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INFLATED MAPPINGS FOR SINGULARITIES OF CODIMENSION ≤ 2

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Abstract: Singularities of an imperfect bifurcation problem $F(u,\lambda,\alpha)=0$ of codimention ≤ 2 are related to simple roots of auxiliary operators (inflated mappings). All generic cases are discussed.

Key words: Imperfect bifurcation problems, organizing centre, classification, numerical approximation.

Classification: 47H15, 65J15, 58C27, 14B05

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1. Introduction. Let U and Y be Banach spaces. We consider an operator $F:U\times\mathbb{R}_1\times\mathbb{R}_k\longrightarrow Y$. In a bifurcation context, the variable x of $F=F(x)$ is a triple $x=(u,\lambda,\alpha)$ where u is the state variable, λ is a control parameter and α is the parameter of an imperfection.

A point $x_0=(u_0,\lambda_0,\alpha_0)$ is called the singular point of F if

(1.1) $$F(x_0)=0$$

(1.2) $$\dim \operatorname{Ker} F_u(x_0)=m\geq 1$$

where F_u is the partial Fréchet derivative of F (at x_0) w.r.t. the variable u, and $\operatorname{Ker} F_u(x_0)$ is the kernel of $F_u(x_0):U\longrightarrow Y$.

We assume $F_u(x_0):U\longrightarrow Y$ to be Fredholm operator with index zero and $F\in C^{\infty}(X,Y)$ where X is a neighbourhood of x_0.

The singular point x_0 satisfying (1.1), (1.2) is not isolated in general. There is an idea to seek for "the most singular" point x_0 which is locally available. Such a point is called an organizing centre of the operator F. In fact, the knowledge of an organizing centre makes it possible to describe (at least qualitatively) the solution sets

$$S_\alpha=\{(u,\lambda):F(u,\lambda,\alpha)=0,\ (u,\lambda)\text{ close to }(u_0,\lambda_0)\}$$

for all parameters α close to α_0.

The organizing centre could be linked with a simple root of an auxiliary operator $\mathcal{F}$ which is defined by means of F and its partial derivatives.

It is used to call such an operator $\mathcal{F}$ the inflated mapping of F.

The classification of organizing centres is developed in terms of singularities of the germs of smooth mappings $g: \mathbb{R}_m \times \mathbb{R}_1 \times \mathbb{R}_k \longrightarrow \mathbb{R}_m$, see [1]. It is assumed that g is the operator of a bifurcation equation to the problem (1.1). In [2], we have suggested a very simple idea to transform the defining conditions for a particular singularity of g into defining conditions of the relevant organizing centre x_0 (i.e. into a definition of $\mathcal{F}$).

In this paper, we demonstrate the idea assuming that the particular singularity of the bifurcation equation has codimension less than or equal to two. This condition means a restriction on the complexity of the structure which is simulated by the equation $F(u, \lambda, \alpha)=0$.

2. Reduction to a bifurcation equation. Let us recall basic ideas of Liapunov-Schmidt reduction of the equation (1.1).

First, we choose a linear bounded operator $L:U \longrightarrow \mathbb{R}_m$, fulfilling the following property: If $v \in \text{Ker } F_u(x_0)$ and $Lv=0$ then $v=0$. Then we define a projection

$$\Pi : U \longrightarrow \text{Ker } F_u(x_0)$$

satisfying the following implication: If $u \in U$ then $\Pi u = v \in \text{Ker } F_u(x_0)$ and $Lv = Lu$. Let Π^c be the complement of Π, i.e. $\Pi^c = I - \Pi$ (I is the identity $U \longrightarrow U$). We set $W = \Pi^c(U)$, i.e. $W = \{ v \in U : Lv = 0 \}$.

Let $\mathcal{R}(F_u(x_0))$ be the range of $F_u(x_0)$; the range is closed. There exists a bounded projection

$$Q: Y \longrightarrow \mathcal{R}(F_u(x_0)).$$

Let Q^c be its complement. Then we can split Y so that

$$Y = \mathcal{R}(F_u(x_0)) \oplus Q^c(Y)$$

where both components are closed and $\dim Q^c(Y) = \dim \text{Ker } F_u(x_0) = m$.

For each $r \in Y$ there exists a unique $z \in U$ such that $F_u(x_0)z = Qr$, $Lz=0$. By setting $F_u^+(x_0)r = z$, a linear bounded operator

$$F_u^+(x_0): Y \longrightarrow W \tag{2.1}$$

is defined. Note that $F_u^+(x_0)$ is an infinite dimensional analogue of the prescribed range/null space generalized inverse, see [6].

The condition (1.1) is reduced to a so called bifurcation equation, see the coming (2.3): Assuming $(v, \lambda, \alpha) \in \text{Ker } F_u(x_0) \times \mathbb{R}_1 \times \mathbb{R}_k$, we define $w \in U$ to be the solution to

$$QF(w+v, \lambda, \alpha) = 0, \quad Lw = 0 \quad (\text{i.e. } w \in W). \tag{2.2}$$

The Implicit Function Theorem yields

$$w=w(v,\lambda,\alpha),\ w\in C^{\infty}(\mathcal{V},\mathcal{W})$$

where $\mathcal{V}\subset \operatorname{Ker} F_u(x_0)\times \mathbb{R}_1\times \mathbb{R}_k$ and $\mathcal{W}\subset W$ are neighbourhoods of $(\Pi u_0, \lambda_0, \alpha_0)$ and $\Pi^c u_0$ respectively.

