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Abstract: It is well-known that if a subcategory of a sufficiently nice category $\mathcal{A}$ has a cowell-powered epireflective hull, then it has a reflective hull in $\mathcal{A}$ .A recent paper of R.E. Hoffmann shows that this amounts to characterize cowell-powered reflective subcategories in $\mathcal{A}$. We improve this result by dealing with a more general class of subcategories. The extension is shown to be particularly relevant in the category of topological spaces through a connection with the work of J.M. Harvey.

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Let $\mathfrak{B}$ be a subcategory (always assumed to be full and isomorphism-closed) of a well-complete (i.e. well-powered and complete) and cowell-powered category $\mathcal{A}$. The following facts are well-known (see [2],[3],[4]):

(1) Any subcategory of $\mathcal{A}$ has an epireflective hull.

(2) A subcategory $\mathfrak{D}$ of $\mathcal{A}$ with $\mathfrak{B} \subseteq \mathfrak{D} \subseteq \overline{\mathfrak{B}}$ (where $\overline{\mathfrak{B}}$ is the epireflective hull of $\mathfrak{B}$ in $\mathcal{A}$) is reflective in $\mathcal{A}$ if and only if it is epireflective in $\overline{\mathfrak{B}}$.

(3) $\overline{\mathfrak{B}}$ is well-complete.

From this it follows that

(4) If $\overline{\mathfrak{B}}$ is cowell-powered, then $\mathfrak{B}$ has a reflective hull in $\mathcal{A}$.

This condition on $\overline{\mathfrak{B}}$ obviously calls for an improvement: it is too unstable (it depends too heavily on $\mathfrak{B}$ itself) and it is suspected to be rather strong (for an example of an epireflective subcategory of $\mathcal{T}op$, the category of topological spaces which is not cowell-powered, see [2]). The first result of this paper gives a refinement of (1). This leads to a corresponding improvement of (4). The fact that this is a significant improvement is shown through a connection with a recent paper of J.M. Harvey ([1]). Note that J. Adámek and J. Rosický have found two reflective subcategories of a well and cowell-complete category having their intersection not reflective, and hence having no reflective hull (see [5]).

First, we need a "local" version of cowell-poweredness. If $\mathcal{B}$ is a subcategory of $\mathcal{C}$, we say that $\mathcal{C}$ is $\mathcal{B}$-<u>cowell-powered</u> if any object of $\mathcal{C}$ is the domain of a representative set of epimorphisms with codomains in $\mathcal{B}$. Denote by $Es(\mathcal{B})$ (respectively $P(\mathcal{B})$) the subcategory of $\mathcal{C}$ having as objects the extremal subobjects (resp. the products) of those in $\mathcal{B}$. The "local form" of (1) says that if $\mathcal{C}$ is an $Es(P(\mathcal{B}))$-cowell-powered well-complete category, then $\mathcal{B}$ has an epireflective hull (which has $Es(P(\mathcal{B}))$ as its class of objects). We improve this result by removing the "P" part:

Theorem. Let $\mathcal{B}$ be a subcategory of a well-complete category $\mathcal{C}$. If $\mathcal{C}$ is $Es(\mathcal{B})$-cowell-powered, then $\mathcal{B}$ has an epireflective hull (which has has $Es(P(\mathcal{B}))$ as its class of objects).

Proof. Remark that $\mathcal{C}$ is an (Epi, Extremal mono) category (see [3]). Any epireflective subcategory of $\mathcal{C}$ being closed for products and extremal subobjects, we have only to show that $Es(P(\mathcal{B}))$ is epireflective.

