s)})ds \text{ for each } t \in \langle 0,\infty).$$

We have to prove (i) and (ii) by the standard way:

(i) $|v_{n+1}(t)|_{\alpha+\varepsilon} \leq Ce^{-at}|h(0)|_{\alpha+\varepsilon}+$

$+C_{\alpha+\varepsilon}\int_0^t e^{-a(t-s)}(t-s)^{-\alpha-\varepsilon}|f(s,v_{n,\omega(s)})|_X\, ds \leq Ce^{-At}|h(0)|_{\alpha+\varepsilon}+$

$+C_{\alpha+\varepsilon}\int_0^t e^{-a(t-s)}(t-s)^{-\alpha-\varepsilon}\max\{|f_1(s,u_{1,\omega(s)})|, |f_1(s,u_{2,\omega(s)})|\}ds \leq C(t)$.

(ii) $|v_{n+1}(t+\tau)-v_{n+1}(t)|_{\alpha} \leq |(e^{-A\tau}-I)e^{-At}h'(0)|_{\alpha}+|\int_0^t (e^{-A\tau}-I)e^{-A(t-s)}.$

$.f(s,v_{n,\omega(s)}))ds|_{\alpha}+|\int_t^{t+\tau}e^{-A(t+\tau-s)}f(s,v_{n,\omega(s)})ds|_{\alpha} \leq C_1\tau^{1-\alpha}+C_2\tau^{\varepsilon}$, where we have used that

$$|f(s,v_{n,\omega(s)})|_X \leq \max\{|f_1(s,u_{1,\omega(s)})|_X,\ |f_1(s,u_{2,\omega(s)})|_X\}.$$

Remarks. 1. In the case of ordinary differential equations and hence when $X^{\infty}=R$ and X^{∞} is a conditionally complete lattice, we could use the Tarski fixed point theorem and omit the assumption (ii) of the theorem.

2. Also, in the general case, the assumption (ii) could be replaced by various others, e.g. $|f(s,z)|_X \leq C(s)$, where C is a real continuous function, or this assumption can be replaced by

$$|f(s,z)|_X \leq k_1(s,|z_1|,|z_2|),\text{ where } k_1 \text{ is a real continuous function.}$$

Then we should need neither an upper solution nor a lower solution. In this case each sequence $u_n=T^n u_1$ contains a subsequence converging to a solution.

3. Since in the theorem we have shown that there is a subsequence of v_n which is convergent in the space $C(\langle 0,T\rangle,X^{\infty})$ and since the sequence v_n is nondecreasing, we have that the whole sequence v_n is convergent to the limit function, which is $\sup_n v_n(t)$. Of course, this limit process could be proceeded also with $T^n u_2$.

The following examples illustrate the last theorem.

Example 4. We shall consider the problem

$$\frac{du}{dt}-u''=u(t+1/2)+u(-1/2)$$
$$u(0,t)=u(\pi,t)=0$$

with the initial condition $u_0=0$.

Hence $f(t,u)=u(0)+u(-1-t)$, $\omega(t)=t+1/2$ and $f(t,u_{t+1/2})=u(t+1/2)+u(-1/2)$. Then we can construct a lower solution $u(t)=0$ and an upper solution

$$u_1(\Theta)=-\Theta \sin x \text{ for each } \Theta\leq 0 \text{ and } u_1(t)=t\sin x \text{ for each } t\geq 0.$$

Hence the assumptions of the previous theorem are fulfilled.

Example 5. We shall consider the problem

$\frac{du}{dt}-u_{xx}=\min\{\sin^3 x,u^3(t-1)\}+u(-1/2)$, which can be written as $\frac{du}{dt}-u_{xx}=f(t,u_{t+1})$, where $f(t,u)=\{\min \sin^3 x,u^3(0)\}+u(-3/2-t)$.

It is easy to see that this function fulfils the assumptions of the previous theorem.

Now, we can take $u(t)=0$ for all $t\in(-\infty,\infty)$ as a lower solution and $v(t)=2cte^t\sin x$ for all $t\geq 0$, $v(-1/2)=x\sin x$ for some $c>1$.

We extend v(t) into $(-\infty,-1/2)\cup(-1/2,0)$ such that v be a nonnegative continuous function on $(-\infty,\infty)$.

We shall show that v(t) is an upper solution of our problem and hence it suffices to show that

$$v(t)\geq e^{-At}v(0)+\int_0^t e^{-A(t-s)}\min\{\sin^3 x, 8c^3(s+1)^3e^{3s+3}\sin^3 x\}\,ds+$$
$$+c\int_0^t e^{-A(t-s)}\sin x\,ds$$

so that

$$v(t)\geq\int_0^t e^{-A(t-s)}\sin^3 x\,ds+c\int_0^t e^{-A(t-s)}\sin x\,ds=3/4\int_0^t e^{-(t-s)}\sin x\,ds-$$
$$-\int_0^t e^{-9(t-s)}(\sin 3x)/4\,ds+c\int_0^t e^{-(t-s)}\sin x\,ds=3/4(1-e^{-t})\sin x+$$
$$+c(1-e^{-t})\sin x-\int_0^t e^{-9(t-s)}\frac{\sin 3x}{4}\,ds.$$

From the fact that $\sin x\geq-\sin 3x$ for each $x\in\langle 0,\pi\rangle$ it suffices to show that

$$2cte^t\sin x=v(t)\geq(1-e^{-t})\sin x(1+c),\ \text{thus}\ 2ct\,e^t\geq(1-e^t)(1+c)$$

for each $t\geq 0$.

But this is true for all sufficiently large c ($c>1$). Hence the assumptions of the previous theorem are fulfilled.

R e f e r e n c e s

[He] D. HENRY: Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math. No 840, Springer-Verlag, 1981.

[Hi,Phi] E. HILLE and R.S. PHILLIPS: Functional Analysis and Semigroups, Amer. Math. Soc, Providence, 1957.

[Mi] J. MILOTA: Stability and saddle point property for a linear autonomous functional parabolic equation, Comment. Math. Univ. Carolinae 27(1986), 87-101.

[Š] V. ŠEDA: Functional differential equations with deviating argument, Preprint.

[Ta] A.E. TAYLOR: Functional Analysis, John Wiley and Sons, Inc., New York 1967 (Czech translation: Academia, Prague, 1973).

ÚAMaVT UK, Mlynská dolina, 842 15, Bratislava, Czechoslovakia

(Oblatum 22.6. 1987)