Define $\mathcal{U}=\{(u,\lambda,\alpha):(\Pi u,\lambda,\alpha)\in\mathcal{V},\ \Pi^c u\in\mathcal{W}$. It can be concluded that

$$F(u,\lambda,\alpha)=0,\ (u,\lambda,\alpha)\in\mathcal{U}$$

if and only if

(2.3) $$g(v,\lambda,\alpha)=0,\ (v,\lambda,\alpha)\in\mathcal{V}$$

where

(2.4) $$g(v,\lambda,\alpha)=Q^cF(v+w(v,\lambda,\alpha),\lambda,\alpha).$$

Since both $\operatorname{Ker} F_u(x_0)$ and $Q^c(Y)$ can be identified with $\mathbb{R}_m$, g could be understood as a germ of a C^{∞}-mapping

$$g:\mathbb{R}_m\times\mathbb{R}_1\times\mathbb{R}_k\longrightarrow\mathbb{R}_m$$

centred at $(v_0, \lambda_0, \alpha_0)$, where $v_0=\Pi u_0$.

3. Classification by codimension. Let us define a germ $h=h(v,\lambda)$ of C^{∞}-mapping

$$h:\mathbb{R}_m\times\mathbb{R}_1\longrightarrow\mathbb{R}_m$$

centred at (v_0,λ_0) such that $h(v,\lambda)=g(v,\lambda,\alpha_0)$. Thus, $g=g(v,\lambda,\alpha)$ is a k-parameter unfolding of the germ h. The unfolding parameter α is assumed to be close to α_0.

We recall the concept of codimension (notation: codim) of the germ h. We refer to [1], p. 121, for the rigorous definition. Roughly speaking, codim equals the least number of the unfolding parameters which could describe all qualitatively substantial perturbations of h.

Codimension is related to the singular point (v_0, λ_0) and the germ h itself. Because of the link between h and g, we relate the same codimension to the singular point $(v_0, \lambda_0, \alpha_0)$ of g.

Let us assume codim ≤ 2. As shown in [1], Theorem 2.1, p. 400, this implies

a) m=1, i.e. $h:\mathbb{R}_1\times\mathbb{R}_1\longrightarrow\mathbb{R}_1$ (nomenclature of the case: "bifurcation from a simple eigenvalue")

b) there are just six classes of singularities (v_0,λ_0) of $h:\mathbb{R}_2\longrightarrow\mathbb{R}_1$ having codim ≤ 2, namely

Table 1: Classification of singularities (codim ≤ 2)

Case No	nomenclature	codim
(i)	limit point	0
(ii)	simple bifurcation or an isola centre	1
(iii)	hyteresis point	1
(iv)	pitchfork	2
(v)	quartic fold	2
(vi)	asymmetric cusp	2

Each of the singularities listed above has to satisfy a set of defining conditions and some nondegeneracy conditions, see [1], Table 2.3, p. 198. These conditions can be viewed as definitions of (v_0, λ_0) and, because of the link between h and g, definitions of $(v_0, \lambda_0, \alpha_0)$ in particular cases (i) - (vi):

Table 2: Defining conditions $G(v_0, \lambda_0, \alpha_0)=0$ for singularities of codim ≤ 2

Case No	operator $G: \mathbb{R}_1 \times \mathbb{R}_1 \times \mathbb{R}_k \longrightarrow \mathbb{R}_\ell$	ℓ
(i)	$G \equiv (g, g_v)$	2
(ii)	$G \equiv (g, g_v, g_\lambda)$	3
(iii)	$G \equiv (g, g_v, g_{vv})$	3
(iv)	$G \equiv (g, g_v, g_\lambda, g_{vv})$	4
(v)	$G \equiv (g, g_v, g_{vv}, g_{vvv})$	4
(vi)	$G \equiv (g, g_v, g_\lambda, \det D^2 g)$	4

where

$$D^2 g = \begin{pmatrix} g_{vv}, g_{v\lambda} \\ g_{v\lambda}, g_{\lambda\lambda} \end{pmatrix}.$$

Table 3: Nondegeneracy conditions for singularities of codim ≤ 2

Case No	nondegeneracy conditions at $(v_0, \lambda_0, \alpha_0)$
(i)	$g_{vv} \neq 0,\ g_\lambda \neq 0$
(ii)	$g_{vv} \neq 0$, $\det D^2 g \neq 0$
(iii)	$g_{vvv} \neq 0,\ g_\lambda \neq 0$
(iv)	$g_{vvv} \neq 0,\ g_{v\lambda} \neq 0$
(v)	$g_{vvvv} \neq 0,\ g_\lambda \neq 0$
(vi)	$g_{vv} \neq 0,\ \frac{d^3}{dt^3} g(v_0+bt, \lambda_0+t, \alpha_0) \neq 0$ at $t=0$

where $b = -g_{v\lambda}/g_{vv}$

We resume that each particular singularity $(v_0, \lambda_0, \alpha_0)$ is a root of some operator

$$G: \mathbb{R}_1 \times \mathbb{R}_1 \times \mathbb{R}_k \longrightarrow \mathbb{R}_\ell .$$

One can check that if k=codim then ℓ=k+2; for, compare ℓ and codim of Table 2 and Table 1.