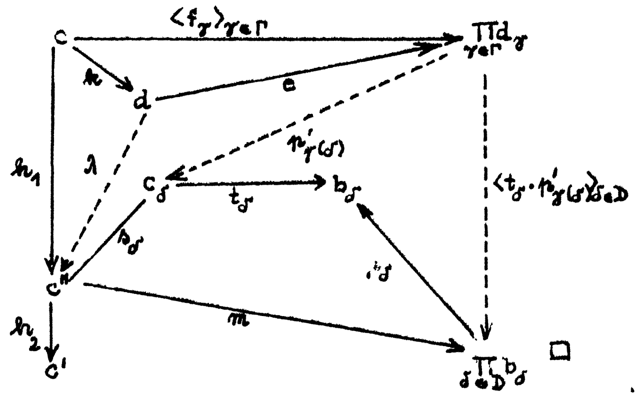
For an object c of $\mathcal{C}$, consider a representative set $\{f_\gamma : c \to d_\gamma \mid \gamma \in \Gamma\}$ of the epimorphisms from c to an object of $Es(\mathcal{B})$, and

$$\langle f_\gamma \rangle_{\gamma\in\Gamma} = ek : c \longrightarrow d \longrightarrow \prod_{\gamma\in\Gamma} d_\gamma$$

the (Epi, Extremal mono)-factorization of the induced morphism from c to the product $\prod_{\gamma\in\Gamma} d_\gamma$. We prove that k is the required reflection morphism.

Let $h : c \longrightarrow c'$ be a morphism with c' in $Es(P(\mathcal{B}))$ and $h_2 h_1 : c \longrightarrow c'' \longrightarrow c$ its (Epi, Extremal mono)-factorization. c" being also in $Es(P(\mathcal{B})$, there exist a set D and an extremal mono $m : c'' \longrightarrow \prod_{\delta\in D} b_\delta$ with each b_δ in $\mathcal{B}$. For each $\delta \in D$, let $t_\delta s_\delta : c'' \longrightarrow c_\delta \longrightarrow b_\delta$ be the (Epi, Extremal mono)-factorization of $p_\delta m$, where p_δ is the canonical projection $\prod_{\delta\in D} b_\delta \longrightarrow b_\delta$. Each $s_\delta h_1$, being an epi with codomain in $Es(\mathcal{B})$, can be identified (via an isomorphism) with $f_{\gamma(\delta)}$ for some $\gamma(\delta) \in \Gamma$.

Then $p'_{\gamma(\delta)} \cdot \langle f_\gamma \rangle_{\gamma\in\Gamma} = s_\delta h_1$ (where $p'_{\gamma(\delta)}$ is the canonical projection $\prod_{\gamma\in\Gamma} d_\gamma \longrightarrow d_{\gamma(\delta)} = c_\delta$). k being an epi and m an extremal mono, the equality $(\langle t_\delta \cdot p'_{\gamma(\delta)} \rangle_{\delta\in D}) \cdot ek = mh_1$ induces a unique $\lambda : d \longrightarrow c''$ such that $\lambda k = h_1$ and $m\lambda = (\langle t_\delta \cdot p'_{\gamma(\delta)} \rangle_{\delta\in D}) \cdot e$. Because k is epi, $h_2\lambda$ is unique such that $h_2 \lambda k = h$.



Corollary. Let $\mathcal{B}$ be a subcategory of a well-complete cowell-powered category $\mathcal{A}$ such that $\overline{\mathcal{B}}$ (the epireflective hull of $\mathcal{B}$) is $\mathcal{B}$-cowell-powered. Then $\mathcal{B}$ is cowell-powered and

a) $\mathcal{B}$ is closed for products and extremal subobjects in $\overline{\mathcal{B}}$ if and only if $\mathcal{B}$ is reflective in $\mathcal{A}$. In this case $\overline{\mathcal{B}}$ is cowell-powered.

b) If $\mathcal{B}$ is the intersection of reflective subcategories of $\mathcal{A}$, then it is reflective in $\mathcal{A}$ and $\overline{\mathcal{B}}$ is cowell-powered.

c) If $\overline{\mathcal{B}}$ is $Es(\mathcal{B})$-cowell-powered, then $\mathcal{B}$ has a reflective hull.