Our aim is to have G regular at $(v_0, \lambda_0, \alpha_0)$ in order to determine $(v_0, \lambda_0 \alpha_0)$ locally uniquely. To this end, we shall assume g to be a universal unfolding of h (for α being close to α_0). We refer to [1], p. 121, for the exact definition. The following remarks can elucidate the definition slightly:

a) k=codim

b) the solutions (v, λ) to the equation

$$h(v, \lambda) + \text{ a small perturbation } (v, \lambda) = 0$$

are "qualitatively the same" (in the sense of the so called contact equivalence) as the solutions (v, λ) to the equation $g(v, \lambda, \alpha)=0$ for an α beirlg close to α_0; note that only those (v, λ)´s are considered which are close enough to (v_0, λ_0).

Proposition 1. Let g be a universal unfolding of h. Let us assume $k \leq 2$ (i.e., m=1 and the singularity (v_0, λ_0) is classified by one of the cases (i) - (vi) in Table 1). Then the appropriate mapping

$$G: \mathbb{R}_1 \times \mathbb{R}_1 \times \mathbb{R}_k \longrightarrow \mathbb{R}$$

see Table 2, is regular at $(v_0, \lambda_0, \alpha_0)$, i.e. the singular point $(v_0, \lambda_0, \alpha_0)$ is defined locally uniquely as a simple root of G=0.

Proof. We are to prove that the Jacobian of G at $(v_0, \lambda_0, \alpha_0)$ does not vanish.

In [1], Table 3.2, p. 204, there are listed the necessary and sufficient algebraic conditions upon $(v_0, \lambda_0, \alpha_0)$ for α to be the parameter of a universal unfolding g of h, provided that the singularity (v_0, λ_0) of h is classified by one of the cases (i) - (vi). (The authors of [1] call it "the recognition problem for universal unfoldings".) What we have to do is very simple: For each of the particular cases (i) - (vi), one has to interpret the adequate condition in [1], p. 204, as the nondegeneracy condition for the Jacobian of the mapping G.

For the sake of brevity, we shall treat only the case (vi). i.e., the asymmetric cusp, which seems to be the most troublesome: According to the quoted result of [1], the germ $g=g(v, \lambda, \alpha)$ is a universal unfolding of h if and only if the 3x3 matrix (mind that $\alpha \in \mathbb{R}_2$)

$$M=\begin{pmatrix} g_v, g_{vv}, g_{v\lambda} \\ g_\alpha, g_{v\alpha}, g_{\lambda\alpha} \end{pmatrix},$$

is nonsingular at $(v_0, \lambda_0, \alpha_0)$.

The gradient of G at $(v_0, \lambda_0, \alpha_0)$ is the following 4x4 matrix:

$$N=\begin{pmatrix} g_v & , & g_\lambda & , & g_\alpha \\ g_{vv} & , & g_{v\lambda} & , & g_{v\alpha} \\ g_{v\lambda} & , & g_{\lambda\lambda} & , & g_{\lambda\alpha} \\ (\det D^2g)_v & , & (\det D^2g)_\lambda & , & (\det D^2g)_\alpha \end{pmatrix}$$

Let us remind that $g_v=g_\lambda=0$, see Table 2. Moreover, since $\det D^2g=0$ and $g_{vv}\neq 0$, we have

$$bg_{vv}+g_{v\lambda}=0$$
$$bg_{v\lambda}+g_{\lambda\lambda}=0$$

where $b=-g_{v\lambda}/g_{vv}$; it is understood that g and its partial derivatives are evaluated at $(v_0, \lambda_0, \alpha_0)$.

Multiply the first column of N by b and add it to the second column which results in the vector $(0,0,0,(\det D^2g)_\lambda+b(\det D^2g)_v)^T$. It can be easily verified that the last component of this vector is just

$g_{vv}\frac{d^3}{dt^3}g(v_0+bt, \lambda_0+t, \alpha_0)$ at $t=0$. Thus,

$$\det N=g_{vv}\frac{d^3}{dt^3}g(v_0+bt, \lambda_0+t, \alpha_0)\Big|_{t=0}\ \det M$$

which implies (see Table 3, (vi)) that $\det N\neq 0$.

It proves the regularity of G in the case (vi). The remaining cases can be treated similarly.

<u>Remark 1.</u> If $g_{vv}\neq 0$ then the condition (vi) of Table 2 can be formulated as follows:

(3.1) $$G(v_0, \lambda_0, \alpha_0, b)=0,\ b\in\mathbb{R}_1$$

for some $b\in\mathbb{R}_1$, where

(3.2) $$G\equiv(g, g_v, g_\lambda, bg_{vv}+g_{v\lambda}, bg_{v\lambda}+g_{\lambda\lambda}).$$

Thus, both the condition (vi) of Table 3 and the validity of (3.1) are equivalent to (vi) of Table 2 and Table 3. Moreover, if the assumptions of Proposition 1 hold then the above mapping $G:\mathbb{R}_1\times\mathbb{R}_1\times\mathbb{R}_2\times\mathbb{R}_1\longrightarrow\mathbb{R}_5$ is regular at $(v_0, \lambda_0, \alpha_0, b)$, where $b=-g_{v\lambda}/g_{vv}$ at $(v_0, \lambda_0, \alpha_0)$.

Remark 2. So far in this Section we have assumed $g: \mathbb{R}_m \times \mathbb{R}_1 \times \mathbb{R}_k \longrightarrow \mathbb{R}_m$ since we have identified both Ker $F_u(x_0)$ and $Q^c(Y)$ with $\mathbb{R}_m$, see Section 2. Naturally, all the conditions of Table 2 and Table 3 and Remark 1 can be formulated in terms of the original variable $v \in \text{Ker } F_u(x_0)$ of the mapping g: $:\text{Ker } F_u(x_0) \times \mathbb{R}_1 \times \mathbb{R}_k \longrightarrow Q^c(Y)$:

Since m=1, we choose a fixed $\eta \in F_u(x_0)$, $\eta \neq 0$, and replace

$$g_v := g_v \eta,\ g_{v\lambda} := g_{v\lambda}\eta,\ g_{vv} := g_{vv}\eta^2$$

(let us write $g_{vv}\eta^2$ instead of $g_{vv}\eta\eta$); g and its partials depend on (v, λ, α). Then the operators G of Table 2 act as follows: $G=G(v, \lambda, \alpha)$, $G:\text{Ker } F_u(x_0) \times \mathbb{R}_1 \times \mathbb{R}_k \longrightarrow [Q^c(Y)]^\ell$. Similar changes should be done in Remark 1.

4. Inflated mappings corresponding to singularities of codim ≤ 2. In this Section we are going to reformulate the defining conditions of Table 2 in terms of F and its partial derivatives. The idea of such a reformulation has already been mentioned in [2].

Convention. Any notion connected with a singular point $(v_0, \lambda_0, \alpha_0)$ of g in Section 3 is naturally transferred to the relevant singular point $(u_0, \lambda_0, \alpha_0)$ of F.

We assume those singular points $x_0=(u_0, \lambda_0, \alpha_0)$ of F which have codim ≤ 2, see the above Convention. Recall that codim ≤ 2 implies m=1, i.e. $L:U \longrightarrow \mathbb{R}_1$ (see Section 2). Thus, by definition, x_0 has to satisfy

(4.1) $$F(x_0)=0$$

and

(4.2) $$\exists\, v \in U: F_u(x_0)v=0,\ Lv \neq 0.$$

We recall $W=\{v \in U: Lv=0\}$. If a point $\eta_0 \in U$, $L\eta_0 \neq 0$ is given then Condition (4.2) is equivalent to

(4.3) $$\exists\, \eta_1 \in W: F_u(x_0)\eta_1 + F_u(x_0)\eta_0 = 0$$

(namely, $v=\eta_1+\eta_0$).

Note that (4.1) and (4.3) imply

(4.4) $$g=g_v\eta=0,\ \eta=\eta_1+\eta_0$$

at $(v_0, \lambda_0, \alpha_0)$.

Proposition 2. A singular point $x_0=(u_0, \lambda_0, \alpha_0)$ of codim ≤ 2 satisfies the defining conditions (3.1) if and only if there exist $b \in R_1$, $\eta_i \in W$, i=1,2, 3,4, such that $(u_0, \lambda_0, \alpha_0, b, \eta_1, \eta_2, \eta_3, \eta_4)$ is a root of an operator

$$\mathcal{F}: U\times\mathbb{R}_1\times\mathbb{R}_k\times\mathbb{R}_1\times[W]^4\longrightarrow[Y]^5$$

which is defined as follows: Let $\eta_0\in U$, $L\eta_0\neq 0$ be given. Then

$$\mathcal{F}(u,\lambda,\alpha,c,\xi_1,\xi_2,\xi_3,\xi_4)= \tag{4.5}$$

$$=\begin{pmatrix} F \\ F_u\xi_1+F_u\eta_0 \\ F_u\xi_2+F_\lambda \\ F_u\xi_3+(cF_{uu}v+F_{uu}\xi_2+F_{u\lambda})v \\ F_u\xi_4+c(F_{uu}\xi_2+F_{u\lambda})v+F_{uu}\xi_2^2+2F_{u\lambda}\xi_2+F_{\lambda\lambda} \end{pmatrix}$$

where $v:=\xi_1+\eta_0$; the values of F, F_u, F_λ, F_{uu} are understood to be evaluated at (u,λ,α).