Proof. The inclusion of $\mathcal{B}$ in $\overline{\mathcal{B}}$ preserves epimorphisms (see [4] for example), from which it follows that $\mathcal{B}$ is cowell-powered.

a) ($\Longleftarrow$) This follows easily from the facts that $\overline{\mathcal{B}}$ is a (Epi, Extremal mono) category and $\mathcal{B}$ is epireflective in $\overline{\mathcal{B}}$.

($\Longrightarrow$) In Theorem 1, take $\mathcal{C} = \overline{\mathcal{B}}$. Then the equalities $Es(\mathcal{B}) = \mathcal{B} = Es(P(\mathcal{B}))$ imply the epireflectivity of $\mathcal{B}$ in $\overline{\mathcal{B}}$, and hence the reflectivity of $\mathcal{B}$ in $\mathcal{A}$. The cowell-poweredness of $\overline{\mathcal{B}}$ follows from [4].

b) Let $\mathcal{B} = \bigcap_{\gamma \in \Gamma} \mathcal{B}_\gamma$ with each $\mathcal{B}_\gamma$ reflective in $\mathcal{A}$. Then $\overline{\mathcal{B}} \subseteq \overline{\mathcal{B}}_\gamma$ for each $\gamma \in \Gamma$, and this, with the fact that they are both reflective in $\mathcal{A}$, implies that an extremal mono of $\overline{\mathcal{B}}$ is an extremal mono of $\overline{\mathcal{B}}_\gamma$ for each $\gamma \in \Gamma$. $\mathcal{B}_\gamma$ being closed for the extremal subobjects in $\overline{\mathcal{B}}_\gamma$, we conclude that $\mathcal{B}$ is closed for the extremal subobjects of $\overline{\mathcal{B}}$. As it is clearly closed for products, we have the result by part a).

c) This follows from the theorem and points (2) and (3) above. □

To make the connection with the terminology and results of [1], we must recall some of its definitions: given a subcategory $\mathcal{B}$ of $\mathcal{A}$, we will say that a morphism $f:Y\rightarrow Y$ in $\mathcal{A}$ is $\mathcal{B}$-generating and that $\mathcal{B}$ is f-generated if for any $r,s:Y \rightrightarrows Z$ in $\mathcal{A}$ with Z in $\mathcal{B}$ and such that rf=sf, we have r=s; a $\mathcal{B}$-generating morphism with co-domain in $\mathcal{B}$ is a $\mathcal{B}$-epi; $\mathcal{B}$ is cowell-powering (in $\mathcal{A}$) if each object in $\mathcal{A}$ is the domain of a representative set of $\mathcal{B}$-epis; $\mathcal{B}_0$ (respectively $\mathcal{B}_1$) is the maximal (full) subcategory of $\mathcal{A}$ which is f-generated for any $\mathcal{B}$-epi (resp. $\mathcal{B}$-generating) f. The following facts are obvious or follow immediately from the remarks at the beginning of [1]:

i) $\mathcal{B}$ is closed under extremal subobjects in $\overline{\mathcal{B}}$ if and only if it is closed under extremal subobjects in $\mathcal{B}_1$ (resp. $\mathcal{B}_0$).

ii) $\mathcal{B}$ is cowell-powering subcategory of $\mathcal{A}$ if and only if $\overline{\mathcal{B}}$ (resp. $\mathcal{B}_1, \mathcal{B}_0$) is $\mathcal{B}$-cowell-powered.

With this in mind, we immediately obtain the proposition 1, the corollary and the proposition 3 of [1] respectively from parts a), b) and c) of our corollary.

There are certain advantages in considering $\overline{\mathcal{B}}$, because it is Es(P($\mathcal{B}$)) (in well-complete cowell-powered categories), but it is the relative insensitivity of $\mathcal{B}_0$ and $\mathcal{B}_1$ to changes in $\mathcal{B}$, pointed out by Harvey through several examples in $\mathcal{T}op$, that shows that the weakening of the condition " $\overline{\mathcal{B}}$ is cowell-powered" to " $\overline{\mathcal{B}}$ is Es($\mathcal{B}$)-cowell-powered" is a significant one.

R e f e r e n c e s

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