The remaining components $b,\eta_1,\eta_2,\eta_3,\eta_4$ of the root have the following interpretations:

$$b=-\frac{g_{v\lambda}\eta}{g_{vv}\eta^2} \tag{4.6}$$

$$\eta_1+\eta_0\in \operatorname{Ker} F_u(x_0),\ L(\eta_1+\eta_0)\neq 0 \tag{4.7}$$

$$\eta_2=w_\lambda \tag{4.8}$$

$$\eta_3=(bw_{vv}\eta+w_{v\lambda})\eta \tag{4.9}$$

$$\eta_4=bw_{v\lambda}\eta+w_{\lambda\lambda}\ , \tag{4.10}$$

where $\eta=\eta_1+\eta_0$; the operator w and its partial derivatives are evaluated at $(\Pi u_0,\lambda_0,\alpha_0)$.

Proof. Let $x_0=(u_0,\lambda_0,\alpha_0)$ be a singular point of codim ≤ 2. Then (4.1), (4.3) and (4.4) hold. Note that (4.1) and (4.3) represent two conditions that a root of $\mathcal{F}$, see (4.5), must fulfil.

Let us fix $\eta\in\operatorname{Ker} F_u(x_0)$ setting $\eta=\eta_1+\eta_0$. The differentiation of (2.2) at $(\Pi u_0,\lambda_0,\alpha_0)$ by λ and v respectively gives

$$Q[F_uw_\lambda+F_\lambda]=0,\ Lw_\lambda=0 \tag{4.11}$$

$$Q[F_u\cdot(I+w_v)]\eta=0,\ Lw_v\eta=0. \tag{4.12}$$

Since $F_u\eta=0$ then (4.12) implies

$$w_v\eta=0. \tag{4.13}$$

Taking into account (4.13), further differentiation of (2.2) yields

$$Q[F_{uu}+F_uw_{vv}]\eta^2=0,\ Lw_{vv}\eta^2=0 \tag{4.14}$$

(4.15) $$Q[F_{uu}w_\lambda + F_{u\lambda} + F_u w_{v\lambda}]\eta = 0, \quad Lw_{v\lambda}\eta = 0$$

(4.16) $$Q[F_{uu}w_\lambda^2 + 2F_{u\lambda}w_\lambda + F_{\lambda\lambda} + F_u w_{\lambda\lambda}] = 0, \quad Lw_{\lambda\lambda} = 0.$$

Similarly, differentiating (2.4) and using (4.13),

(4.17) $$g_\lambda = Q^c[F_u w_\lambda + F_\lambda]$$

(4.18) $$g_{vv}\eta^2 = Q^c[F_{uu} + F_u w_{vv}]\eta^2$$

(4.19) $$g_{v\lambda}\eta = Q^c[F_{uu}w_\lambda + F_{u\lambda} + F_u w_{v\lambda}]\eta$$

(4.20) $$g_{\lambda\lambda} = Q^c[F_{uu}w_\lambda^2 + 2F_{u\lambda}w_\lambda + F_{\lambda\lambda} + F_u w_{\lambda\lambda}].$$

By virtue of (4.11) and (4.17), the condition $g_\lambda = 0$ is equivalent to the following one:

(4.21) $$\exists \eta_2 \in W: F_u\eta_2 + F_\lambda = 0$$

with the interpretation (4.8).

The last two conditions of (3.1) read (see Remark 2) as

(4.22) $$(bg_{vv}\eta + g_{v\lambda})\eta = 0$$

(4.23) $$bg_{v\lambda}\eta + g_{\lambda\lambda} = 0.$$

It is simple to conclude from (4.14) - (4.20) that (4.22) and (4.23) respectively are equivalent to the following conditions

(4.24) $$\exists \eta_3 \in W: F_u\eta_3 + (bF_{uu}\eta + F_{uu}\eta_2 + F_{u\lambda})\eta = 0$$

and

(4.25) $$\exists \eta_4 \in W: F_u\eta_4 + b(F_{uu}\eta_2 + F_{u\lambda})\eta + F_{uu}\eta_2^2 + 2F_{u\lambda}\eta_2 + F_{\lambda\lambda} = 0.$$

The interpretation of both η_3 and η_4 is clearly that of (4.9) and (4.10).

Note that (4.1),(4.3),(4.21), (4.24),(4.25) mean that $(u_0, \lambda_0, \alpha_0, b, \eta_1, \eta_2, \eta_3, \eta_4)$ is a root of $\mathcal{F}$, see (4.5). The equivalence of (3.1) and the mentioned conditions were checked throughout.

The proof is completed.

Let us review the inflated mappings for the remaining cases (i) - (v) of Table 2. We shall omit the proofs because they are similar to that of Proposition 2.

<u>Proposition 3.</u> A singular point $x_0 = (u_0, \lambda_0, \alpha_0)$ of codim ≤ 2 satisfies the defining conditions (i) of Table 2 if and only if there exists $\eta_1 \in W$ such that $(u_0, \lambda_0, \alpha_0, \eta_1)$ is a root of an operator

$$\mathcal{F}: U \times \mathbb{R}_1 \times \mathbb{R}_k \times W \longrightarrow [Y]^2$$

which is defined as follows: Let $\eta_0 \in U$, $L\eta_0 \neq 0$ be given. Then

$$(4.26) \qquad \mathcal{F}(u, \lambda, \alpha, \xi_1) = \begin{pmatrix} F \\ F_u \xi_1 + F_u \eta_0 \end{pmatrix}$$

where F and F_u are evaluated at (u, λ, α). The component η_1 is interpreted as (4.7).

Proposition 4. A singular point $x_0=(u_0, \lambda_0, \alpha_0)$ of codim ≤ 2 satisfies the defining conditions (ii) and (iii) of Table 2 respectively if and only if there exist $\eta_1 \in W$, $\eta_2 \in W$ such that $(u_0, \lambda_0, \alpha_0, \eta_1, \eta_2)$ is a root of an operator

$$\mathcal{F}: U \times \mathbb{R}_1 \times \mathbb{R}_k \times [W]^2 \longrightarrow [Y]^3$$

which is defined as follows: Let $\eta_0 \in U$, $L\eta_0 \neq 0$ be given. Then, for the cases (ii) and (iii) respectively,

$$(4.27) \qquad \mathcal{F}(u, \lambda, \alpha, \xi_1, \xi_2) = \begin{pmatrix} F \\ F_u \xi_1 + F_u \eta_0 \\ F_u \xi_2 + F_\lambda \end{pmatrix}$$

and

$$(4.28) \qquad \mathcal{F}(u, \lambda, \alpha, \xi_1, \xi_2) = \begin{pmatrix} F \\ F_u \xi_1 + F_u \eta_0 \\ F_u \xi_2 + F_{uu} v^2 \end{pmatrix}$$

where $v := \xi_1 + \eta_0$; F, F_u, F_λ, F_{uu} are evaluated at (u, λ, α). In both cases (ii) and (iii), the component η_1 is interpreted as (4.7). The component η_2 means

$$(4.29) \qquad \eta_2 = w_\lambda$$

and

$$(4.30) \qquad \eta_2 = w_{vv}\eta^2, \quad \eta = \eta_1 + \eta_0$$

for the cases (ii) and (iii) respectively; the derivatives of w are evaluated at $(\Pi u_0, \lambda_0, \alpha_0)$.

Proposition 5. A singular point $x_0=(u_0, \lambda_0, \alpha_0)$ of codim ≤ 2 satisfies the defining conditions (iv) and (v) of Table 2 respectively if and only if there exist $\eta_i \in W$, i=1,2,3 such that $(u_0, \lambda_0, \alpha_0, \eta_1, \eta_2, \eta_3)$ is a root of an operator

$$\mathcal{F}: U \times \mathbb{R}_1 \times \mathbb{R}_k \times [W]^3 \longrightarrow [Y]^4$$

where $\mathcal{F}$ is defined as follows: Let $\eta_0 \in U$, $L\eta_0 \neq 0$ be given. Then, for the cases (iv) and (v) respectively,

$$\mathcal{F}(u, \lambda, \alpha, \xi_1, \xi_2, \xi_3) = \begin{pmatrix} F \\ F_u \xi_1 + F_u \eta_0 \\ F_u \xi_2 + F_\lambda \\ F_u \xi_3 + F_{uu} v^2 \end{pmatrix}, \tag{4.31}$$

and

$$\mathcal{F}(u, \lambda, \alpha, \xi_1, \xi_2, \xi_3) = \begin{pmatrix} F \\ F_u \xi_1 + F_u \eta_0 \\ F_u \xi_2 + F_{uu} v^2 \\ F_u \xi_3 + (3F_{uu} \xi_2 + F_{uuu} v^2) v \end{pmatrix}, \tag{4.32}$$

where $v := \xi_1 + \eta_0$; $F, F_u, F_\lambda, F_{uu}, F_{uuu}$ are evaluated at (u, λ, α). In both cases (iv) and (v), the component η_1 is interpreted via (4.7). The component η_2 means (4.29) and (4.30) respectively. The interpretation of η_3 is as follows:

$$\eta_3 = w_{vv} \eta^2, \quad \eta = \eta_1 + \eta_0 \tag{4.33}$$

and

$$\eta_3 = w_{vvv} \eta^3, \quad \eta = \eta_1 + \eta_0 \tag{4.34}$$

respectively.

A numerical application of Propositions 2 - 5 can follow the next recommendation: Guess the kind of a singularity x_0 of codim ≤ 2 which is to be found, then choose the corresponding mapping $\mathcal{F}$ and find its root. To this end it is vital to know that the root is simple (for Newton method to be applicable), i.e. that the gradient $D\mathcal{F}$ of $\mathcal{F}$, being evaluated in the root, is continuously invertible, i.e. the mapping $\mathcal{F}$ is regular in the root.

Theorem 1. The inflated mappings $\mathcal{F}$ given by (4.5),(4.26),(4.27),(4.28), (4.31) and (4.32) respectively are regular at a root (u_0, λ_0, α_0, plus the relevant auxiliary variables) if and only if the corresponding mappings G (see (3.2) and the cases (i) - (v) of Table 2, respectively) are regular at $(\Pi u_0, \lambda_0, \alpha_0)$.

Proof. We have already proved the above statement for the case (ii), see [2], Proposition 2. The remaining cases can be treated in the same way. Namely, the inverse to $D\mathcal{F}$ can be constructed explicitly by means of the inverse to DG. Since it involves just straightforward calculations, we believe, we could omit it here.

Remark 3. Both Proposition 1 and Theorem 1 provide sufficient conditions for a root of $\mathcal{F}$ to be simple.

5. Roots of inflated mappings. Suppose $x_0=(u_0, \lambda_0, \alpha_0)$ to be a singularity of codim ≤ 2. Let $(u_0, \lambda_0, \alpha_0$, plus the auxiliary variables) be the relevant root of the inflated mapping $\mathcal{F}$. The aim of this Section is to show that the auxiliary components of the root provide a useful piece of information concerning the bifurcation diagram of the equation $F(u, \lambda, \alpha_0)=0$, i.e. the solution set

$$S=\{(u,\lambda)\in U\times \mathbb{R}_1 : F(u,\lambda,\alpha_0)=0\}. \tag{5.1}$$

Namely, a parametric description of S in a neighbourhood of (u_0, λ_0) can be obtained.

We recall the notion of bifurcation equation (2.3). Since $\dim \operatorname{Ker} F_u(x_0)=1$, choose $\eta \in \operatorname{Ker} F_u(x_0)$, $\eta \neq 0$, and substitute $v:=v_0+t\eta$, $v_0=\Pi^c u_0$, $\lambda:=\lambda_0+\mu$, $\alpha:=\alpha_0$ into (2.3) for t and $\mu \in \mathbb{R}_1$, assuming that $|t|$ and $|\mu|$ are small enough. It is natural to set $\eta:=\eta_1+\eta_0$, see Propositions 2 - 5.

We have to specify Q^c, see Section 1. It could be defined as follows: Let $\langle\cdot,\cdot\rangle$ be the pairing of Y and its dual Y^*. Denote $F_u^*(x_0)$ the adjoint operator to $F_u(x_0)$. Scale $\eta^* \in \operatorname{Ker} F_u^*(u_0)$ such that $\langle \eta, \eta^*\rangle=1$. If $y \in Y$ then define

$$Q^c y=\langle y, \eta^*\rangle \eta .$$

The means of identification of both $\operatorname{Ker} F_u(x_0)$ and $Q^c(Y)$ with $\mathbb{R}_1$ are obvious now. It is natural to define $h: \mathbb{R}_1\times\mathbb{R}_1 \longrightarrow \mathbb{R}_1$ such that

$$h(t,\mu)=\langle g(v_0+t\eta, \lambda_0+\mu, \alpha_0), \eta^*\rangle .$$

Clearly, the solution set $\mathcal{Y}$ to the equation

$$h(t,\mu)=0 \tag{5.2}$$

is locally isomorphic with the solution set S, (5.1). The isomorphism is represented by the following formulas:

$$u=\Pi u_0+t\eta+w(\Pi u_0+t\eta, \lambda_0+\mu, \alpha_0) \tag{5.3}$$

$$\lambda=\lambda_0+\mu \tag{5.4}$$

for $t, \mu \in \mathbb{R}_1$, $|t|$ and $|\mu|$ small enough.

The point is that some terms of Taylor expansion

$$(5.5)\qquad w(\Pi u_0+t\eta,\ \lambda_0+\mu,\alpha_0)=\Pi^c u_0+tw_v\eta+\mu w_\lambda+\frac{t^2}{2}w_{vv}\eta^2+$$

$$+t\mu w_{v\lambda}\eta+\frac{\mu^2}{2}w_{\lambda\lambda}+\text{higher order terms (h.o.t.)}$$

are given by components of the root to $\mathcal{F}$. Here and in the sequel, the operators w and g and their partials are understood to be evaluated at $(\Pi u_0, \lambda_0, \alpha_0)$. The operator F and its partials are evaluated at $(u_0, \lambda_0, \alpha_0)$.

We shall demonstrate the idea on an example only: Suppose that x_0 is a point of a pitchfork bifurcation. Thus, due to our definition in Section 3, the conditions (iv) of both Table 2 and Table 3 hold. They are equivalent to the assumption

$$(5.6)\qquad h(t,\mu)=at^3+b\mu t+c\mu^2+\text{h.o.t.}$$

where $a\cdot b\neq 0$,

$$(5.7)\qquad a=\frac{1}{6}\langle g_{vvv}\eta^3,\eta^*\rangle\ ,\ b=\langle g_{v\lambda}\eta,\eta^*\rangle\ ,\ c=\frac{1}{2}\langle g_{\lambda\lambda},\eta^*\rangle\ .$$

Qualitative analysis (see e.g. [1], Proposition 9.2, p. 95) yields that the solution set $\mathcal{S}$ to (5.6) consists (locally) of two smooth branches $\mathcal{S}_1=\{(t,\mu):t=t(\mu),\ |\mu|\ \text{small}\}$ and $\mathcal{S}_2=\{(t,\mu):\ \mu=\mu(t),\ |t|\ \text{small}\}$ where both functions $t(\mu)$ and $\mu(t)$ are smooth, $t(0)=\mu(0)=0$. Substituting the functions $t=t(\mu)$ and $\mu=\mu(t)$ respectively into (5.2), one obtains the following expansions:

$$(5.8)\qquad t(\mu)=-\frac{c}{b}\mu+O(\mu^3)$$

$$(5.9)\qquad \mu(t)=-\frac{a}{b}t^2+O(t^3)$$

The local isomorphism (5.3), (5.4) implies that S consists (locally) of two smooth branches S_1, S_2; the branch S_i being isomorphic with $\mathcal{S}_i$. Applying Proposition 5 (namely the interpretations of η_1,η_2,η_3) to (5.5), and taking into account (4.13), one can read (5.3) as follows:

$$(5.10)\qquad u=u_0+t\eta+\mu\eta_2+\frac{t^2}{2}\eta_3+O(t\mu)+O(\mu^2)+\text{h.o.t.}$$

Formulas (5.10), (5.4) and the expansions (5.8), (5.9) are ready to supply a local approximation to branches S_1 and S_2. Of course, the approximation relies upon the constants a, b, c (see (5.6)) we have not fixed yet.

Let us note at this place that the constants a, b, c participate in the gradient DG which is connected with the gradient $D\mathcal{F}$ of the inflated mapping

$\mathcal{F}$. Since the gradient $D\mathcal{F}$ is evaluated in the course of Newton iterations, the constants a, b, c (namely, their approximations) are cheeply available as a by-product. Thus the following calculations could be economised.

By definition, see (2.4), and taking (4.13) into account,

$$(5.11)\quad \begin{cases} g_{vvv}\eta^3 = Q^c(F_{uuu}\eta + 3F_{uu}\eta w_{vv})\eta^2 \\ g_{v\lambda}\eta = Q^c(F_{u\lambda} + F_{uu}w_\lambda)\eta \\ g_{\lambda\lambda} = Q^c(F_{\lambda\lambda} + F_{uu}w_\lambda^2 + 2F_{u\lambda}w_\lambda) \end{cases}$$

Again, both $w_{vv}\eta^2$ and w_λ are to be replaced by η_3 and η_2 respectively. Resuming (5.7) and (5.11),

$$(5.12)\quad \begin{cases} a = \frac{1}{6}\langle F_{uuu}\eta^3 + 3F_{uu}\eta\eta_3, \eta^*\rangle \\ b = \langle F_{u\lambda}\eta + F_{uu}\eta\eta_2, \eta^*\rangle \\ c = \frac{1}{2}\langle F_{\lambda\lambda} + F_{uu}\eta_2^2 + 2F_{u\lambda}\eta_2, \eta^*\rangle . \end{cases}$$

We just proved

Proposition 6. Suppose $x_0=(u_0, \lambda_0, \alpha_0)$ to be a point of a pitchfork bifurcation; let $(u_0, \lambda_0, \alpha_0, \eta_1, \eta_2, \eta_3)$ be the relevant root of the mapping $\mathcal{F}$, see (4.31). Then the solution set S, see (5.1), consists (locally) of two smooth branches S_1 and S_2:

$$S_1 = \{(u,\lambda): u=u_0-(\tfrac{c}{b}\eta-\eta_2)\mu+O(\mu^2),\ \lambda=\lambda_0+\mu,\ \text{for } \mu\in \mathbb{R}_1,\ |\mu| \text{ small}\}$$

and

$$S_2 = \{(u,\lambda): u=u_0+t\eta-(\tfrac{a}{b}\eta_2-\tfrac{1}{2}\eta_3)t^2+O(t^3),\ \lambda=\lambda_0-\tfrac{a}{b}t^2+O(t^3),\ \text{for } t\in\mathbb{R}_1,\ |t| \text{ small}\}$$

where $\eta=\eta_1+\eta_0$ and the constants a, b, c are given by (5.12).

Solution sets S for the remaining singularities of codim ≤ 2 can be approximated similarly following the above example, see [7].

Remark 4. As we have already hinted, the gradient $D\mathcal{F}$ provides the leading terms of the equation (5.2) which contributes to a description of the solution set S. Let us note that $D\mathcal{F}$ is a source of much more complex information. For example, one can approximate transition sets (see [1], Chapter III) of the bifurcation problem $F(u,\lambda,\alpha)=0$ or one can (qualitatively) describe all the solution sets $S_\alpha=\{(u,\lambda):F(u,\lambda,\alpha)=0\}$ for α being close to

α_0, etc. We shall discuss it elsewhere (we refer to [7] for a preliminary version of the mentioned results).

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